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RECORDS OF PROCEEDINGS AT MEETINGS

SESSION NOVEMBER, 1913–JUNE, 1914.

Thursday, November 13th, 1913.

ANNUAL GENERAL MEETING.

Prof. A. E. H. LOVE, President, in the Chair.

Present twenty-three members and a visitor.

The minutes of the last meeting were read and confirmed.

Messrs. A. Korn and R. E. Powers were elected members.

Profs. A. Hurwitz, M. Noether, P. Painlevé, C. Segre, W. Voigt were elected honorary members.

The Treasurer, Sir Joseph Larmor, presented his Report, and on the motion of Lt.-Col. Cunningham, seconded by Mr. A. B. Grieve, it was received.

Dr. J. G. Leathem was reappointed Auditor.

On the motion of the President, seconded by Sir Joseph Larmor, it was agreed that a letter of condolence be sent to the relations of the late Mr. S. Roberts, a former President of the Society.

The President moved, and Dr. Hobson seconded, the following resolution, which was carried unanimously :—

> That the London Mathematical Society hereby places on record its sense of the deep debt of gratitude which it owes to Sir Joseph Larmor for his management of its financial affairs during the twenty-one years of his tenure of the office of Treasurer, and tenders to him its hearty thanks for the work that he has done in that capacity.

The members of Council and Officers for the new Session were elected as follows :—President, Prof. A. E. H. Love; Vice-Presidents, Dr. H. F. Baker and Prof. W. Burnside; Treasurer, Dr. A. E. Western; Secretaries, Mr. J. H. Grace and Dr. T. J. I'A. Bromwich; other members of the

Council, Mr. S. Chapman, Mr. E. Cunningham, Mr. A. L. Dixon, Prof. L. N. G. Filon, Prof. E. W. Hobson, Mr. J. H. Jeans, Mr. J. E. Littlewood, Prof. H. M. Macdonald, Major P. A. MacMahon, Mr. H. W. Richmond.

The following papers were communicated :—

*The Skew Isogram Mechanism : Mr. G. T. Bennett.

*Tauberian Theorems concerning Power Series and Dirichlet's Series whose Coefficients are Positive : Messrs. G. H. Hardy and J. E. Littlewood.

Note on Lambert's Series : Mr. G. H. Hardy.

*(i) The Connexion between Surfaces whose Lines of Curvature are Spherical and Surfaces whose Inflectional Tangents belong to Linear Complexes, (ii) Surfaces whose Systems of Inflectional Tangents belong to Systems of Linear Complexes : Mr. J. E. Campbell.

*On Integration with respect to a Function of Bounded Variation : Prof. W. H. Young.

The Computation of Cotes's Numbers, and their Values up to $n = 20$: Prof. W. W. Johnson.

*Some Ruler Constructions for the Covariants of a Binary Quantic : Mr. S. G. Soal.

†Analogues of Orthocentric Tetrahedra in Higher Space : Mr. T. C. Lewis.

*Closed Linkages and Poristic Polygons : Col. R. L. Hippisley.

ABSTRACTS.

Tauberian Theorems concerning Power Series and Dirichlet's Series whose Coefficients are Positive : Messrs. G. H. Hardy and J. E. Littlewood.

It was proved by Lasker and Pringsheim that, if

$$f(x) = \Sigma a_n x^n,$$

and

$$s_n = a_0 + a_1 + \dots + a_n \sim A n^{\alpha} L(n),$$

where

$$L(n) = (\log n)^{\alpha_1} (\log \log n)^{\alpha_2} \dots,$$

* Printed in this volume.

† Printed in *Proceedings*, Ser. 2, Vol. 12

the indices $a, a_1, a_2, \ldots$ being such that $n^{\alpha}L(n) \to \infty$, then

$$f(x) \sim \frac{A\Gamma(a+1)}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right)$$

as $x \to 1$. The principal object of the paper is to show that *the converse also is true when the coefficients a_n are positive.* Various generalisations are made, and analogous theorems proved for ordinary Dirichlet's series.

Note on Lambert's Series: Mr. G. H. Hardy.

Generalisation of theorems proved by the author (*Math. Annalen*, Vol. 64) and Knopp (*Crelle's Journal*, Vol. 142) concerning the behaviour of a "Lambert's series"

$$\Sigma a_n \frac{x^n}{1-x^n},$$

convergent for $|n| < 1$, as x tends to a rational point $e^{2p\pi i/q}$ on the circle of convergence.

(i) The Connexion between Surfaces whose Lines of Curvature are Spherical and Surfaces whose Inflectional Tangents belong to Linear Complexes, (ii) Surfaces whose Systems of Inflectional Tangents belong to Systems of Linear Complexes: Mr. J. E. Campbell.

Much study has been devoted to the system of surfaces characterized by the property that their lines of curvature are plane or spherical, and references will be made in Forsyth's *Differential Geometry*, Chapter IX. My own knowledge of the subject is almost entirely derived from Darboux's *Théorie Générale des Surfaces*, Livre IV, Ch. IX and XI, and I have been guided in my investigations by the results there presented. It is well known that Lie's contact transformation, by which spheres are transformed into straight lines, connects the geometry of lines of curvature on the surface with the geometry of asymptotic lines on another surface. I do not know, however, that any attempt has been made to apply this theory to the investigation of the properties of surfaces, whose lines of curvature are spherical, by investigating the properties of surfaces, whose inflectional tangents belong to systems of linear congruences. The first of the two papers here presented treats of Lie's contact transformation, and shows how the knowledge of either class of surface involves that of the other. The second investigates the properties of surfaces whose inflectional tangents belong to systems of linear complexes. The two papers are connected, but either can be read independently of the other. I have

tried to write them that they may be intelligible to one who has very little knowledge of the theory of surfaces, and therefore there is much in them, and especially in the first, that lays no claim to originality, except possibly in method.

The Computation of Cotes's Numbers and their Values up to $n = 20$: Prof. W. W. Johnson.

This paper contains the computed values of Cotes's numbers for values of n from 11 to 20, in continuation of the values up to $n = 10$ given by Cotes, and published in the *Harmonia Mensurarum.*

Attention is called to some curious cases, occurring in the computation of the divisibility of certain sums, whereby the denominators of the fractional values are rendered considerably smaller than might have been expected.

Some Ruler Constructions for the Covariants of a Binary Quantic: Mr. S. G. Soal.

In this paper a binary quantic and its covariants are represented by points which are taken to lie on a conic S.

It is first shown how to determine by ruler the polars of any point P of S with respect to the system of conics which pass through the marking points of the $C_{2,4}$ of a sextic.

One conic of the system has for a pole and polar pair any point P of S, and the line which denotes the $C_{2,2}$ of the polar 5-points of P with respect to the sextic.

The polar reciprocal with respect to S of this conic intersects S in the marking points of the covariant $C_{4,4}+I_2 C_{2,4}$.

The next step is to construct a certain joint quartic covariant of a quadratic and a sextic linear in the coefficients of each.

If for the quadratic is taken the unique $C_{3,2}$ of the sextic, this joint covariant becomes the $C_{4,4}+I_2 C_{2,4}$ determined above.

The reversal of this process leads to a ruler construction for the unique $C_{3,2}$ of the sextic.

The remaining quadratic and quartic covariants are then readily constructed.

In connection with the sextic the following result is of interest:—If the connector of the polar 2-points of each of the marking points of a sextic with respect to the remaining five points be constructed, then the

six lines so obtained touch a conic which is harmonically circumscribed to a system of covariant conics associated with the Hessian of the sextic.

Next, using the septimic and octavic as illustrations, there is indicated a chain of ruler constructions for the $C_{2,2}$ of a $(2n+1)$-ic based upon the two following results:—

(i) If P be one of the marking points of a $2n$-ic, and p be the line which represents the $C_{2,2}$ of the polar $(2n-3)$-points of P with respect to the remaining $(2n-1)$-points, then $[P, p]$ is a pole and polar pair with respect to a member of the system of conics through the unique $C_{2,4}$ of the $2n$-ic.

(ii) If P be one of the marking points of a $(2n+1)$-ic, and Q be the conjugate of P with respect to the four-point system through the unique $C_{2,4}$ of the remaining $2n$ points, then Q is a point of the line which represents the $C_{2,2}$ of the $(2n+1)$-ic.

Prof. Morley's elegant construction for the $C_{2,2}$ of a quintic may be exhibited as a special case of this result.

Analogues of Orthocentric Tetrahedra in Higher Space: Mr. T. C. Lewis.

The results obtained geometrically in the author's paper published in the September and October numbers of the *Proceedings* of the Society, Vol. 12, pp. 474–483, are now proved analytically by the use of penta-spherical coordinates, or corresponding coordinates when the space considered is of more than three dimensions. These coordinates are explained by M. Gaston Darboux in *La Théorie Générale des Surfaces*, Livre II, Ch. VI, p. 213; see also the same writer's *Systèmes Orthogonaux et les Coordonnées Curvilignes*, Livre I, Ch. VI, p. 121.

An independent investigation of this system of coordinates is given, based on its connexion with an orthocentric tetrahedron or higher analogue, a connexion not noted by M. Gaston Darboux, but one which makes the system naturally suitable for application to the study of the geometry of such orthocentric figures in space of any dimensions. The application of the method for this purpose is believed to be new.

While the homogeneous equation of the first degree represents an n-sphere or n-plane, the homogeneous equation of the second degree will in general represent a (hyper)-cyclide which reduces to a (hyper)-quadric on the fulfilment of certain conditions. This opens out a further field of investigation, which is being pursued.

Closed Linkages and Poristic Polygons: Col. R. L. Hippisley.

This is an article pointing out certain consequences arising out of the connection between closed linkages and poristic polygons which was briefly alluded to by the author in a previous paper (*Proc. London Math. Soc.*, Ser. 2, Vol. 11, p. 29). The results arrived at have a special bearing on the "Theory of the Transformation of Elliptic Functions." It is shown that the subsidiary polygons formed by joining the vertices of the original polygon in every possible way are connected together by an endless chain of linkages and also by a double system of inversions.

Thursday, December 11*th*, 1913.

Prof. A. E. H. LOVE, President, in the Chair.

Present twenty members and a visitor.

The minutes of the last meeting were read and confirmed.

Mr. S. B. Maclaren was elected a member.

The President presented the Auditor's Report. On the motion of Lt.-Col. Cunningham, seconded by Prof. Nicholson, the Treasurer's Report was adopted and the thanks of the Society were voted to the Treasurer and to the Auditor.

The President alluded to the deaths of Sir R. S. Ball, Prince Camille de Polignac, and Mr. Morgan Jenkins, the last of whom was for nearly thirty years Secretary to the Society. It was unanimously agreed that a letter of condolence should be sent to the widow of Mr. Jenkins.

The following papers were communicated:—

*On the Linear Integral Equation: Prof. E. W. Hobson.

*Generalized Hermite Functions and their Connexion with the Bessel Functions: Mr. H. E. J. Curzon.

*Limiting Forms of Long Period Tides: Mr. J. Proudman.

*On the Number of Primes of the same Residuacity: Lt.-Col. Cunningham.

*Some Results on the Form near Infinity of Real Continuous Solutions of a certain Type of Second Order Differential Equation: Mr. R. H. Fowler.

* Printed in this volume.

The Potential of a Uniform Convex Solid possessing a Plane of Symmetry with Application to the Direct Integration of the Potential of a Uniform Ellipsoid: Dr. S. Brodetsky.

The Dynamical Theory of the Tides in a Polar Basin: Mr. G. R. Goldsbrough.

Proof of the Complementary Theorem: Prof. J. C. Fields.

ABSTRACTS.

On the Linear Integral Equation: Prof. E. W. Hobson.

In this paper the integral equation

$$f(s) = \phi(s) - \lambda \int_a^b K(s, t)\, \phi(t)\, dt$$

is treated of with a view to removing as far as possible the restrictions on the nature of the nucleus $K(s, t)$ and the given function $f(s)$. Throughout, the definition of a definite integral due to Lebesgue is employed. When $K(s, t)$ is summable and limited in the square for which it is defined, it is shewn that Fredholm's solution is the only summable solution of the integral equation. The case in which $K(s, t)$ and a finite number of the repeated nuclei are unlimited is investigated by a method in which Lalesco's theory of the canonical sub-groups of the resolvant of a limited nucleus is employed; a more general result than that obtained by Poincaré is thus established. Certain cases are considered in which Fredholm's solution is applicable when the nucleus and all the repeated nuclei are unlimited.

Generalized Hermite Functions and their connexion with the Bessel Functions: Mr. H. E. J. Curzon.

This forms the sequel to a preceding paper in which a connexion was found between the Hermite functions and the Legendre functions. In a memoir, read before the Royal Society in 1908, Mr. Cunningham discusses the properties of certain ω-functions occurring as the result of searching for all solutions of

$$\nabla^2 u = \frac{\partial u}{\partial t}$$

that have the form $f(t)\, \phi(r^2/t)\, \Theta$, where Θ depends only on angular co-

ordinates. Or, slightly transforming the fundamental differential equation for these ω-functions, an equation that arises is

$$\frac{d^2z}{d\xi^2}+\frac{2\mu}{\xi}\frac{dz}{d\xi}-2\xi\frac{dz}{d\xi}+2(\nu-\mu)z=0, \tag{i}$$

an equation that deforms into Hermite's equation on making μ zero. In the present paper this equation (i) is solved by means of two definite integrals which I call $H_{\nu,\mu}(\xi)$ and $K_{\nu,\mu}(\xi)$, a function $W_{\nu,\mu}(\xi)$ being related to these functions when they are not independent solutions of (i) in the same way that the function $Y_n(x)$ is related to the ordinary Bessel functions when n is integral. Connexions are established between the generalised Hermite functions and the Bessel functions of the type

$$H_{\nu,\mu}(x)=\frac{e^{\frac{1}{2}\pi i(\nu-\frac{1}{2})}\,\Gamma\left(\dfrac{z+\mu+1}{\nu}\right)}{\pi x^{\mu-\frac{1}{2}}}\int_{\infty}^{-\infty}e^{-t^2}t^{-\nu-\frac{1}{2}}J_{\mu-\frac{1}{2}}(-2xti)\,dt,$$

the path of integration avoiding the origin by means of a small semicircle above the real axis. The relation is worked out between the H and K functions and Mr. Cunningham's ω-function, and various other properties of these generalised Hermite functions are discussed, apart from their connexions with the Bessel functions.

Limiting Forms of Long Period Tides : Mr. J. Proudman.

This paper contains a general discussion of the limiting forms of forced tides on a rotating globe, as the period of the disturbing force tends to become infinite, and of the use of such as approximations to tides of long period.

The method adopted for the former purpose is to use the limiting forms of the general equations of forced motion, and to add as conditions any properties of the general motion which are independent of the period, so long as it is finite, but do not follow as consequences of the limiting forms of the equations themselves. These properties are obtained and used in a form which involves "relative circulations."

The area of the surface of the water is divided into regions of three different types, according to the properties of $h\sec\theta$ (h being the depth and θ the co-latitude). These regions are first discussed separately, and then the solutions are combined.

In all cases but one the limiting forms are found to exist uniquely.

A few examples are given in which the limiting forms are calculated

for such disturbing forces as occur in terrestrial tides. The chief of these refers to a polar sea bounded by any parallel of latitude.

In the discussion of the possibility of applying limiting forms as approximations, an attempt is made to apply to tidal theory results which have been established only for systems with a finite number of degrees of freedom.

Using the tide-tables for the Indian ports it is concluded that free oscillations of long period may exist, as the theory is found not to apply.

A suggestion is made to use the Lake Victoria Nyanza to determine the amount of the earth's yielding to tidal forces.

An appendix is added on the general equations of tidal motion for seas on a yielding nucleus.

On the Number of Primes of the same Residuacity: Lt.-Col. Cunningham.

This paper presents the results of a count of the numbers (μ, m) of odd primes (p), which satisfy the congruences

$$y^{1/\nu(p-1)} \equiv +1, \qquad y^{1/n(p-1)} \equiv +1 \quad (\text{mod } p),$$

for certain bases (y), through certain ranges (R) of the natural numbers, where

ν = the maximum factor of $(p-1)$ possible to y, n = *any* factor of ν;

μ is the number with ν, m is the number with n.

The counts of μ were made for all values of ν for the eight bases

$$y = 2, 3, 5, 6, 7, 10, 11, 12,$$

for the range $R = 1$ to 10^4; and also for $y = 2$ and 10 for each successive range of 10000 up to 10^5. The counts of m were made for all values of $n = 1$ to 40 for the same bases and ranges as for μ.

Comparing μ, m with the number (M) of primes (p) of form

$$p = (\nu\varpi + 1),$$

$(n\varpi + 1)$ in the same range, the chief results were the following as

approximate rules (in *large ranges* of numbers)

$\nu = 1$ (y a primitive root), $\mu = \frac{1}{2.5}$ to $\frac{1}{2.8} M$.

$\nu = 2$ (y a quadratic root), $\mu = \frac{1}{3.2}$ to $\frac{1}{3.6} M$.

$m = \frac{1}{n} M$, nearly $\left(\text{but usually} < \frac{1}{n} M\right)$ [except as below].

$m = \frac{1}{n} 2M$, nearly $\left(\text{but usually} < \frac{1}{n} 2M\right)$, with $n = 8i$, for base 2.

$m = \frac{1}{n} 2M$, nearly; in some cases, with $n = ky$, when $y > 2$.

Some Results on the Form near Infinity of Real Continuous Solutions of a certain Type of Second Order Differential Equations: Mr. R. H. Fowler.

This paper is the outcome of an attempt to obtain for the differential equation

$$P(xyy'y'') = 0,$$

where P is a polynomial with real coefficients, results analogous to those obtained by Mr. Hardy (*Proc. London Math. Soc.*, Ser. 2, Vol. 10, pp. 451–468) for the equation

$$P(xyy') = 0.$$

The following theorem is proved:—*If $y = y(x)$ is any real continuous solution (with real continuous differential coefficients of the first two orders) of the equation*

$$y'' = P(x, y),$$

where P is a polynomial in x and y, then either there exists a K such that

$$y(x) = O(x^k),$$

or else there exists real numbers A and ρ, of which ρ is rational, such that

$$y(x) = e^{Ax^\rho(1+\epsilon)} \quad (\epsilon \to 0).$$

The latter type cannot exist unless the degree of P in y is unity.

Various generalizations of this theorem are considered; in particular it is shown that the theorem remains true when a rational function of x and y is substituted for $P(x, y)$.

The Potential of a Uniform Convex Solid possessing a Plane of Symmetry, with Application to the Direct Integration of the Potential of a Uniform Ellipsoid: Dr. S. Brodetsky.

The difficulty of finding the potential of a uniform solid at an external point consists in the fact that the integral representing the potential involves limits which are complicated functions of position. This renders the direct integration impossible, except in very few special cases. Thus the potential at an external point of a uniform ellipsoid has hitherto been calculated only by means of special methods and devices. The nearest approach to a direct method is the discontinuous factor used by Dirichlet, Kronecker, and Hobson (see *Proc. London Math. Soc.*, Old Series, Vol. XXVII). The object of this paper is to devise a method of integration that shall overcome this difficulty and give us a general direct integrating process.

The potential of a uniform straight rod of line density m is

$$m \log (1+e)/(1-e),$$

where e is the excentricity of the ellipse having the ends of the rod as foci and passing through the point at which the potential is calculated. We split up the solid into elementary rods perpendicular to the plane of symmetry, and we find its potential at any point in the form

$$V = \sigma \iint r\,dr\,d\phi \log (1+e)/(1-e),$$

the double integral being taken for all the rods in the body, and r, ϕ being measured in the plane of symmetry. The body under consideration being convex, it follows easily that there is only one maximum value for e corresponding to any given point at which the potential is to be found; further, that the rods giving any particular value of e less than the maximum lie on a convex cylinder surrounding the rod giving the maximum value of e. Still measuring in the plane of symmetry with the projection of the rod giving maximum e as a new origin for r', ϕ', we get

$$V = -\sigma \int_0^E \int_0^{2\pi} r'\,dr'\,d\phi' \log(1+e)/(1-e),$$

E being the maximum value of e. We now take e as an independent variable instead of r', and integrate partially with respect to e. We obtain, after dropping a vanishing term,

$$V = \sigma \int_0^E \int_0^{2\pi} \frac{r'^2}{1-e^2}\,de\,d\phi'.$$

For an internal point (ξ, η, ζ), $E = 1$, and the equation for r'^2 in terms of e and ϕ' is

$$\frac{1}{e^2} f(r' \cos \phi' + \xi,\ r' \sin \phi' + \eta) = \zeta^2 + r'^2/(1-e^2),$$

the equation to the surface of the solid being

$$z^2 = f(x, y).$$

For an external point, we first have to find the coordinates (ξ_0, η_0) of the rod giving maximum E, from the equations

$$\frac{1}{e^2} f(x, y) = \xi^2 + \{(x-\xi)^2 + (y-\eta)^2\}/(1-e^2),$$

$$\frac{1}{e^2}\, \partial f/\partial x = 2(x-\xi)/(1-e), \qquad \frac{1}{e^2}\, \partial f/\partial y = 2(y-\eta)/(1-e^2).$$

These equations give us E, ξ_0, η_0, and we get for r'^2 the equation

$$\frac{1}{e^2} f(r' \cos \phi' + \xi_0,\ r' \sin \phi' + \eta_0)$$
$$= \xi^2 + \{(r' \cos \phi' + \xi_0 - \xi)^2 + (r' \sin \phi' + \eta_0 - \eta)^2\}/(1-e^2).$$

The method does not simplify the process in the case of an internal point. But for an external point the method of this paper introduces appreciable simplification. Having solved the equations giving E, ξ_0 and η_0, and having found r'^2 in terms of e and ϕ', the integration is quite straightforward, and the limits are simple and well defined. It is worth noticing that E is really a solution of

$$\int_0^{2\pi} r'^2 d\phi' = 0.$$

The uniform ellipsoid can be treated very simply indeed by this method, as it is found that in this special case we do not need the actual values of ξ_0, η_0, and E is found by equating $\int_0^{2\pi} r'^2 d\phi'$ expressed in terms of e to zero.

Incidentally, the analysis used in this paper leads to some interesting properties of confocal surfaces, including confocal conicoids. In the case of confocal conicoids these properties can be obtained by elementary methods, but I do not think they have been noticed before.

Proof of the Complementary Theorem: Prof. J. C. Fields.

In a recent paper (*Proc. London Math. Soc.*, Ser. 2, Vol. 12, pp. 218–

235) the writer deduced, with reference to an arbitrary fundamental equation $f(z, u) = 0$, the expression

$$ni_\kappa + \sum_{s=1}^{r_\kappa} \tau_s^{(\kappa)} \nu_s^{(\kappa)} - \tfrac{1}{2} \sum_{s=1}^{r_\kappa} \left(\mu_s^{(\kappa)} - 1 + \frac{1}{\nu_s^{(\kappa)}}\right) \nu_s^{(\kappa)},$$

for the number of the conditions imposed by a set of orders of coincidence $\tau_1^{(\kappa)}, \ldots, \tau_{r_\kappa}^{(\kappa)}$ on the general rational function of (z, u), which can be represented in the form $(z-a_\kappa)^{-i_\kappa}((z-a_\kappa, u))$, the integer i_κ being taken sufficiently large. Representing any pair of complementary bases by (τ) and $(\bar{\tau})$, we know that 0 is the value of the principal residue relative to the value $z = \infty$ in the products of the general rational function built on the basis $(\bar{\tau})$ by the general rational functions conditioned by the partial bases $(\tau)'$ and $(\tau)^{(\infty)}$ respectively. This fact, combined with the result cited above, enables us to derive the inequality

$$N_{\bar{\tau}} \leqslant N_\tau - n + \sum_\kappa \sum_{s=1}^{r_\kappa} \tau_s^{(\kappa)} \nu_s^{(\kappa)} - \tfrac{1}{2} \sum_\kappa \sum_{s=1}^{r_\kappa} \left(\mu_s^{(\kappa)} - 1 + \frac{1}{\nu^{(\kappa)}}\right) \nu_s^{(\kappa)},$$

where N_τ and $N_{\bar{\tau}}$ represent the numbers of arbitrary constants involved in the general rational functions built on the bases (τ) and $(\bar{\tau})$ respectively. Interchange of (τ) and $(\bar{\tau})$ in this inequality gives us a second inequality. Addition of the corresponding sides of the two inequalities shews us the inadmissibility of the unequal sign in either of the inequalities. The two inequalities then both become equalities from whose combination we immediately derive the complementary theorem

$$N_\tau + \tfrac{1}{2} \sum_\kappa \sum_{s=1}^{r_\kappa} \tau_s^{(\kappa)} \nu_s^{(\kappa)} = N_{\bar{\tau}} + \tfrac{1}{2} \sum_\kappa \sum_{s=1}^{r_\kappa} \bar{\tau}_s^{(\kappa)} \nu_s^{(\kappa)}.$$

Thursday, January 22nd, 1914.

Prof. A. E. H. LOVE, President, in the Chair.

Present ten members and a visitor.

The minutes of the last meeting were read and confirmed.

Mr. T. L. Wren was elected and admitted a member of the Society.

The following papers were communicated:—

(i) A Generalisation of the Euler-Maclaurin Sum Formulæ, (ii) The

Deduction of the Formulæ of Mechanical Quadrature from the Generalised Euler-Maclaurin Sum Formulæ, (iii) A Generalisation of certain Sum Formulæ in the Calculus of Finite Differences: Mr. S. T. Shovelton.

*On Binary Forms: Dr. A. Young.

On Darboux's Method of Solution of Partial Differential Equations of the Second Order: Mr. J. R. Wilton.

The President made an informal communication "On the Potential due to the Distribution on an Electrified Circular Disc."

ABSTRACTS.

A Generalisation of the Euler-Maclaurin Sum Formulæ: Mr. S. T. Shovelton.

The formula is obtained by using the operator $\phi(\Delta)$, where

$$\phi(\Delta) \equiv 1 - \frac{\Delta}{2} + \frac{\Delta^2}{3} - \frac{\Delta^4}{4} + \ldots,$$

and it is shown that

$$\Sigma_a^b v_x = \int_a^b v_x + \phi_1(0)(v_b - v_a) + \frac{\phi_2(0)}{2!}(v_b' - v_a') + \ldots + \frac{\phi_n(0)}{n!}(v_b^{(n-1)} - v_a^{(n-1)}) \ldots,$$

where $$\phi_n(0) = \phi(\Delta)\, 0^n.$$

A more general formula of summation is then proved in the form

$$h(v_{a+\kappa h} + v_{a+h+\kappa h} + \ldots + v_{b+\kappa h-h})$$
$$= \int_a^b v_x\, dx + h\phi_1(\kappa)(v_b - v_a) + \frac{h^2\phi_2(\kappa)}{2!}(v_b' - v_a') + \ldots,$$

where the remainder after $2(n+1)$ terms is

$$-\frac{h^{2n+1}}{(2n)!}\left[\int_0^{1-\kappa} \phi_{2n}(z+\kappa) \sum_{r=1}^{r=(b-a)h} v_{a+rh-zh}^{(2n)}\, dz \right.$$
$$\left. + \int_{1-\kappa}^{1} \phi_{2n}(z+\kappa-1) \sum_{r=1}^{r=(b-a)h} v_{a+rh-zh}^{(2n)}\, dz\right],$$

$\phi_n(\kappa)$ is $\phi(\Delta)\kappa^n$, and $\phi_n(\kappa) - \phi_n(0)$ is shown to be the Bernoullian function

* Printed in this volume.

of degree n. By putting $v_x = e^x$ it follows that $\phi_n(\kappa)$ is the coefficient of h^n in $\dfrac{he^{\kappa h}}{e^h-1}$, and that

$$\phi_{2n}(0) = (-1)^{n-1} B_n.$$

Defining a new function $\phi_n^r(\kappa)$ by the equation

$$\phi_n^r(\kappa) = [\phi(\Delta)]^r \kappa^n,$$

it is shown that

$$h^r D^r u_{x+\kappa h} = \Delta^r u_x + h\phi_1^r(\kappa)\Delta^r u_x' + \frac{h^2\phi_2^r(\kappa)}{2!}\Delta^r u_x'' + \ldots,$$

from which we get

$$h^r \Sigma^r u_{x+\kappa h} = \int^{(r)} u_x (dx)^r + h\phi_1^r(\kappa)\int^{(r-1)} u_x (dx)^{r-1} + \ldots.$$

The function $\phi_m^r(\kappa)$ possesses properties analogous to those of the Bernoullian function—amongst others $\phi_m^r\left(\frac{r}{2}\right) = 0$ if m is odd,

$$\phi \quad (\kappa) = (-1)^m \phi_m^r(r-\kappa),$$

and $(-1)^m \phi_{2m}^r\left(\frac{r}{2}\right)$ is positive. Thus taking $\kappa = \frac{r}{2}$ in the series for $D^r u_{x+\kappa h}$, we have

$$h^r D^r u_{x+\frac{1}{2}rh} = \Delta^r u_x + \frac{h^2\phi_2^r\left(\frac{r}{2}\right)}{2!}\Delta^r u_x'' + \ldots,$$

and it is shown that

$$x^r \operatorname{cosec}^r x = 1 - \frac{2^2\phi_2^r\left(\frac{r}{2}\right)}{2!}x^2 + \frac{2^4\phi_4^r\left(\frac{r}{2}\right)}{4!}x^4 - \ldots.$$

If u_x is put equal to e^x we find

$$\frac{h^r e^{\kappa h}}{(e^h-1)^r} = 1 + h\phi_1^r(\kappa) + \frac{h^2}{2!}\phi_2^r(\kappa) + \ldots,$$

and it follows that

$$\frac{x^r \cos(2\kappa - r)x}{\sin^r x} = 1 - \frac{2^2\phi_2^r(\kappa)}{2!}x^2 + \frac{2^4\phi_4^r(\kappa)}{4!}x^4 - \ldots,$$

and

$$\frac{x^r \sin(2\kappa - r)x}{\sin^r x} = 2\phi_1^r(\kappa)\,x - \frac{2^3\phi_3^r(\kappa)}{3!}x^3 + \ldots.$$

The function $\phi_{2m}^{r}\left(\frac{r}{2}\right)$ satisfies the reduction formula

$$(r-1)(r-2)\,\phi_{2m}^{r}\left(\frac{r}{2}\right)$$
$$=(r-2m-1)(r-2m-2)\,\phi_{2m}^{r-2}\left(\frac{r}{2}-1\right)-\frac{m(2m-1)}{2}(r-2)^2\phi_{2m-2}^{r-2}\left(\frac{r}{2}-1\right),$$

from which a table of its values can be formed. Also

$$r\phi_m^{r+1}(\kappa)=(r-m)\,\phi_m^{r}(\kappa)+m(\kappa-r)\,\phi_{m-1}^{r}(\kappa),$$

from which a table of the values of $\phi_m^r(0)$ can be formed.

The expansions of $\tan x$ and $\sec x$ are found, and from the latter it appears that

$$E_{2n}=(-1)^{n-1}\frac{2^{4n+2}\phi_{2n+1}(\frac{1}{4})}{(2n+1)}$$
$$=\frac{(-1)^{n-1}}{(2n+1)}\left\{1-2(2n+1)+{}^{2n+1}C_2\,4^2B_1-{}^{2n+1}C_4\,4^4B_2+\dots\right\}.$$

Other expansions are

$$x^r\cot^r x=1+\Sigma(-1)^n\frac{x^{2n}}{(2n)!}\left\{2^{2n}\phi_{2n}^{r}(0)+r2^{2n-1}.\,2n\phi_{2n-1}^{r-1}(0)\right.$$
$$\left.+{}^{r}C_2.\,2^{2n-2}(2n)^{(2)}\,\phi_{2n-2}^{r-2}(0)+\dots\right\},$$

and also can be expressed in terms of $\phi_{2n}^{2m}(m)$, $\phi_{2n-2}^{2n-2}(m-1)$, ..., when r is of the form $2m$ or $2m+1$,

$$\tan^2 x=\left\{\Delta^r-2r\frac{\Delta^{r+1}}{4}+\frac{2r(2r+3)}{2!}\frac{\Delta^{r+2}}{4^2}\right.$$
$$\left.-\frac{2r(2r+4)(2r+5)}{3!}\frac{\Delta^{r+3}}{4^3}\dots\right\}\begin{matrix}(-1)^{\frac{1}{2}r}\cos(4x.0)\\(-1)^{\frac{1}{2}(r-1)}\sin(4x.0)\end{matrix}$$

according as r is even or odd, the Δ's operating on the powers of zero,

$$\sec^r x=\left\{1-r\frac{\Delta}{4}+\frac{r(r+3)}{2!}\frac{\Delta^2}{4^2}-\dots\right\}\left[1+r\frac{\Delta}{4}+\frac{r(r-4)}{2!}\frac{\Delta^2}{4^2}+\dots\right]$$
$$\times\cos(4x.0).$$

The Deduction of the Formulæ of Mechanical Quadrature from the Generalised Euler-Maclaurin Sum Formulæ: Mr. S. T. Shovelton.

Several of the well-known formulæ, such as Weddle's and G. F.

Hardy's, are very easily obtained by the use of the sum formula, and new ones are given, one of which is

$$\int_0^{10} v_x dx = \tfrac{5}{126}\left[8(v_0+v_{10})+35(v_1+v_2+v_7+v_9)+15(v_2+v_4+v_6+v_8)+36v_5\right],$$

which has an error of less than (1—7)-th of Weddle when applied to the same range.

The generalised sum formula can be used to give approximate values of definite integrals in almost endless variation, but few of the results are of much practical value. A formula which is useful when fourth differences are small is

$$\int_0^{10} v_x dx = \tfrac{5}{6}\left[13(v_2+v_4+v_6+v_8)-20(v_3+v_7)\right].$$

A Generalisation of certain Sum Formulæ in the Calculus of Finite Differences: Mr. S. T. Shovelton.

In this paper an investigation is given for the expression of $\Sigma^r a^x\phi(x+\kappa h)$ in terms of the differential coefficients of $\phi(x)$. The result can be best expressed in the form

$$(a^h-1)^r a^x\phi(x+\kappa h)$$
$$= \Delta^r a^x\phi(x)+h\theta_1^r(\kappa)\Delta^r a^x\phi'(x)+\ldots+\frac{h^n}{n!}\theta_n^r(\kappa)\Delta^r a^x\phi^{(n)}(x)+R_n,$$

where Δ refers to intervals of h in x, and $\theta_n^r(\kappa)$ is defined by the equation

$$\theta_n^r(\kappa) = [\theta(\Delta)]^r\kappa^n = \left[1-\frac{a^h}{a^h-1}\Delta+\left(\frac{a^h}{a^h-1}\right)^2\Delta^2-\ldots\right]^r\kappa^n,$$

in which the differences refer to intervals of unity in κ. The remainder assumes a somewhat untractable form for general values of r, but when r is unity is equal to

$$-\frac{h^{n+1}}{n!}\left[\int_0^{1-\kappa}\theta_n(\kappa+z)\,a^{x+h}\phi^{(n+1)}(x+h-zh)\,dz\right.$$
$$\left.+\int_{1-\kappa}^1\theta_n(\kappa+z-1)\,a^x\phi^{(n+1)}(x+h-zh)\,dz\right].$$

When $a=-1$ and h is an odd integer, $\theta_n^r(\kappa)$ possesses properties analogous to those of $\phi_n^r(\kappa)$ defined in my paper on the Euler-Maclaurin

formula. With these values we obtain

$$2\left[\phi(a+\kappa h)-\phi(a+h+\kappa h)+\phi(a+2h+\kappa h)+\ldots\right.$$
$$\left.+(-1)^{(p-1)}\phi(a+\overline{p-1+\kappa}\,h)\right]$$
$$=\{\phi(a)-(-1)^p\phi(a+ph)\}+\ldots+\frac{h^n\theta_n(\kappa)}{n!}\left[\phi^{(n)}(a)-(-1)^p\phi^{(n)}(a+ph)\right]+R_n,$$

and R_n may be made to vanish by suitably choosing n when $\phi(x)$ is a polynomial in x.

From the general result given above it follows that

$$\frac{(a-1)^r e^{\kappa h}}{(ae^h-1)^r}=1+h\theta_1^r(\kappa)+\frac{h^2}{2!}\theta_2^r(\kappa)+\ldots,$$

where a is written for a^h. When $a=-1$, the most useful value of κ is $\frac{r}{2}$, for $\theta_{2n+1}^r\left(\frac{r}{2}\right)=0$ and $\theta_{2n}^r\left(\frac{r}{2}\right)$ can be expressed in terms of the corresponding functions of $2n$ and $(2n+2)$ of order $(r-1)$. $\theta_{2n}^r\left(\frac{r}{2}\right)$ can also be calculated directly from

$$\theta_{2n}^r\left(\frac{r}{2}\right)=\frac{1}{2^{2n}}\left[1-r\,\frac{\delta^2}{2}+\frac{r(r+1)}{2!}\,\frac{\delta^4}{4}-\ldots\right]O^{2n},$$

where $$\delta=\frac{\Delta}{E^{\frac{1}{2}}}.$$

We readily deduce that

$$\sec^r x=\left[1-\frac{r}{1}\,\frac{\delta^2}{2}+\frac{r(r+1)}{2!}\,\frac{\delta^4}{4}-\ldots\right]\cos(x\,.\,0),$$

and that

$$\tan^r x=\delta^r\left[1-\frac{r}{2}\,\frac{\delta^2}{4}+\frac{r(r+2)}{2\,.\,4}\,\frac{\delta^4}{16}-\ldots\right]\left[\frac{x^r\,.\,O^r}{r!}-\frac{2^2x^{r+2}O^{r+2}}{(r+2)!}+\ldots\right].$$

Hence $$E_{2n}=(-1)^n\left[1-\frac{\delta^2}{2}+\frac{\delta^4}{4}-\ldots\right]O^{2n},$$

and $$B_n=\frac{(-1)^{n-1}n}{2(2^{2n}-1)}\left[\delta-\frac{1}{2}\,\frac{\delta^3}{4}+\frac{1.3}{2.4}\,\frac{\delta^5}{16}-\ldots\right]O^{2n-1}.$$

Tables of the values of $\delta^{2m}O^{2n}$ and $\delta^{2m+1}O^{2n+1}$ can be readily constructed from the equation

$$\delta^p O^q=\frac{p^2}{4}\,\delta^p O^{q-2}+p(p-1)\,\delta^{p-2}O^{q-2}.$$

On Binary Forms: Dr. A. Young.

In a paper on perpetuants, Grace applied the symbolical method to discover the irreducible types; his result was that all perpetuants can be expressed in terms of the forms

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \dots (a_1 a_\delta)^{\lambda_\delta},$$

where

$$\lambda_r \geqslant 2^{\delta - r} \quad (r = 2, 3, \dots, \delta).$$

The attempt is made to solve the same problem here for irreducible covariant types of quantities of finite order, by multiplying all such covariants by the symbolical expression $a_{1_x}^{w-n_1}$, where w is the weight of the covariant $\gamma a_{1_x}^{n_1}$ is one of the quantics. We can thus express all our covariants as linear functions of the forms

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \dots (a_1 a_\delta)^{\lambda_\delta}.$$

By means of this expression we find a certain set of covariant types of degree δ, which we write

$$(\lambda_2, \lambda_3, \dots, \lambda_\delta)$$

completely defined by the arguments, which are linearly independent, and in terms of which all can be linearly expressed.

The method used by Grace is not suitable here. So a suitable method of dealing with the covariants is developed. For the sake of the analysis the case of perpetuants is first discussed, and Grace's results are obtained. Also the complete system of syzygies of the first kind is obtained for perpetuants.

On Darboux's Method of Solution of Partial Differential Equations of the Second Order: Mr. J. R. Wilton.

By a slight modification of Darboux's method of obtaining intermediate integrals of any given order in the case of the equation

$$r + f(x, y, z, p, q, s, t) = 0,$$

we may assume, as the form of an integral of the n-th order ($n > 2$),

$$p_{1, n-1} + m p_{0, n} + v = 0,$$

where

$$p_{hk} = \frac{\partial^{h+k} z}{\partial x^h \partial y^k},$$

m is either root of the characteristic equation, and v is a function of

$x, y, z, p, q, \ldots, p_{1, n-2}, p_{0, n-1}$. If, however, $n = 2$, the form of the integral is

$$u + v(x, y, z, p, q) = 0,$$

where
$$\frac{\partial u}{\partial s} = \mu, \quad \frac{\partial u}{\partial t} = m\mu.$$

It is found that the number of independent equations which must be satisfied by v is two if n is greater than two, but when $n = 2$ the number depends on the nature of the given equation.

The assumption of this form of the intermediate integral leads directly to the determination of special integrals, such as Serret's integral of

$$rt - s^2 + a^2(1 + p^2 + q^2)^2 = 0.$$

It will also be found to involve rather less labour than is usual in the determination of general intermediate integrals when such exist.

If, in the two equations which v must satisfy, when $n > 2$, we put

$$\frac{\partial v}{\partial \xi} = -\frac{\partial u}{\partial \xi} \Big/ \frac{\partial u}{\partial v},$$

where ξ is any one of the independent variables, they become homogeneous and may be denoted by $\Delta_1 u = 0$, $\Delta_2 u = 0$. If $\Delta_{12} u = \Delta_1 \Delta_2 u - \Delta_2 \Delta_1 u$, it is found that, if $n > 3$,

$${}_1\Delta_{12} u \equiv \Delta_1 \Delta_{12} u - \Delta_{12} \Delta_1 u = 0,$$

where $\Delta_1 u = 0$ is that one of the two equations which involves no differential coefficients of u other than $\frac{\partial u}{\partial v}, \frac{\partial u}{\partial p_{0, n}}, \frac{\partial u}{\partial p_{1, n-1}}$. This fact considerably diminishes the number of independent conditions which must be satisfied in order that the equations for v should possess a common integral.

Thursday, February 12th, 1914.

Prof. H. F. BAKER, Vice-President, in the Chair.

Present seventeen members and a visitor.

The minutes of the last meeting were read and confirmed.

Messrs. W. E. H. Berwick and A. G. Veitch were elected members.

Professor S. B. McLaren was admitted into the Society.

The Secretaries reported that during the last Session the Society had lost seven members and that eight new members had been elected; thus the total number of members was 306 at the end of that Session as against 305 at the beginning.

The following papers were communicated :—

Exhibition and Explanation of some Models illustrating Kinematics: Mr. G. T. Bennett.

*Formulæ for the Spherical Harmonic $P_n^{-m}(\mu)$, when $1-\mu$ is a Small Quantity: Prof. H. M. Macdonald.

The Representation of the Symmetrical Nucleus of a Linear Integral Equation: Prof. E. W. Hobson.

*Fitting of Polynomial by the Method of Least Squares (Second Paper): Dr. W. F. Sheppard.

The Differential Geometry of Point-Transformations between Two Planes: Mr. H. Bateman.

ABSTRACTS.

On the Representation of the Symmetrical Nucleus of a Linear Integral Equation: Prof. E. W. Hobson.

This paper is concerned with the relation between the symmetrical nucleus of the integral equation

$$f(s) = \phi(s) - \lambda \int_a^b K(s, t)\, \phi(t)\, dt,$$

and the series
$$\sum_{n=1} \frac{\phi_n(s)\, \phi_n(t)}{\lambda_n};$$

where λ_n denotes a characteristic number, and $\phi_n(s)$ the corresponding normal characteristic function. Cases are considered in which $K(s, t)$ has discontinuities.

It is shewn that a nucleus $K(s, t)$, such that $\int_a^b \{K(s, t)\}^2 dt$ is a limited function of s, exists, such as to have prescribed characteristic functions and numbers $\{\phi_n(s)\}$, $\{\lambda_n\}$; provided the series $\sum\limits_{n=1} \left\{\frac{\phi_n(s)}{\lambda_n}\right\}^2$ converges for each value of s to a value which is a limited function of s, in (a, b).

* Printed in this volume.

It is also shewn that the function $K(s, t)$, determined in accordance with the prescribed conditions is unique, except for equivalent functions not differing essentially from it. Several theorems are deduced from this result, and a simple proof of Mercer's theorem is obtained, that when $K(s, t)$ is continuous, and all the numbers $\{\lambda_n\}$ are positive, the series $\sum_{n=1} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n}$ converges uniformly to $K(s, t)$.

By means of an extension of a well known theorem due to Hilbert, to the case of a discontinuous nucleus, an extension of Mercer's theorem is obtained, which applies to all nuclei that are not infinitely discontinuous on the line $s = t$.

It is shewn that, in the case in which the repeated function of $K(s, t)$ is continuous, the nucleus $K(s, t)$ may be expressed as the sum of two functions $K^{(1)}(s, t)$, $K^{(2)}(s, t)$ that are orthogonal to one another, and are such that $K^{(1)}(s, t)$ has for its sole characteristic numbers those characteristic numbers of $K(s, t)$ that are positive, whereas $K^{(2)}(s, t)$ has for its sole characteristic numbers those belonging to $K(s, t)$ that are negative. The characteristic function corresponding to a characteristic number of $K^{(1)}(s, t)$, or of $K^{(2)}(s, t)$, is the same as that of $K(s, t)$ corresponding to the same characteristic number.

Fitting of Polynomials by Method of Least Squares: Dr. W. F. Sheppard.

If $\delta_1, \delta_2, \ldots, \delta_m$ are quantities containing errors, and ϵ_f is the improved value of δ_f, obtained by adding to it a linear compound of the δ's after δ_j, and choosing the coefficients so as to make the mean square of error of the total expression a minimum, then (i) the mean product of errors of ϵ_f and δ_t $(t > j)$ is zero, and (ii) the improved value of any linear compound of the δ's is the corresponding linear compound of the ϵ's. These two propositions give, by general reasoning, certain relations between the ϵ's for different values of j and f, and also a simple formula for the mean product of errors of ϵ_f and ϵ_g as the sum of $j-f+1$ or $j-g+1$ terms. Also, if $u_1, u_2, \ldots, u_m$ are connected with the δ's by linear relations, there are linear relations connecting the coefficients in the expression for ϵ_f in terms of the u's with the mean products of error of ϵ_f and the ϵ's.

These results are applied to the case (*Proceedings*, Ser. 2, Vol. 12, p. xliv) in which the u's satisfy the condition that their errors are independent and have all the same mean square, and the δ's are the advancing or central differences of the u's. Some questions in reference to the more general case are also investigated.

The Differential Geometry of Point-Transformations between Two Planes: Mr. H. Bateman.

In the neighbourhood of a given place any such transformation is equivalent to a projective one, in general, *i.e.*, unless the Jacobian vanishes or is infinite. The directions through a point are, in fact, altered in a projective manner. In this paper the approximation is carried a step further.

The pencil of lines through a point P in the original plane Π are mapped into a system of curves through the corresponding point p in the new plane π, and for three of the lines through P the curve arising has an inflexion at p. These are called inflexional lines, and from them spring inflexional curves everywhere tangent to an inflexional line. The properties of these are discussed and transformations are found for which the inflexional curves have assigned properties, *e.g.*, are such that at every point two of them coincide. Transformations which leave areas unaltered are discussed, and the results compared with the theory of the motion of an incompressible fluid in two dimensions.

Thursday, March 12th, 1914.

Prof. A. E. H. LOVE, President, in the Chair.

Present sixteen members and a visitor.

The minutes of the last meeting were read and confirmed.

The following papers were communicated:—

On the Rational Solutions of the Equation $X^3+Y^3+Z^3=0$ in Quadratic Fields: Prof. W. Burnside.

On Orthoptic and Isoptic Loci of Plane Curves: Prof. H. Hilton and Miss R. E. Colomb.

Normal Coordinates in Dynamical Systems: Dr. T. J. I'A. Bromwich.

On the Zeroes of Riemann's Zeta-Function: Mr. G. H. Hardy.

ABSTRACTS.

On the Rational Solutions of the Equation $X^3+Y^3+Z^3=0$ in Quadratic Fields: Prof. W. Burnside.

In this note attention was drawn to the fact that every solution of the

equation

$$X^3+Y^3+Z^3=0,$$

in which X, Y, Z belong to the same quadratic field, while no one of them is zero, is obtained from the identity

$$(6k)^3+[3+\sqrt{-3(1+4k^3)}]^3+[3-\sqrt{-3(1+4k^3)}]^3\equiv 0,$$

on multiplying each term by $[\alpha+\beta\sqrt{-3(1+4k^3)}]^3$, and then taking for α, β, k any rational numbers, the value -1 for k being omitted.

If two solutions for which the relation

$$\frac{X}{X'}=\frac{Y}{Y'}=\frac{Z}{Z'}$$

do not hold, in whatever order X', Y', Z' are taken, be called distinct, the equation either admits no solution in an assigned quadratic field, or it admits an infinite number of distinct solutions.

On Orthoptic and Isoptic Loci of Plane Curves: Prof. Harold Hilton and Miss R. E. Colomb.

Little appears to have been written on this subject except that the degree of the orthoptic locus of a given curve has been found both in the general case and when the curve has two- or three-point contact with the line at infinity. In this paper are found all Plücker's numbers for the orthoptic locus in these cases. The nature of the intersections of the orthoptic locus with the given curve, and of the multiple points of the orthoptic locus are also investigated. For instance, if the given curve is the general curve of degree n and class m, with ι inflexions, the orthoptic locus is of degree $m(m-1)$ and class $m(m+n-3)$ with $m\iota$ cusps.

It is obvious that in general the orthoptic locus of a given curve is complicated, but, if the curve is specialized, the orthoptic locus may be fairly simple. This is especially the case if the curve has the line at infinity as a multiple tangent, so that pairs of points of contact form a harmonic range with the circular points. There are respectively 2, 3, 3, 3, 14, 38 types of curve with orthoptic locus, a straight line, a circle, a parabola, a conic, a cubic, a quartic.

A brief account is given of similar properties of isoptic loci in the simplest cases which can occur.

It is readily seen that we can deduce the properties of the locus of the intersection of two tangents, one drawn to each of two given curves and

inclined at a given angle, and of the locus of the intersection of two normals to a given curve inclined at a given angle.

The results of the investigation are illustrated by several examples.

Normal Coordinates in Dynamical Systems: Dr. T. J. I'A. Bromwich.

Let the motion of a dynamical system performing small oscillations be given by the differential equations

$$\left.\begin{aligned} e_{11}x_1+e_{12}x_2+\ldots+e_{1n}x_n &= 0 \\ e_{21}x_1+e_{22}x_2+\ldots+e_{2n}x_n &= 0 \\ \ldots \quad \ldots \quad \ldots \quad & \ldots \\ e_{n1}x_1+e_{n2}x_2+\ldots+e_{nn}x_n &= 0 \end{aligned}\right\}. \tag{1}$$

The notation is that used by Lord Rayleigh (*Theory of Sound*, Vol. 1, § 82), namely, e_{rs} denotes the differential operator

$$e_{rs} = a_{rs}D^2+b_{rs}D+c_{rs};$$

but the symmetrical relation $e_{rs} = e_{sr}$ is not required, so that gyrostatic terms may be present, and the forces acting need not be conservative.

Then the solution of (1) is given by contour integrals

$$x_r = \frac{1}{2\pi\iota}\int e^{\lambda t}\xi_r\,d\lambda, \tag{2}$$

where $\xi_1, \xi_2, \ldots, \xi_n$ are functions of λ derived by solving the equations

$$\lambda_{11}\xi_1+\lambda_{12}\xi_2+\ldots+\lambda_{1n}\xi_n = p_1, \ \ldots, \tag{3}$$

where $$\lambda_{rs} = a_{rs}\lambda^2+b_{rs}\lambda+c_{rs},$$

and $p_1, p_2, \ldots, p_n$ are linear combinations of the initial displacements and velocities, such that

$$p_1 = \{a_{11}(\lambda+\delta)+h_{11}\}\,x_1+\{a_{12}(\lambda+\delta)+h_{12}\}\,x_2+\ldots \\ +\{a_{1n}(\lambda+\delta)+h_{1n}\}\,x_n, \ \ldots. \tag{4}$$

In the formulæ (4), $x_1, x_2, \ldots$ denote the *initial* displacements, $\delta x_1, \delta x_2, \ldots$ the *initial* velocities; and in the integrals (2), the path of integration in the λ-plane surrounds all the poles of the functions ξ, which are, of course, the roots of the determinant $|\lambda_{rs}| = 0$.

If the system is subject to additional impressed forces of the type $P_1e^{\mu t}, P_2e^{\mu t}, \ldots, P_ne^{\mu t}$, so that equations (1) become

$$e_{11}x_1+e_{12}x_2+\ldots+e_{1n}x_n = P_1e^{\mu t}, \ \ldots \tag{5}$$

(where μ may be real, imaginary, or complex), the solution corresponding to *zero* initial displacements and velocities is given by writing

$$p_1 = P_1/(\lambda-\mu), \ \ldots, \tag{6}$$

the path of integration now enclosing $\lambda = \mu$ in addition to all the roots of $|\lambda_{rs}| = 0$.

The solutions (2) can be shewn to reduce to the known solutions in terms of normal coordinates, when the equations (1) are specialised by the omission of one of the sets of letters (*a*), (*b*), (*c*); and thus (2) gives the extension to the general problem of the method of normal coordinates, which (as ordinarily presented) can only be used in special cases.

The extension to a continuous system (in which n tends to infinity) is also considered: the functions ξ_r then appear as integrals, instead of sums, and the resulting contour-integral reduces when $t = 0$ to the integral used by Prof. A. C. Dixon (*Proc. London Math. Soc.*, Ser. 2, Vol. 3, 1905; *Phil. Trans.*, 1911).

On the Zeroes of Riemann's Zeta-Function: Mr. G. H. Hardy.

It has been shown recently by Bohr and Landau (*Comptes Rendus*, January 12th, 1914) that, however small be the positive number δ, the majority of the zeroes of $\zeta(s) = \zeta(\sigma+ti)$ lie in the region $\frac{1}{2}-\delta < \sigma < \frac{1}{2}+\delta$. The object of this paper is to show that an infinity of the zeroes lie on the line $\sigma = \frac{1}{2}$, and so to advance one step further towards the proof of the "Riemann hypothesis" that *all* the zeroes, except the real negative zeroes $-2, -4, -6, \ldots$, lie on this line.

It may be observed that there is a numerical method, due to Lindelöf and Gram, and developed by Backlund (*Öfversigt af Finska Vetenskaps-Societetens Förhandlingar*, Vol. 54), by which it has been shown that there are exactly twenty-nine zeroes on the line $\sigma = \frac{1}{2}$ between $t = 0$ and $t = 100$; and that Backlund has also shown that there are no other complex zeroes whose imaginary part is positive and less than 100. These results are of great interest as evidence of the probable truth of Riemann's hypothesis; but they do not prove even that there are an infinity of zeroes on the critical line. Such a proof is supplied in the present paper.

The method adopted depends on Mellin's formula

$$e^{-y} = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \Gamma(u)\, y^{-u}\, du,$$

where $\mathfrak{R}(y) > 0$ and $\kappa > 0$. This formula is applied to the evaluation

of a definite integral containing a function $\Xi(t)$ introduced by Riemann and closely connected with $\zeta(s)$. If $\zeta(s)$ has not infinitely many roots for which $\sigma = \frac{1}{2}$, $\Xi(t)$, which is real for real values of t, is ultimately of constant sign; and it is shown that this hypothesis leads to a contradiction.

Thursday, April 23rd, 1914.

Prof. A. E. H. LOVE, President, in the Chair.

Present seven members.

Messrs. C. Jordan and J. Proudman were elected members.

The President referred to the death of Mr. G. M. Minchin, who had been a member of the Society since 1875.

The following paper was communicated:—

On a Modified Form of Pure Reciprocants possessing the Property that the Algebraical Sum of the Coefficients is Zero: Major P. A. MacMahon.

And Major MacMahon made an informal communication on "Lattice and Prime Lattice Permutations."

ABSTRACTS.

On a Modified Form of Pure Reciprocants possessing the Property that the Algebraical Sum of the Coefficients is Zero: Major P. A. MacMahon.

Instead of writing $$a_s = \frac{1}{(s+2)!}\,\frac{d^s y}{dx^s},$$

the author writes $$b_s = \frac{(s+1)!}{(2s+2)!}\,\frac{d^s y}{dx^s},$$

and represents a pure reciprocant as a homogeneous and isobaric function of

$$b_0,\ b_1,\ b_2,\ \ldots.$$

The algebraic sum of the coefficients in every pure reciprocant now vanishes, and the actual coefficients have much smaller numerical expressions.

Lattice and Prime Lattice Permutations: Major P. A. MacMahon.

We consider any assemblage of letters

$$\alpha^p\beta^q\gamma^r\ldots,$$

where $p, q, r, \ldots$ are in descending order of magnitude.

Certain permutations possess the property that if a line be drawn between *any two* letters, the letters to the left of the line are in some permutation of the assemblage

$$\alpha^{p'}\beta^{q'}\gamma^{r'}\ldots,$$

where $p', q', r', \ldots$ are in descending order of magnitude. These have been termed "lattice permutations" for a reason that has been given by the author in previous papers.

Consider in particular the assemblage

$$\alpha^p\beta^q.$$

It has been shewn that there are

$$\frac{(p+q)!}{(p+1)!\,q!}(p-q+1)$$

lattice permutations. This number in the coefficients of $x^p y^q$ is the expansion of

$$\frac{1-\dfrac{y}{x}}{1-x-y},$$

a crude generating function because it involves many terms which are not required. The exact generating function is obtained from the notion of prime lattice permutations. If we write down the lattice permutations of the assemblage

$$\alpha^2\beta^2,$$

we find them to be

$$\alpha\alpha\beta\beta,$$
$$\alpha\beta.\alpha\beta,$$

and it is noticed that the second of these is divisible into two lattice permutations of smaller assemblages (as shewn by the dot), whereas the first $\alpha\alpha\beta\beta$ is not so divisible. $\alpha\alpha\beta\beta$ is a prime lattice permutation, and $\alpha\beta.\alpha\beta$ a composite lattice permutation.

Similarly of the assemblage $\alpha^3\beta^3$ we find five lattice permutations, two of which are prime and three composite, viz.,

$$\alpha\alpha\alpha\beta\beta\beta,$$
$$\alpha\alpha\beta\alpha\beta\beta,$$

are prime, and

$$\alpha\alpha\beta\beta.\alpha\beta,$$
$$\alpha\beta.\alpha\alpha\beta\beta,$$
$$\alpha\beta.\alpha\beta.\alpha\beta,$$

composite.

If N_{xy} denote the generating function of the lattice permutations, and P_{xy} that of the prime permutations, slight reflection shews that

$$N_{xy} = \frac{1}{1-P_{xy}}.$$

Consider first the case $p = q$, it is seen that every lattice permutation of the assemblage

$$\alpha^p\beta^p$$

produces a prime permutation of the assemblage

$$\alpha^{p+1}\beta^{p+1},$$

by prefixing α and affixing β. Thus

Lattice Permutations.	Prime Permutations.
1	$\alpha\beta$
$\alpha\beta$	$\alpha\alpha\beta\beta$
$\alpha\alpha\beta\beta$	$\alpha\alpha\alpha\beta\beta\beta$
$\alpha\beta\alpha\beta$	$\alpha\alpha\beta\alpha\beta\beta$
$\alpha\alpha\alpha\beta\beta\beta$	$\alpha\alpha\alpha\alpha\beta\beta\beta\beta$
$\alpha\alpha\beta\alpha\beta\beta$	$\alpha\alpha\alpha\beta\alpha\beta\beta\beta$
$\alpha\alpha\beta\beta\alpha\beta$	$\alpha\alpha\alpha\beta\beta\alpha\beta\beta$
$\alpha\beta\alpha\alpha\beta\beta$	$\alpha\alpha\beta\alpha\alpha\beta\beta\beta$
$\alpha\beta\alpha\beta\alpha\beta$	$\alpha\alpha\beta\alpha\beta\alpha\beta\beta$
$\vdots$	$\vdots$

if therefore for the case $p = q$, u_{xy} be the generating function for the lattice permutations, xyu_{xy} is the generating function of the prime permutations.

Hence
$$u_{xy} = \frac{1}{1-xyu_{xy}},$$

leading to
$$u_{xy} = \frac{1}{2xy} - \frac{1}{2xy}\sqrt{(1-4xy)}.$$

If $p \neq q$, there is only one additional prime permutation, viz., α, so that

$$x + xyu_{xy},$$

enumerates the prime permutations, and if v_{xy} be the function which enumerates the lattice permutations

$$v_{xy} = \frac{1}{1-x-xyu_{xy}},$$

where $$u_{xy} = \frac{1}{1-xyu_{xy}}.$$

Eliminating u_{xy} or substituting the found expression for u_{xy}, we find

$$v_{xy} = \frac{1}{2x}\,\frac{\sqrt{(1-4xy)}+2x-1}{1-x-y}.$$

In the case of the assemblage $\alpha^p\beta^q\gamma^r$,

the crude enumerating generating function is

$$\frac{\left(1-\frac{y}{x}\right)\left(1-\frac{z}{x}\right)\left(1-\frac{z}{y}\right)}{1-x-y-z}.$$

The attempt to form the exact generating function through the medium of the prime permutations has not yet proved successful. The theory of the prime permutations requires investigation.

Thursday, May 14th, 1914.

Prof. A. E. H. LOVE, President, in the Chair.

Present seventeen members and two visitors.

The following papers were communicated :—

Diffraction by a Straight Edge: Prof. H. M. Macdonald.

Quadratic Forms and Factorization of Numbers: Hon. H. F. Moulton.

On the Reduction of Sets of Intervals: Prof. W. H. Young and Mrs. Young.

Diffraction of Tidal Waves on Flat Rotating Sheets of Water: Mr. J. Proudman.

The Algebraic Theory of Modular Systems: Dr. F. S. Macaulay.

ABSTRACTS.

Quadratic Forms and Factorization of Numbers: Hon. H. F. Moulton.

This paper describes a method for greatly diminishing the labour of finding whether a number M, which has one known representation X^2+DY^2, has any other $X'^2+DY'^2$, and of so finding the factors of M.

Take first the case where $X^2+4Y^2=M$. The ordinary methods of elimination show that if $M\equiv p^2 \bmod k$, where k is a prime, then X and also $X'\equiv \pm n \bmod k$, where n has one of a limited number of values, at most $\frac{k+1}{4}+1$. This paper shows that these values can be divided into two groups, and that if it is known that k is a residue of each factor of M, the value of $X' \bmod k$ must lie in the same group as that of $X \bmod k$.

It is also shown that the same method may apply to representations $M=X^2+DY^2$ if $D/k=1$, but in this case the various possible expressions of the factors m, m' of M by forms of determinant $-D$ must be considered, *e.g.*, $m=ax^2+2bxy+cy^2$, $m'=a'x'^2+2b'x'y'+c'y'^2$, and the criterion for whether the X's lie in the same or different groups, is whether am and $a'm'$ are both residues or both non-residues of k. It is shown that the grouping is the same for all values of D such that $D/k=1$ if k is an odd prime, but if k is composite and $=k_1k_2$, these are different groupings according as $D/k_1=D/k_2=\pm 1$. The method can also be applied with certain modifications to representations of the form $M=AX^2+BY^2$, where $AB/k=1$. The grouping need only be tabulated for cases where $M\equiv 1 \bmod k$, as that for other cases can be derived by multiplication.

The fundamental theorem on which the method is based is that if a number M has two factors m, m' expressible as $ax^2+2bxy+cy^2$, ..., where $ac-b^2=a'c'-b'^2=D$, then $aa'M$ has two representations

$$X^2+DY^2,\quad X'^2+DY'^2,$$

and
$$am=\frac{1}{4(a'x'+b'y')^2}\left[(X+X'^2)+D(Y-Y'^2)\right]$$

$$=\frac{1}{4Dy'^2}\left[(X-X'^2)+D(Y+Y'^2)\right],$$

$a'm'$ has two similar values. If then

$$D/k=1 \quad\text{and}\quad am/k=a'm'/k=\pm 1,$$

then
$$\left[(X\pm X'^2)+D(Y\pm Y'^2)\right]/k=\pm 1,$$

according as $$am/k = \pm 1.$$

The grouping for odd primes up to 25 is as follows, a semicolon showing the division of the groups.

Number ...	3	5	7	11	13	17	19	23
Grouping...	1; 0	1; 0	1, 0; 2	1, 5; 0, 3	1, 6; 0, 2	0, 1, 3; 4, 6	1, 2, 7; 0, 3, 4	0, 1, 8, 11; 4, 9, 10

Diffraction of Tidal Waves on Flat Rotating Sheets of Water: Mr. J. Proudman.

The methods of this paper are very similar to those already well known for two-dimensional problems in the diffraction of sound and electric waves.

Only free tidal motion of sheets of uniform depth is considered.

A complete solution is obtained for the case of the diffraction of a plane wave by a circular island, but the remaining solutions are all approximations. They are based on Lord Rayleigh's approximate theory of diffraction, and the method of conjugate functions is introduced so that Schwarz's method for conformal transformations becomes available. The problems considered are those of the diffraction of a plane wave by an elliptic island, by a semi-elliptic cape, by a rectangular bay, and by a passage between one sea and another.

On the Reduction of Sets of Intervals: Prof. W. H. Young and Mrs. Young.

It has been already pointed out that before sets of intervals were studied for their own sake, various writers had had occasion to make use of them, and had in this way virtually obtained the Heine-Borel theorem. Without entering again into these matters, a new moment is introduced, and it is shown how the careful study of the early documents leads to a new theorem, including the Heine-Borel theorem as a special case. The proof of this theorem is obtained by retaining all that is essential in Heine's proof of the property of uniform continuity, and rejecting that which is accessory.

The new theorem is not only more general than the Heine-Borel theorem, it leads to results unobtainable by application of the latter theorem alone. From it we deduce as easy corollaries, beside that already mentioned, Lusin's kindred theorem, used in the proof of his result that

a continuous function cannot have an infinite differential coefficient at a set of points of positive content; Young's first lemma, used in our proof that a function with bounded derivates is the integral of any one of them; the tile theorem, used in various theorems on integration, as well as the earliest theorem on the reduction of a perfectly general set of overlapping integrals, given in 1902 in the *Proceedings* of this Society. These various and scattered theorems, and perhaps others also, are in this way classified systematically together, and proved without any use being made of the idea of transfinite ordinal types, or Cantor's numbers.

In the course of the work it was found that there was a flaw in the deduction of Lebesgue's lemma from Young's first lemma, given in a former number of the Society's *Proceedings*. Lebesgue's lemma remains therefore dependent on Cantor's numbers. The same error occurred in that one of the two proofs of Young's second lemma which was independent of Contor's numbers; a new proof is accordingly here supplied of a simpler nature. Thus the proof of all Lebesgue's results on derivates and their integrals and the extensions of these results already given without Cantor's numbers remain valid.

On the Algebraic Theory of Modular Systems: Dr. F. S. Macaulay.

The principal object of this paper is to supplement the account of the theory of modular systems given in the *Encyclopädie der Mathematischen Wissenschaften*. It also includes some properties which have not been published previously.

Thursday, June 11*th*, 1914.

Prof. A. E. H. LOVE, President, in the Chair.

Present thirteen members.

The President announced that the Council had awarded the De Morgan medal to Prof. Sir J. Larmor.

The following papers were communicated :—

A Problem of Diophantine Approximation : Mr. R. H. Fowler.

Some Theorems by Mr. S. Ramanujan : Mr. G. H. Hardy.

Proof of the general Borel-Tauber Theorem: Messrs. G. H. Hardy and J. E. Littlewood.

Theorems relating to Functions defined implicitly, with Applications to the Calculus of Variations: Prof. E. W. Hobson.

On the Differentiation of a Surface Integral at a Point of Infinity: Dr. J. G. Leathem.

On Mersenne's Numbers: Mr. R. E. Powers.

Free and Forced Longitudinal Tidal Motion in a Lake: Mr. J. Proudman.

ABSTRACTS.

A Problem of Diophantine Approximation: Mr. R. H. Fowler.

In a recent paper,* Messrs. G. H. Hardy and J. E. Littlewood have investigated the distribution of the points

$$(a^{\nu}\theta) \quad (\nu = 1, 2, \ldots, n),$$

where $(a^{\nu}\theta)$ is the fractional part of $a^{\nu}\theta$, and a is an integer, over the interval (0, 1) for large values of n. This is almost the same problem as the investigation of the distribution of the first n digits in the decimal expression for θ in the scale of a, and it is from this point of view, which is in that case the more interesting, that they solve the problem.

In this paper, I extend some of their results to the set of points $(\lambda_{\nu}\theta)$, resulting from any sequence of positive numbers $\lambda_1\lambda_2 \ldots \lambda_n \ldots$ which satisfy the inequalities

$$\lambda_n/\lambda_{n-1} \geqslant \beta^{n^{-1+\xi}} \quad (\xi > 0, \ \beta > 1)$$

for all values of $n \geqslant n_0$. Roughly speaking, it may be said that my results hold for any sequence for which the increase of the n-th term is sufficiently regular, and faster than that of

$$\exp(n^{\xi}),$$

for some positive value of ξ.

The following is the principal theorem of the paper:—

THEOREM.—*If* $\{\lambda_n\}$ *be any sequence of positive numbers satisfying*

$$\lambda_n/\lambda_{n-1} \geqslant \beta^{n^{-1+\xi}} \quad (\xi > 0, \ \beta > 1),$$

if δ *be the length of any interval included in the interval* (0, 1), *and if* Δ_n

* "Some Problems of Diophantine Approximation," *Acta Math.*, Vol. 37, pp. 155–190.

be the number of the first n numbers $(\lambda_\nu \theta)$ *that fall inside* δ, *then*

$$\Delta_n \sim \delta n,$$

for all θ's *between* 0 *and* 1 *which do not belong to a set of measure zero.*

I conclude the paper by considering the extension of this theorem to the m-dimensional set of points

$$(\lambda_n \theta_1),\ (\lambda_n \theta_2),\ \ldots,\ (\lambda_n \theta_m).$$

Theorems relating to Functions defined implicitly, with Applications to the Calculus of Variations: Prof. E. W. Hobson.

This paper contains extensions of the known theorems relating to the determination of values of a set of real variables $u_1, u_2, \ldots, u_n$, defined explicitly by means of a set of equations connecting them with m real variables $v_1, v_2, \ldots, v_n$; when one system of corresponding values of the two sets of variables is known.

The cases are treated in which the equations involve a set of parameters as well as the variables, and in which a set of systems of corresponding values of the two sets of variables is known, instead of only one such system. The results obtained are applied to proofs of theorems concerning the existence of Weierstrassian fields in the problems of the calculus of variations.

On the Differentiation of a Surface Integral at a Point of Infinity: Dr. J. G. Leathem.

The subject is the differentiation of a surface integral on a curved surface with respect to a displacement $\delta\lambda$ of a point O at which the subject of integration f has an infinity. Use is made of a lemma which constitutes a somewhat general relation between a surface integral and a line integral round the boundary curve T. Subject to certain restrictions on the form of f a differentiation formula is obtained in the shape

$$\frac{d}{d\lambda}\int^T f\,dS = \lim_{\eta\to 0}\left\{\int_\eta^T \frac{\partial f}{\partial\lambda}\,dS - \int_\eta f\cos(\delta\lambda, \nu)\,ds\right\},$$

where η is a vanishing cavity round O, and ν is the direction of the normal to η. If the original integral is absolutely convergent, the form of η is arbitrary; otherwise it is subject to restriction.

By way of illustration the theorem is employed to determine the tangential force at a point (i) in a sheet of gravitating matter of given surface density, (ii) in a double sheet. Incidentally a method of discussing the

convergence and discontinuities of the potential due to a double sheet is explained.

On Mersenne's Numbers: Mr. R. E. Powers.

The purpose of Mr. Powers's paper is to show that the Mersenne number $2^{107}-1$ is prime, four mistakes having now been found in Mersenne's classification, viz., 2^p-1 proved composite for $p = 67$, and prime for $p = 61$, 89 and 107, contrary to his assertion. That $2^{107}-1$ is a prime number is shown by means of the following theorem, which was proved by Lucas in 1878:—

If $N = 2^{4q+3}-1$ ($4q+3$ prime, $8q+7$ composite), and we calculate the residues (modulo N) of the series

$$3, \ 7, \ 47, \ 2207, \ \ldots,$$

each term of which is equal to the square of the preceding, diminished by two units: the number N is prime if the residue 0 occurs between the $(2q+1)$-th and the $(4q+2)$-th term; N is composite if none of the first $(4q+2)$ residues is 0.

The 106th term of the above series is congruent to 0 (modulo $2^{107}-1$), consequently the latter is a prime number.

Free and Forced Longitudinal Tidal Motion in a Lake: Mr. J. Proudman.

Following Prof. Chrystal, the solution of this problem is made to depend on that of the differential equation

$$\frac{d^2V}{dx^2} + \frac{\lambda}{p(x)} V = F(x), \tag{1}$$

between $x = 0$ and $x = a$, when V vanishes at these limits. Here λ is a constant, and $p(x)$, $F(x)$ are functions of x subject to certain very general conditions.

Take

$$I_n(\xi, \eta) = \frac{1}{a} \int_{s_n=\xi}^{\eta} \int_{s_{n-1}=\xi}^{s_n} \cdots \int_{s_2=\xi}^{s_3} \int_{s_1=\xi}^{s_2} \frac{(\eta-s_n)(s_n-s_{n-1}) \ldots (s_2-s_1)(s_1-\xi)}{p(s_1)\, p(s_2) \ldots p(s_{n-1})\, p(s_n)} \times ds_1\, ds_2 \ldots ds_{n-1}\, ds_n, \tag{2}$$

for $n > 0$, with $I_0(\xi, \eta) = (\eta-\xi)/a$, where $0 \leqslant \xi \leqslant \eta \leqslant a$.

Take also $$R(\xi, \eta, \lambda) = \sum_{n=0}^{\infty} (-\lambda)^n I_n(\xi, \eta). \tag{3}$$

Sufficient conditions will be given in the paper for the existence of (2) and the convergence of (3).

The *free modes* are then given by

$$V = R(0, x, \lambda_n), \tag{4}$$

where λ_n is such as to satisfy

$$R(0, a, \lambda_n) = 0, \tag{5}$$

and the *forced motion* is given by

$$V = -\frac{a}{R(0, a, \lambda)} \left\{ R(x, a, \lambda) \int_0^x R(0, s, \lambda) F(s) \, ds + R(0, x, \lambda) \int_x^a R(s, a, \lambda) F(s) \, ds \right\}. \tag{6}$$

In the paper these statements will be proved, and the solution shown to arise naturally when the equation (1) is regarded as the limiting form of a certain difference equation. The connection of the solution with that of a particular Fredholm's equation will be indicated and various other relations proved.

It appears probable that, with a reasonable amount of labour, the solution can be applied, by means of approximate methods, to such forms of $p(x)$ as are provided by a bathymetric survey of a lake. This is at present being investigated.

LIBRARY

Presents.

BETWEEN August, 1913, and July, 1914, the following presents were made to the Library, from their respective Authors or Publishers :—

Elliott, E. B.—" Algebra of Quantics."
Royal Society.—" Catalogue of Periodical Publications in the Library."
Webb, W. L.—" Brief Biography and . . . . unparalleled Discoveries of T. J. J. See."

Calcutta : Indian Engineering, vol. 54, nos. 9–26, 1913 ; vol. 55, 1914.
Cape Town : South African Journal of Science, vol. 10, nos. 1, 2, 3, 5, 7, 1913 ; 1914, vol. 10, no. 8.
Coimbra : Accademia Polyt. de Porto, Ann. Scientificos, vol. 8, nos. 3, 4, 1913.
Hamburg : Math. Gesellschaft, Mittheilungen, bd. 5, heft 3, 1914.
Kansas : University of Kansas Science Bulletin, vol. 6, nos. 2–7, 1913.
Kyoto : College of Science and Engineering, Imperial University, Memoirs, vol. 4, 1912 ; vol. 5, nos. 1–9, 1913 ; vol. 6, nos. 1–3.
London : Educational Times, vol. 66, nos. 629–632, 1913 ; vol. 67, nos. 633–638, 1914.
London : Educational Times, Math. Questions and Solutions, new series, vol. 24, 1913.
London : Mathematical Gazette, vol. 7, nos. 107–109, 1913.
London : Nautical Almanac for 1916.
Madras : Indian Mathematical Society, Journal, vol. 5, nos. 3–5, 1913.
Paris : L'Enseignement Math., ann. 15, no. 6, 1913 ; ann. 16, nos. 1, 2, 1914.
Sendai : Tôhoku Imperial University, Science Reports, vol. 1, 1912 ; vol. 2, nos. 1–5, 1913 ; vol. 3, nos. 1–3, 1914.
Sendai : Tôhoku Mathematical Journal, vol. 3, nos. 2–4 ; vol. 4, nos. 1–4, 1913 ; vol. 5, nos. 1, 2, 1914.
Tokyo : Sûgaku-Butsurigaku, vol. 7, nos. 6–14, 1913.
Western Australian Astronomical Society, Proceedings, vol. 1, 1914.

Exchanges.

Between August, 1913, and July, 1914, the following exchanges were received :—

American Journal of Mathematics, vol. 35, no. 3, 1913.
American Mathematical Society, Transactions, vol. 14, no. 4, 1913 ; vol. 15, no. 1, 2, 1913.
American Mathematical Society, Bulletin, vol. 20, nos. 1–4, 6–8, 1913–14.
American Philosophical Society, Proceedings, vol. 52, nos. 211, 212, 1913.
Amsterdam : Nieuw Archiev, deel 10, stuk 3, 4, 1913.
Amsterdam : Revue Semestrielle, tome 21, pt. 2, 1913.
Amsterdam : Tables des Matières, tomes 16–20, 1913.
Amsterdam : Wiskundige Opgaven, deel 11, stuk 4, 5, 1913.

Belgique: Académie Royale des Sciences, Bulletin, 1913, nos. 7, 8.
Berlin: Jahrbuch über die Fortschritte, bd. 42, hefte 1-3, 1913.
Berlin: Journal für die Mathematik, bd. 143, 1913.
Berlin: Sitzungsberichte der K. Preuss. Akademie, 1913, nos. 23-53; 1914, nos. 1-18.
Bologna: R. Accademia delle Scienze, Classe Scienze Fisiche Rendiconti, vol. 16, 1912; Memorie, ser. 6, tom. 9, 1912.
Bordeaux: Société des Sciences, Procès-Verbaux, 1911-12.
Boston: American Academy of Arts and Sciences, vol. 48, nos. 18-21; vol. 49, nos. 1-11. 1913-14.
Calcutta: Mathematical Society, Journal, vol. 1, no. 4, 1913.
Cambridge Philosophical Society, Proceedings, vol. 17, pt. 5, 1914.
Cambridge Philosophical Society, Transactions, vol. 22, pts. 3, 4, 1913.
Cambridge, Mass.: Annals of Mathematics, vol. 15, nos. 2, 3, 1913.
Canadian Institute, Transactions, no. 23, 1913.
Canadian Institute, Year Book and Annual Report, 1912-13.
Catania: Accademia Gioenia, Bollettino, ser. 2, fasc. 26-30, 1913.
Dublin: Royal Irish Academy, Proceedings, vol. 29, no. 6, 1912; vol. 30, nos. 1-6, 1912.
Edinburgh: Mathematical Society, Proceedings, vol. 30, 1912.
Edinburgh: Royal Society, Proceedings, vol. 32, 1912.
France: Société Mathématique, Bulletin, tome 41, fasc. 3, 1913.
Göttingen: Königl. Gesell. der Wissenschaften, Nachrichten, Math. Klasse, 1913; hefte 3, 4.
Göttingen: Königl. Gesell. der Wissenschaften, Nachrichten, Geschaft. Mitth., 1913, heft 1.
København: Kgl. Danske Vidensk. Selsk., Forhandlinger, Oversigt, 1913, nos. 3-6; 1914, nos. 1, 2.
København: Kgl. Danske Vidensk. Selsk., Skrifter, ser. 7, tome 10, no. 3.
Leipzig: Beiblätter zu den Annalen der Physik, bd. 37, hefte 12-24, 1913; bd. 38, hefte 1-11, 1914.
Leipzig: K. Sächsische Gesell., Math. Klasse, Berichte, 1913, nos. 2, 3.
Livorno: Periodico di Matematica, anno 29, fasc. 1, 1913.
Livorno: Periodico di Matematica, Supplemento, anno 17, fasc. 2, 1914.
London: Nature, vol. 92, 1913-14; vol. 93, nos. 2314-2338, 1914.
London: Royal Society, Proceedings, Series A, vol. 89, nos. 610-614; vol. 90, nos. 615-618, 1913-14.
London: Royal Society, Transactions, Series A, vol. 213, 1913.
London: Physical Society, Proceedings, vol. 26, pts. 1-3, 1913.
London: Institution of Naval Architects, Transactions, vols. 52 and 53, 1911-12.
London: Institute of Actuaries, Journal, vol. 56, pts. 2-4, 1912; vol. 57, pt. 1, 1913.
Manchester Literary and Philosophical Society, Memoirs, vol. 56, 1912.
Modena: Regia Accademia, Memorie, Ser. 3, vol. 10, pt. 2, 1913.
München: K. Bayer Akad. d. Wissen., Math.-Phys. Kl., Abhandlungen, bd. 26, abh. 2-10, 1913.
München: K. Bayer Akad. d. Wissen. Math.-Phys. Kl., Sitzungsberichte, 1913, abh. 1-23.
München: K. Bayer Akad. d. Wissen. Math.-Phys. Kl., Register, 1860-1910, 1807-1913.
Palermo: Rendiconti del Circolo Matematico, tome 36, 1913.
Paris: Bulletin des Sciences Mathématiques, Nov.-Déc., 1913; Jan.-Mai, 1914.
Paris: Des Publications Mathématiques, tome 22, no. 1.
Porto: Academi Polytechnica de Porto, vol. 9, no. 1.
Roma: Reale Accademia dei Lincei, Rendiconti, vol. 22, sem. 2, 1913; vol. 23, sem. 1-9, 1914.
Stockholm: Acta Mathematica, bd. 37, pts. 1-3, 1913.
Torino: R. Accademia delle Scienze, Atti, vol. 48, disp. 10-15, 1913; vol. 49, disp. 1-7, 1914.
Washington: Naval Observations for Year 1913.
Wien: Monatshefte für Mathematik, jahr. 25, viertel. 1, 2, 1914.
Zurich: Vierteljahrsschrift, hefte 3, 4, 1913.

OBITUARY NOTICES

GEORGE MINCHIN MINCHIN.

By the death of George Minchin Minchin on March 16th, 1914, there was lost to science a man of many-sided ability, which had borne fruit in original investigation and model exposition, and of charming personality, which had endeared him to a large circle of friends.

He was born in Valentia Island, Co. Kerry, Ireland, on May 25th, 1845. After the death of his mother, and from the time when he was about nine years of age, he was brought up in Dublin by his materna uncle by marriage, Mr. David Bell, a distinguished Shakespearean scholar. Mr. Bell kept a school; and among the scholars contemporary with Minchin, and also living in Mr. Bell's house, was another nephew, Alexander Graham Bell, afterwards of telephone fame. Minchin's mathematical bent was soon discovered, and special tuition in the subject was procured for him; but he never regretted the time spent on literary studies, and there can be no doubt that the admirable simplicity and lucidity of his writings are to be attributed largely to his classical and linguistic training. He entered Trinity College, Dublin, in 1862, and during the next few years gained many scholarships and prizes for mathematics. In 1875 he was appointed to the chair of mathematics in the Royal Indian Engineering College, Cooper's Hill, where he remained until the College was closed (1907), when he took up his residence in Oxford. He joined this Society in 1875, and was elected a Fellow of the Royal Society in 1895.

His most original scientific work was that on photo-electricity. His interest in experimental physics dated from his student days, and was revived after his appointment to the Cooper's Hill College. The fundamental observation from which his work grew was the fact that electric currents are produced by the action of light on plates of certain metals coated with various substances when the plates are immersed in suitable liquids. After many years of work he exhibited cells of this construction to the Physical Society in 1891. The paper describing the apparatus and experiments then shown will be found in the *Phil. Mag.*, Ser. 5,

Vol. 31 (1891). One form embodied the principle of the coherer, and another has since been adapted to tele-photography. But with a determination which surprised even those who knew him best, Minchin shrank from the steps that must be taken to turn a scientific discovery into a commercial success. It was of much greater interest to him to utilise his cells for measuring the light emitted by the stars. Observations for this purpose were made for him first by W. H. S. Monck, and later by W. E. Wilson, and the results recorded in a paper on "The Electrical Measurement of Starlight," *Proc. Roy. Soc.*, Vol. 58 (1895). The "Seleno-aluminium bridges" described by him in a later paper published in *Proc. Roy. Soc.*, Ser. A, Vol. 81 (1908), were meant to be substituted for the previously devised cells, as they achieve the same ends by simpler means. One need not be an experimental physicist to admire the patience and ingenuity displayed in these researches, but it is much to be hoped that a complete account of them will be given by someone who is more familiar with such work than the present writer.

Minchin was a natural philosopher: he wished to understand. Hence his interest in theoretical physics and applied mathematics was no less genuine than his interest in experimental work, and if in these departments his discoveries are less conspicuous, yet a deep debt of gratitude is owed to him for his books. It appears from letters written by G. F. Fitzgerald, with whom he maintained a regular correspondence for many years, that he at one time contemplated a general treatise on theoretical physics. Several of these letters are occupied with discussions of the best mode of treating thermodynamics. It seems that Minchin always rebelled against the doctrine of the degradation of energy. His thought on this matter is expressed in a little volume of prose and verse entitled *Naturae Veritas*, published in 1887. The plan for a general treatise never came to maturity, but the well known *Treatise on Statics* was published in 1877—a comparatively small single volume. This treatise passed through several editions, each containing more numerous applications to physics than its predecessor. Other treatises from his pen are *Uniplanar Kinematics of Solids and Fluids* (1882), *Hydrostatics and Elementary Hydrokinetics* (1892), which developed into a two-volume *Treatise on Hydrostatics* (1912); *Geometry for Beginners* (1898), which has been described as an early and very favourable specimen of the methods of those who sought after the improvement of geometrical teaching, *The Student's Dynamics* (1900), and *Mathematical Drawing* (1906), the last named having been written in conjunction with J. B. Dale. Apart from books his most important mathematical writings are a paper on "Astatic Equilibrium" in these *Proceedings*, Ser. 1, Vol. IX (1878), and three

papers on the magnetic field in the neighbourhood of a wire carrying an electric current and the calculation of coefficients of self-induction in the *Phil. Mag.*, Ser. 5, Vols. 36 and 37 (1893–4). The impression produced by his writings is one of remarkable geometrical elegance combined with distinct analytical power and profound thinking on physical subjects, while for clearness and apt expression they might well serve as models of what mathematical writings ought to be. Those who knew him know that the range of his knowledge was much wider than the range of his writings, wide as this was; there were few matters coming within the domain of mathematics and physics on which he had not something interesting to say. In Oxford especially, where he was a constant worker in the Electrical Laboratory, and frequently read papers before the Mathematical and Physical Society, expending much trouble and thought upon their preparation, his loss is felt deeply.

A. E. H. L.

MORGAN JENKINS.

By the death of Mr. Morgan Jenkins, the Society has lost its oldest officer. Appointed Secretary within a year of its foundation, he served the Society long and well, holding this office for twenty-nine years.

Mr. Jenkins was the fifth son of Mr. James Jenkins, of Coventry, and was born there on November 10th, 1841. He was educated at the Coventry Grammar School, Mr. Sheepshanks, the father of the late Bishop of Norwich, being the Head Master at the time. He thence proceeded to Christ's College, Cambridge, where he obtained a scholarship and graduated as bracketed 34th Wrangler in 1864. After leaving Cambridge he was a student at Lincoln's Inn, and passed the examinations, but was not called to the Bar. From 1867 to 1870, he was a mathematical tutor at Wren's establishment in Powis Square, W. (preparation for the Government Examinations, &c.). From 1870 or 1871 to 1885, he took pupils for the Army or Cooper's Hill, at his house, 50 Cornwall Road, W. He then lectured at Queen's College, London, for a year. In 1886, he was appointed lecturer at King's College, London, a post which he held till he left London in 1894 to reside at Tunbridge Wells. He was also for some years Assistant Examiner at South Kensington. On July 14th, 1880, he married Jane, daughter of Mr. William Potter, of Kirby Moorside, Yorkshire, who survives him. There were no children. He died at Tunbridge Wells, on October 21st, 1913. For the last three years of his life he suffered from severe heart trouble, but he was always bright and cheerful and enjoyed working in his garden.

The first meeting of the London Mathematical Society was held on January 16th, 1865, when Augustus De Morgan gave an address. Mr. Benjamin Kisch acted as Secretary, but at that meeting Mr. H. M. Bompas and Mr. Cozens Hardy were elected Secretaries, the latter being also Treasurer. Mr. Cozens Hardy held office for only a month, and was succeeded by Mr. W. Jardine. Both Secretaries retired on November 10th, 1865, and Mr. Jenkins (who had been elected a member of the Society on May 15th) was appointed deputy Secretary and Treasurer until the Annual Meeting, on January 15th, 1866, when he and Mr. G. C. De Morgan (son of Augustus De Morgan) were elected Secretaries. Mr. Jenkins held his office until November, 1894, but Mr. De Morgan died in October, 1867, and was succeeded by Mr. Tucker, who was elected Secretary in the following month, and held this office for thirty-five years.

Mr. Tucker had been elected a member of the Society on October 16th, 1865, five months after Mr. Jenkins, and he became Secretary fourteen months after Mr. Jenkins. Thus, following so many changes in the Secretaryship dnring the first two years of the life of the Society, there was aperiod of twenty-eight years (1867–1895) without a change; and (except for the first fourteen months) Mr. Jenkins had Mr. Tucker as his colleague during the whole of his tenure of office.

Mr. Tucker, as is well known to many of us, was a most indefatigable Secretary, and the Society was the one great interest of his life; but in a quiet way Mr. Jenkins was no less devoted. In the division of the work Mr. Tucker's department included everything relating to the papers read before the Society: he procured the papers for the meetings, often extracting them from apathetic or reluctant authors (as otherwise, in early days, there would have been some meetings without a paper), he sent them to the referees and was persistent in his efforts to obtain their reports before the next meeting; he transmitted extracts from referees' reports to authors, and superintended the publication of the *Proceedings;* he also kept the minutes. All this formed much the larger and more difficult part of the secretarial duties; and Mr. Jenkins's share was not only smaller in amount but less exacting. All matters relating to the election of members belonged to his province, and he kept the list of members up to date with conscientious care, sparing no pains to obtain suitable details to be added to the mere list of names and addresses. The ballot box also was in his department both in regard to the election of members and the Council's decision on the printing of a paper. The Society possessed no servant or paid officer, and the whole of the work, down to the smallest details, was done by the Secretaries themselves. For many years the *Proceedings* were sent out personally to the members by the Secretaries, each Secretary taking one half of the alphabet. The present writer's copy fell in Mr. Jenkins's half, and it always came to him addressed in his careful and accurate handwriting, for, unlike Mr. Tucker, whose handwriting was rapid and sometimes hard to decipher, Mr. Jenkins always wrote slowly, forming every letter with care, and correcting errors by an erasure with a penknife.

It is difficult for anyone who knew the Society in the seventies and eighties to think of it without picturing Mr. Jenkins, the senior Secretary, on the President's left, and Mr. Tucker on his right. It is said that Mr. Tucker only missed three meetings of the Society while he was Secretary, and Mr. Jenkins seemed quite as regular; at all events the writer cannot remember an occasion on which he was absent. In all the years that he was Secretary he seemed to change very little in appearance, having to

the last a somewhat youthful look. He was unassuming in his manner, almost self-effacing, and rarely spoke at the Council meetings, except on matters concerning his own secretarial business; and he scarcely ever took part in the discussions on papers at the meetings. Mr. Tucker was so zealous for the Society that, as has been mentioned, he took upon himself the whole of the business relating to the papers—which, of course, were the life's blood of the Society—and Mr. Jenkins was content to occupy himself with all that was left for him, but he would have willingly taken a larger share of work had his colleague been less omnivorous. The debt the Society owes to its two early Secretaries, who held office for so long, is very great. They placed it on a secure foundation. The rapid progress of the Society was no doubt mainly due to Mr. Tucker, but Mr. Jenkins rendered useful and continuous aid in matters that were essential to its efficiency. Mr. Jenkins only retired from the Secretaryship in 1895, after he had left London. He was then elected Vice-President, and held this office for the usual term of two years, but his attendance was irregular, and after a time the journey to London became too much for him.

Neither Mr. Jenkins nor Mr. Tucker were famous mathematicians, nor had they obtained high honours at Cambridge, or won fellowships, but they yielded to no one in their interest in mathematical science or their desire for its advancement. The Mathematical Society was not one which could confer public distinction on its office-holders, and both Secretaries gave their services simply for love of the subject. One of the earliest papers read to the Society (in 1865) was by Mr. Jenkins on "The Regular Hypocycloidal Tricusp," which, however, does not appear among the few papers printed in that year. His contributions to mathematics were not numerous or lengthy, but all were the result of careful study.

The subject that interested him most was Theory of Numbers, and a paper of his relating to Legendre's Law of Reciprocity was published in Vol. II of the *Proceedings*. He also wrote several papers on Spherical Trigonometry, which also had a special interest for him. Other subjects on which he published occasional papers in the *Quarterly Journal* and *Messenger* were the Geometry of the Triangle, Combinations, &c.

He retained his interest in the Theory of Numbers to the end, and in a letter written to the present writer in June, 1910, he mentioned that he was afraid he was too old to do any useful mathematics, but that there was a theorem concerning the resolution of a number into the sum of three squares which he was still most anxious to prove. This letter was written in the same careful handwriting as of old.

J. W. L. G.

SAMUEL ROBERTS.

SAMUEL ROBERTS was the second son of the Rev. Griffith Roberts, Presbyterian Minister, and Anna his wife, who was the eldest daughter of Mr. Samuel Churchill of Exeter, merchant. He was born on December 15th, 1827, and was educated at Queen Elizabeth's Grammar School, Horncastle, Lincolnshire, his father being then minister of the Presbyterian Chapel at Kirkstead, near Horncastle, and resident at Horncastle.

He entered Manchester New College in June, 1844. He matriculated at the University of London in 1845, with Honours in Classics and Mathematics, and took the degree of B.A. in 1847, with Honours in Mathematics, and the degree of M.A. in 1849, when he was first in Mathematics and Natural Philosophy, and obtained the gold medal.

He was admitted a solicitor in January, 1853, having served his articles of clerkship with Mr. Richard Mason, Town Clerk of Lincoln.

After some years he gave up practice and removed to London, where he devoted himself to mathematical studies.

He was twice married: in 1858 to Mary Ann Astley, only child of the Rev. Richard Astley, formerly of Shrewsbury, who died in 1895; and, in 1896, to Lucy Elizabeth Holland, second daughter of Philip Henry Holland, surgeon, who survives him. By his first wife he had three children, the eldest of whom, Samuel Oliver Roberts, was born in 1859, and died in 1899. He was a scholar of St. John's College, Cambridge, and 7th Wrangler in 1882; he was afterwards Mathematical Master at Merchant Taylors School. His third son died in infancy. His second son, Mr. H. A. Roberts, a solicitor, survives him.

Mr. Roberts was one of the oldest members of the Society, having been elected on June 19th, 1865, five months after the foundation of the Society. It cannot be said of him as it was said of Mr. Thomas Cotterill, that the Society "discovered" him, for he had published a paper in the *Philosophical Magazine* as early as 1848, and had contributed more than a dozen papers to the *Quarterly Journal* before the Society was founded: still it is certainly true that the Society was from the first a powerful incentive to him, and that it always afforded him great encouragement. It brought him into contact with other mathematicians whom he otherwise might not have known personally, for, not having been at Oxford or Cambridge, he was a somewhat solitary worker until the Society came into existence. Within six months of his becoming a member he was

placed on the Council, on which he served from 1866 to 1892, except for a break of one year (1870–71). He was also Treasurer for eight years (1872–80), was twice Vice-President, each time for two years (1871–73, 1882–84), and President (1880–82). He thus rendered valuable and constant service to the Society, and for a number of years was almost as much identified with it as the Secretaries, and even more so in one respect, as he was a frequent contributor of papers.

Mr. Roberts was the second Treasurer of the Society after the separation of the Treasurership from the Secretaryship. As was mentioned in the obituary notice of Mr. Jenkins, one of the Secretaries acted also as Treasurer during the first year of the Society's life, Mr. Jenkins holding the double office from November, 1865, till January, 1866. The Treasurer then became an independent officer, and Dr. Hirst held the position from 1866 to 1872, when he was succeeded by Mr. Roberts, who in 1880 was succeeded by Mr. Merrifield.

Mr. Roberts received the De Morgan Medal in 1896. He was elected a Fellow of the Royal Society in 1878.

Mr. Roberts's contributions to mathematics were numerous and valuable, and they covered a somewhat wide range. Among the subjects to which his principal papers related were plane and solid Geometry, Theory of Numbers, and link motion. He also wrote on the Calculus of operations, interpolation, &c. His writings on Geometry included several important papers on Parallel Curves and Surfaces. In Theory of Numbers he was interested in the Pellian equation and similar problems. A noticeable paper of his related to the proof that the theorems which express the product of 2^m squares by 2^m squares as the sum of 2^m squares do not admit of extension beyond $m = 3$. The greater number of his papers were published in our *Proceedings*, but he contributed about thirty papers to the *Quarterly Journal* and several to the *Messenger*. Cayley once said to the writer that he regretted that joint papers were so rare in mathematics, and that he should have liked to co-operate with Mr. Roberts in several pieces of work. This remark, which was made towards the close of Cayley's life, had special reference to Mr. Roberts's paper "On the Motion of a Plane under given Conditions," published in the third volume of our *Proceedings*, of which Cayley makes use in a paper "On the Kinematics of a Plane," &c., published in 1894 (Vol. XIII, p. 505, of his *Collected Papers*).

Although mathematics was the main occupation of Mr. Roberts's life, he had various other interests, such as geology, microscopy, and working with the lathe. In the last ten years of his life his eyesight almost completely failed him.

Mr. Roberts is an interesting type of mathematician, for he was strictly an amateur—that is to say, he never held any office, directly or indirectly, connected with mathematics or other branch of science or any application of science. He was simply a private gentleman who pursued researches from pure love of investigation and desire to extend the boundaries of subjects that were attractive to him. And he never sought any recognition of his work; for him it was entirely its own reward. Owing mainly to the course of instruction at Cambridge, there are numerous cases in which ecclesiastical dignitaries, barristers, and judges have contributed to our science: indeed, all Cayley's early work was done while he was at the bar, and it was only the foundation of the Sadlerian Professorship at Cambridge that tempted him to forgo his legal career. But instances are rare of men who, not having been Fellows of a College or having had any official connection with mathematics, have yet worked assiduously and with adequate knowledge for its advancement.

In the seventies and eighties the members who attended the Society's meetings with some regularity formed almost a family party, and among them Mr. Roberts occupied a conspicuous position. He was always present at the Council meetings and those of the Society, and was most willing to do any work which he was asked to undertake, but he was somewhat reserved and reticent, and did not take a leading part in the business of the Council. The value and amount of his work prove his devotion to mathematics, but he never gave way to any manifestation of enthusiasm, nor did his undemonstrative manner seem to indicate any very keen interest, but the meetings of the Society were evidently very congenial to him.

It is perhaps, at this distance of time, of some interest to recall the diversity of manner in which some of the regular frequenters of the Society communicated their papers to the audience. All were no doubt greatly interested in their subject, but they differed fundamentally in the way they they gave an account of their work to the Society, and in the importance which they attached to the oral explanation of their results. Cayley regarded the reading of a paper as a disagreeable formality that had to be gone through before it could be printed. Nevertheless he endeavoured to give a clear account of the contents of his paper, and was never lengthy. Still, unless one knew the subject beforehand, his method of running over the leading formulæ was not instructive. Henry J. S. Smith once said to the writer that he thought that the account given of a paper ought to end where the paper began. He carried out this principle and the reading of a paper by him was an admirable exposition, gracefully delivered, of the nature of the subject to which the paper belonged. He started *ab initio*,

and when the point of departure of the paper was reached, he briefly indicated the kind of problem he was dealing with, which had then become comprehensible to all. It was always possible for anyone to learn something of a new subject from Henry Smith's reading of a paper; and if the subject was already familiar, his clearness of mind and charm of expression gave it new interest. Clifford's expositions or discourses were perhaps even more attractive than Henry Smith's, but he allowed himself more latitude and they were even less closely connected with the subject of the paper, or perhaps they related to work he had merely proposed to himself or ideas that were not yet worked out. So much, however, depended upon Clifford's manner, his versatility, and his imagery, that after an address from him it was difficult to recall what really were the essential features of a discourse which had been so fascinating at the time. Mr. Roberts followed Cayley's method, only he gave details of working as well as results. With his paper in his hand he would go steadily through it, writing down the principal formulæ on the blackboard. Thus unless the subject was elementary, which was only occasionally the case, not much could be gained by an auditor who had no previous acquaintance with the state of the subject. Mr. J. J. Walker was careful and slow in his explanations, but the subjects he dealt with were usually more or less known to the audience. Sylvester very rarely attended a meeting of the Society, but when he did his excited enthusiasm was apparent in every word and action. Whatever interested him at the moment seemed to him of consummate importance to the science. Looking back on the twenty years (1870–90) no other very distinct personalities seem to stand out, except perhaps Hirst, whose papers related to subjects which at that time had not been much studied in this country. There was no attempt among the younger members to imitate Henry Smith or Clifford: as a rule we were content to do our best to describe our papers as well as we could without being too technical or tiring the audience.

It may be here remarked that very different opinions were held as to the actual utility of reading mathematical papers at the meetings. The views expressed depended very much upon the advantages the speaker himself was able to derive. Sylvester and Cayley were at opposite extremes. The former always set great store on verbal communications and personal intercourse, but Cayley said he rarely gained anything appreciable except from the printed page. It is unquestionable that Mr. Roberts received benefit from the reading of papers. He was an attentive listener, and probably more than any other member always kept his mind on the paper being read. For him the reading of a paper was very far from a formality.

For a number of years the Council met at 7.30, the Society's meeting being at 8 and lasting about two hours, sometimes more, but generally less. At length the time came when the half-hour allowed for the Council was found to be insufficient, and it then met at 7. When the Society moved to Albemarle Street in 1870, and for a number of years afterwards, the Council met in the small room at the back of the building on the first floor, and sat at the round table which, though it occupied most of the space in the room, afforded barely sufficient accommodation for a full Council, the meeting of the Society taking place in the large front room, in which the Council met in later years. All the associations of the early Council meetings in Albemarle Street are connected with this small room at the back.

It has always seemed to the writer very surprising that so young a Society should have been so precise with respect to reporting upon papers. Two independent referees were always appointed, their separate reports obtained and read, and the question of the publication of the paper was decided by ballot. In no case in the writer's experience was there any bias; nor was any distinction made in favour of distinguished mathematicians or on personal grounds. All papers were adjudicated upon by exactly the same procedure and with the same impartiality. In this respect the new Society was a pattern to many older and more dignified institutions, and perhaps no Society has ever been more strict or thorough in its attempt to maintain a definite standard and be equally fair to all. It is also probable that no Society has ever done so much for its subject with so small an income. Except for the rent of rooms and a trifling expenditure on the Library, all the income was spent on the *Proceedings*, and, until Lord Rayleigh made his gift of £1,000 to the Society, the means of the Society were very narrow.

In connection with the system of appointing referees, the writer may mention that his first introduction to Mr. Roberts was when he attended a meeting of the Society as a visitor in 1868. The meeting was held in the rooms of the Chemical Society at Burlington House, which had not then been rebuilt; and, while waiting in the meeting room, Mr. Roberts entered from the Council room and explained that he had temporarily left the Council while referees were being appointed for a paper of his.

Mathematicians are usually long lived and happily several of our earliest members are still with us. Prof. Clifton was an original member, and Prof. Crofton and Lord Justice Stirling served on the Council in 1867–68; but with the death of Mr. Jenkins and Mr. Roberts, all who were regular attendants at the Society's meetings during the first ten years of its existence have passed away. J. W. L. G.

PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

SERIES 2.—VOL. 13.—PART 1.

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THE LONDON MATHEMATICAL SOCIETY is instituted for the promotion and extension of Mathematical Knowledge.

It was founded in 1865, and incorporated under Section 23 of the Companies Act 1867 in 1894.

Every Candidate for Membership must be proposed and recommended, according to a form, which the Secretaries will supply, by not less than three Members, of whom one at least, except in special cases to be submitted for the decision of the Council, must certify his personal knowledge of the Candidate.

This form is read at one of the Ordinary or Annual General Meetings, and the Candidate is balloted for at the next ensuing meeting, provided that seven Members are present thereat.

The Candidate, if elected, is informed of his election by one of the Secretaries, and supplied with a copy of the Memorandum and Articles of Association and By-Laws. He must pay the contribution which is due from him within six months after the day of his election, otherwise his election shall be void.

An entrance fee of one guinea is required to be paid by each newly elected Member.

The Annual Subscription to be paid by each Member is one guinea: any Member may compound for his *annual* subscriptions by the payment of fifteen (or in certain cases ten) guineas in one sum.

Every Member is considered liable for his annual subscription until he has signified in writing his desire to resign, and has returned all books and property belonging to the Society.

Annual subscriptions are due at the beginning of each Session (the second Thursday in November of each year), and may be paid to the account of the London Mathematical Society, London County and Westminster Bank, 1 St. James's Square, London, S.W., or to the Treasurer, Dr. A. E. Western, 27 Pembridge Square, Notting Hill Gate, London, W.

The affairs of the Society are directed by the Council and Officers.

The Council consists of sixteen Members, including the Officers, and is chosen from among the Ordinary Members of the Society at the Annual General Meeting, held on the second Thursday in November.

The Officers are a President, Vice-President, a Treasurer, and Secretaries.

The Ordinary Meetings of the Society are held at its Rooms, 22 Albemarle Street, and commence at 5.30 o'clock in the evening. The dates of meeting for the Session 1913–1914 are the *second* Thursdays in November, December, February, March, May, June, and the *fourth* Thursdays in *January* and *April*. Cards giving the Programmes of the meetings as they occur can be had by Members a few days beforehand on application to C. F. Hodgson & Son, 2 Newton Street, Kingsway, London, W.C.

At these meetings papers are read and communications made: upon each paper or communication the Chairman invites discussion.

The Council alone decides whether any paper proposed for reading shall or shall not be read.

After a paper has been presented to the Society, it is referred by the Council to two or more Members, who report to the Council on its fitness for publication in the *Proceedings*. After hearing the reports, the Council decides by ballot whether it shall be printed or not.

Authors of Papers intended for communication to the Society are requested to furnish to the Secretaries short abstracts of their Papers, indicating the nature of the methods employed and the character of the results obtained.

Communications for the Secretaries may be forwarded to them at the following addresses:—

London Mathematical Society, 22 Albemarle Street, W. { J. H. GRACE. / T. J. I'A. BROMWICH.

Peterhouse, Cambridge.—J. H. GRACE.

1 Selwyn Gardens Cambridge.—T. J. I'A. BROMWICH.

PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

A THEOREM CONCERNING POWER SERIES

By Harald Bohr.

Communicated by G. H. Hardy.*

[Received February 5th, 1913.—Read February 13th, 1913.]

1. The theory of Diophantine approximation has, as I have shown in a series of previous papers,† important applications to the theory of Dirichlet's series. It becomes clear in the course of these investigations that the theory of the absolute convergence of Dirichlet's series of the type $\Sigma a_n n^{-s}$ is very closely connected with the theory of power-series in an infinite number of variables. As an illustration of this I may mention the following theorem:—

Let $\Sigma a_n n^{-s}$ be a Dirichlet's series absolutely convergent for $\sigma = \mathfrak{R}(s) = \sigma_0$, and let

$$P(x_1, x_2, \ldots) = c_0 + \sum_{\alpha} c_\alpha x_\alpha + \sum_{\alpha, \beta} c_{\alpha, \beta} x_\alpha x_\beta + \ldots$$

be the power-series which may be obtained formally from the Dirichlet's series by writing

$$p_1^{-s} = x_1, \quad p_2^{-s} = x_2, \quad \ldots,$$

where p_r denotes the r-th prime number. Let θ denote the set of points

* Extracted from letters of the author.

† See, for example, *Acta Mathematica*, Vol. 36, pp. 197–240.

in the plane of the complex variable corresponding to the values taken by the Dirichlet's series when the variable $s = \sigma + it$ *describes the line* $\sigma = \sigma_0$, *and* Θ *the set of points corresponding to the values taken by the power-series when the variables* $x_1, x_2, \ldots$ *describe independently the circles*

$$|x_1| = p_1^{-\sigma_0}, \quad |x_2| = p_2^{-\sigma_0}, \quad \ldots .$$

Then the set θ, *which is obviously part of* Θ, *is everywhere dense in* Θ.

In particular the solution of what is called the "absolute convergence problem" for Dirichlet's series of the type $\Sigma a_n n^{-s}$ must be based upon a study of the relations between the absolute value of a power-series in an infinite number of variables on the one hand, and the sum of the absolute values of the individual terms of this series on the other. It was in the course of this investigation that I was led to consider a problem concerning power-series in one variable only, which we discussed last year, and which seems to be of some interest in itself.

2. The question which we discussed was as follows. *Let* x_1 *be a positive number between* 0 *and* 1. *Is it always possible to find a power-series* $\Sigma a_n x^n$ *such that*

(1) $f(x) = \Sigma a_n x^n$ *is regular for* $|x| < 1$ *and continuous for* $|x| \leqslant 1$,

(2) $|f(x)| < 1$ *for* $|x| \leqslant 1$,

(3) $\Sigma |a_n| x_1^n > 1$?

It is obvious that the question is to be answered in the affirmative when x_1 is sufficiently near to 1: in fact, the hypotheses (1) and (2) are perfectly consistent with

$$\Sigma |a_n| x_1^n \to \infty,$$

as $x_1 \to 1$. I can, however, now prove that the *general* question must be answered *negatively*. This is shown by the following theorem.

THEOREM.—*If the conditions* (1) *and* (2) *are satisfied, then*

$$\Sigma |a_n x^n| < 1,$$

for $|x| \leqslant \frac{1}{6}$; *i.e.*, $\Sigma |a_n| 6^{-n} < 1$.

It is plain that we may, without loss of generality, suppose a_0 real and positive. Then

$$a_0 = f(0) < 1.$$

Let
$$g(x) = f(x) - a_0 = \sum_1^\infty a_n x^n,$$
$$R = \max_{|x|=1} \mathfrak{R}\{g(x)\},$$
$$\mathbf{m} = \max_{|x|=\frac{1}{2}} |g(x)|.$$

We use the inequality of Carathéodory*

$$\max_{|x|=\rho} |F(x)| \leqslant |\gamma| + |\beta| \frac{r-\rho}{r+\rho} + \frac{2\rho}{r-\rho} \max_{|x|=r} \mathfrak{R}\{F(x)\},$$

in which $F(x)$ is supposed regular for $|x| \leqslant r$,† and $0 < \rho < r$, and
$$F(0) = \beta + i\gamma.$$

Taking $F(x) = g(x)$, so that $\beta = \gamma = 0$, and $r = 1$, $\rho = \frac{1}{2}$, we obtain
$$\mathbf{m} \leqslant 2R.$$

Now $R \geqslant 0$, and so

$$a_0 + R = a_0 + \max_{|x|=1} \mathfrak{R}\{g(x)\} \leqslant \max_{|x|=1} |a_0 + g(x)| = \max_{|x|=1} |f(x)| < 1;$$

i.e.,
$$R < 1 - a_0,$$
and
$$\mathbf{m} = \max_{|x|=\frac{1}{2}} |g(x)| < 2(1-a_0).$$

But
$$a_n = \frac{1}{2\pi i} \int_{|x|=\frac{1}{2}} \frac{g(x)}{x^{n+1}}\, dx \quad (n \geqslant 1),$$

and so
$$|a_n| \leqslant 2^n \mathbf{m} < 2^{n+1}(1-a_0).$$

Accordingly, for $|x| \leqslant \frac{1}{6}$, we have

$$\sum_0^\infty |a_n x^n| = a_0 + \sum_1^\infty |a_n x^n| < a_0 + 2(1-a_0) \sum_1^\infty 2^n |x|^n$$
$$= a_0 + \frac{4(1-a_0)|x|}{1-2|x|} \leqslant a_0 + (1-a_0) = 1.$$

Thus the theorem is proved.

3.‡ If k is any positive number less than unity, it either is or is not

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* See, for example, Landau, *Handbuch*, p. 299.

† The inequality obviously still holds if $F(x)$ is regular for $|x| < r$, and continuous for $|x| \leqslant r$.

‡ This section is extracted from a later letter dated June 19th, 1913.

true that the hypotheses (1) and (2) involve

$$\Sigma \mid a_n x^n \mid < 1$$

for $|x| \leqslant k$. It follows from my theorem that the numbers k for which the implication holds have a positive upper limit K, and that $K \geqslant \frac{1}{6}$.

The problem remains of the exact determination of the value of K. I have learnt recently that Messrs. M. Riesz, Schur, and Wiener, whose attention had been drawn to the subject by my theorem, have succeeded independently in solving this problem completely. Their solutions show that $K = \frac{1}{3}$. Mr. Wiener has very kindly given me permission to reproduce here his very simple and elegant proof of this result.

Wiener proves first that if the hypotheses (1) and (2) are satisfied, and if, as above, we assume that $0 \leqslant a_0 < 1$, then the inequality

$$|a_n| < 2^{n+1}(1-a_0)$$

established above may be replaced by

(1) $$|a_n| < 1-a_0^2.$$

For $n = 1$ this is a known result, which follows at once from the fact that the function

$$\frac{f(x)-a_0}{x\{1-a_0 f(x)\}}$$

is regular for $|x|<1$, continuous for $|x|\leqslant 1$, numerically less than 1 for $|x| = 1$, and takes for $x = 0$ the value $a_1/(1-a_0^2)$. Now let n be any integer greater than 1, and let ρ be a primitive n-th root of 1. Then

$$F(x) = f(x)+f(\rho x)+\ldots+f(\rho^{n-1} x) = \sum_{m=0}^{\infty} n a_{mn} x^{mn}$$

is regular for $|x|<1$ and continuous and numerically less than n for $|x| \leqslant 1$. Hence it follows that the function

$$\phi(x) = a_0 + a_n x + a_{2n} x^2 + \ldots$$

satisfies the conditions originally imposed upon $f(x)$; and thus (1) is proved.

A fortiori, for $n \geqslant 1$,

(2) $$|a_n| < (1+a_0)(1-a_0) < 2(1-a_0),$$

and so $$\sum_1^{\infty} 3^{-n}|a_n| < a_0 + 2(1-a_0)\sum_1^{\infty} 3^{-n} = a_0+1-a_0 = 1.$$

Thus $$K \geqslant \tfrac{1}{3}.$$

That, on the other hand, $K \leqslant \frac{1}{3}$, follows immediately from the con-

sideration of the special function

$$\psi(x) = \frac{1-x}{1-ax} = 1+x(a-1)+x^2(a^2-a)+\ldots = \Sigma a_n x^n,$$

where $0 < a < 1$. Here

$$\max_{|x|=1} |\psi(x)| = \frac{2}{1+a},$$

and $$\Sigma |a_n x^n| = 1+\frac{(1-a)|x|}{1-a|x|}.$$

Hence $$\Sigma |a_n x^n| > \max_{|x|=1} \left|\frac{1-x}{1-ax}\right|,$$

for $$1+\frac{(1-a)|x|}{1-a|x|} > \frac{2}{1+a},$$

i.e., for $$|x| > \frac{1}{1+2a}.$$

As $a \to 1$, the last expression tends to the limit $\frac{1}{3}$. Thus $K \leqslant \frac{1}{3}$, and so $K = \frac{1}{3}$.

ON SOME PROPERTIES OF GROUPS WHOSE ORDERS ARE POWERS OF PRIMES

(SECOND PAPER.)

By W. BURNSIDE.

[Received April 29th, 1913.—Read May 8th, 1913.]

I. *The Independent Generating Operations of a Group of Prime Power Order.*

1. If G is a group of prime power order and G_1 its derived group, G/G_1 is an Abelian group of type $(m_1, m_2 \ldots, m_n)$, say. It is immediately obvious that G cannot be generated by fewer than n operations. In fact, G/G_1 arises from G by supposing that the generating operations of G are permutable with each other, and this supposition cannot increase their number.

Let $S_1', S_2', \ldots, S_n'$ of orders $p^{m_1}, p^{m_2}, \ldots, p^{m_n}$ be a set of independent generating operations of G/G_1, and, in the multiple isomorphism between G and G/G_1, let

$$S_1 G_1, \quad S_2 G_1, \quad \ldots, \quad S_n G_1$$

be the sets of operations of G which correspond to the operations

$$S_1', \quad S_2', \quad \ldots, \quad S_n'$$

of G/G_1.

Finally, let

$$S_1, \quad S_2, \quad \ldots, \quad S_n$$

be n operations of G chosen one from each of the above n sets of operations in any way. Then it will be shewn that these n operations form a set of independent generating operations of G.

Assuming the contrary, let T be one of the further operations which it is necessary to take in addition to $S_1, S_2, \ldots, S_n$, in order to generate G. It may be assumed that T belongs to G_1; for, if it does not, it must belong to one of the sets

$$S_1^{a_1} S_2^{a_2} \ldots S_n^{a_n} G_1,$$

and then
$$S_1^{-a_1} S_2^{-a_2} \dots S_n^{-a_n} T$$
belongs to G_1.

Denote $\{S_1, S_2, \dots, S_n, T\}$ by H. Let K be the self-conjugate sub-group of H generated by T and its conjugates, so that K belongs to the derived group of H. If K and $\{S_1, S_2, \dots, S_n\}$ have a common sub-group it is self-conjugate in H, and it does not contain T. Denote this self-conjugate sub-group by h, and let
$$S_1', \quad S_2', \quad \dots, \quad S_n', \quad T'$$
be the operations of H/h which correspond to the operations
$$S_1, \quad S_2, \quad \dots, \quad S_n, \quad T$$
of H. Then H/h is not generated by $S_1', S_2', \dots, S_n'$; and the self-conjugate sub-group L of H/h which is generated by T' and its conjugates has no operation in common with $\{S_1', S_2', \dots, S_n'\}$. The sub-group L must contain a sub-group L' of index p, self-conjugate in H/h, in which T' does not enter; and $\{S_1', S_2', \dots, S_n', L'\}$ is a sub-group of index p in H/h in which T' does not enter. Such a sub-group, however, must contain the derived group of H/h to which T' belongs.

Hence the supposition that such an operation as T enters among the independent generating operations of G leads to a contradiction.

THEOREM.—If G is a group of prime power order, G_1 the derived group of G, $S_1', S_2', \dots, S_n'$ any set of independent generating operations of G/G_1, and $S_1G_1, S_2G_1, \dots, S_nG_1$ the corresponding sets of operations of G, then any n operations chosen one from each of these sets is a set of independent generating operations of G, and in this way all possible sets of independent generating operations of G arise.

2. If H is any self-conjugate sub-group of G, and if G/H be called G' and its derived group G_1', the number of independent generators of G/H is equal to the number of independent generators of G'/G_1'. If H is contained in G_1, then G/G_1 and G'/G_1' are simply isomorphic; and if H is not contained in G_1 the number of independent generators of G'/G_1' is equal to or less than the number for G/G_1. Hence, in any case, the number of independent generators in G/H is equal to or less than the number of independent generators of G.

II. *The Series of Derived Groups of a Group of Prime Power Order.*

3. Denote by G a group of prime power order, and by $G_1, G_2, \dots$, its

series of derived groups. Let P be an operation of G_1, which in G_1 is one of $p^m (m > 0)$ conjugate operations. Denote by G_P the greatest sub-group of G within which P is self-conjugate and by G' any sub-group of G of index p which contains G_P. If the index of G_P in G is p^μ, then in G' the operation P is one of a conjugate set of $p^{\mu-1}$ operations. Now G' necessarily contains G_1, and in G_1 the operation P is one of a conjugate set of p^m operations. Hence

$$p^{\mu-1} \geqslant p^m,$$

i.e.,

$$\mu \geqslant m+1.$$

From this it follows that, if P is one of $p^m (m > 0)$ conjugate operations in G_i, it is one of at least p^{m+1} conjugate operations in G.

4. If C_i is the central of a non-Abelian group G_i, the i-th derived group of G, then G_i must contain a sub-group $\{P, C_i\}$, which is self-conjugate in G, while P^p belongs to C_i. Now P is one of at least p conjugate operations in G_i, and therefore, by the preceding paragraph, is one of at least p^{i+1} conjugate operations in G. But, since both $\{P, C_i\}$ and C_i are self-conjugate in G, all the operations of G which are conjugate to P belong to the set PC_i. It follows that the order of C_i must be p^{i+1} at least. Since G_i is not Abelian G_{i+1} is not E. Let H be a sub-group of index p in G_{i+1}, which is self-conjugate in G, and denote G/H by G'. If $G'_1, G'_2, \ldots,$ are the sub-groups of G' which correspond to the sub-groups $G_1, G_2, \ldots,$ of G, then

$$G'_1, \quad G'_2, \quad \ldots, \quad G'_i, \quad G'_{i+1}, \quad E$$

is the derived series of G', and the order of G'_{i+1} is p.

The order of the central of G'_i is p^{i+1} at least, and therefore, since G'_i is not Abelian, the order of G'_i is p^{i+3} at least. It follows that the order of G'_i/G'_{i+1}, and therefore the order of G_i/G_{i+1}, is p^{i+2} at least. Hence

THEOREM.—If the non-Abelian group G_i is the i-th derived group of a prime power group G, then the order of the central of G_i is at least p^{i+1}, and the order of G_i/G_{i+1} is at least p^{i+2}.

It is an immediate consequence of this theorem that $p^{\frac{1}{2}(n+1)(n+2)}$ is a lower limit for the order of a prime power group whose n-th derived group is not identity. It will, however, be shewn in the next section that this lower limit is not attained except when n is 1 or 2.

5. Let C_1 be the central of G, C_2 the sub-group of G which corresponds to the central of G/C_1, C_3 the sub-group of G that corresponds to

the central of G/C_2, and so on, while G/C_m is Abelian. Using the previous notation for the series of derived groups of G, we consider the relations between the two sets of sub-groups

$$C_m, \quad C_{m-1}, \quad \ldots, \quad C_2, \quad C_1,$$

and

$$G_1, \quad G_2, \quad \ldots, \quad G_n, \quad E.$$

Since G/C_m is Abelian, C_m must contain G_1. Similarly, since C_{m-1} contains the derived group of C_m, while G_2 is the derived group of G_1, C_{m-1} contains G_2. In this way it is shown * that C_{m-i+1} contains G_i.

On the other hand, if P_2 is any operation of C_2 and S, T are any operations of G, then

$$S^{-1}P_2S = P_2P_1, \quad T^{-1}P_2T = P_2P_1',$$

where P_1, P_1' belong to C_1. It follows that $S^{-1}T^{-1}ST$ is permutable with P_2, *i.e.*, every operation of G_1 is permutable with every operation of C_2.

Similarly, if P_3 is any operation of C_3 and S', T' any operations of G_1,

$$S'^{-1}P_3S' = P_3P_2, \quad T'^{-1}P_3T' = P_3P_2',$$

where P_2, P_2' are operations of C_2 and are, therefore, permutable with S' and T'. Hence $S'^{-1}T'^{-1}S'T'$ is permutable with P_3, *i.e.*, every operation of G_2 is permutable with every operation of C_3. In the same way it is shewn that every operation of G_i is permutable with every operation of C_{i+1}.

Combining these results, it follows that if

$$i+1 \geqslant m-i+1$$

every operation of G_i is contained in and is self-conjugate in C_{i+1}. Now G_{n-1} is not Abelian, and, therefore, the above inequality is not true when $n-1$ is written for i. It follows that

$$m > 2n-2.$$

III. *On the Maximum Abelian Self-conjugate Sub-groups.*†

6. Let A_ν be a "maximum" sub-group of G, $A'_{\nu-1}$ a "maximum" sub-group of G/A_ν, and $A_{\nu-1}$ the corresponding sub-group of G; $A'_{\nu-2}$ a "maximum" sub-group of $G/A_{\nu-1}$, and $A_{\nu-2}$ the corresponding sub-group

* *Theory of Groups*, 2nd edition, p. 121.

† This phrase is too long for constant repetition. In the present section, since there is no risk of confusion, such a group will be called a "maximum" sub-group.

of G, and so on, G/A_1 being Abelian. Such a series as

$$A_1, \quad A_2, \quad \ldots, \quad A_\nu$$

can, in general, be formed in more than one way. If

$$G_1, \quad G_2, \quad \ldots, \quad G_n, \quad E$$

is the series of derived groups of G (it will be said to consist of n terms), G_n is a self-conjugate Abelian sub-group of G, and G will have at least one "maximum" sub-group which contains G_n. Suppose A_ν chosen so as to contain G_n. It does not contain G_{n-1} because the latter is not Abelian. Hence the series of derived groups of G/A_ν consists of $n-1$ terms.

Suppose $A'_{\nu-1}$ chosen so as to contain the last derived group of G/A_ν. It cannot contain the last but one, and $A_{\nu-1}$ therefore contains G_{n-1} and does not contain G_{n-2}. This reasoning can obviously be repeated, so that A_1 contains $G_{n-\nu+1}$ and does not contain $G_{n-\nu}$. Now, G/A_1 being Abelian, A_1 must contain G_1, and therefore, when the series of groups

$$A_1, \quad A_2, \quad \ldots, \quad A_\nu$$

is chosen in the way indicated, ν must be n. If A_ν does not contain G_n the series of derived groups of G/A_ν consists of n terms, and ν must be equal to or greater than $n+1$. Hence, if the n-th derived group of G is Abelian, and not identity, the series of groups

$$A_1, \quad A_2, \quad \ldots, \quad A_\nu$$

consists of n terms at least, and one such series can certainly be chosen to consist of just n terms.

7. In my preceding paper with the same title* I have shewn that groups exist for which G_1 is not Abelian, while the orders of G/G_1 and G_1/G_2 are p^2 and p^3 respectively. It is a result of § 4 that if G_2 is not Abelian the order of G_2/G_3 cannot be less than p^4. It will now be shewn that a prime power group in which G_2 is not Abelian, while the orders of G/G_1, G_1/G_2, and G_2/G_3 are p^2, p^3 and p^4 respectively is non-existent.

Assuming the contrary, let H be a sub-group of index p in G_3, which is self-conjugate in G, and consider G/H. Its order is p^{10}, the orders of its first, second, and third derived groups are p^8, p^5 and p; and it has two independent generators.

The order of a "maximum" sub-group must be equal to or greater than p^4. If it were greater than p^4, the series of groups

$$A_1, \quad A_2, \quad \ldots$$

would consist of two terms only, whereas by § 6 it must consist of three at least. The order of the "maximum" sub-group is therefore p^4. Denoting this by A and the group itself by B, B/A of order p^6 is simply isomorphic with a group of isomorphisms of an Abelian group of

* *Proc. London Math. Soc.*, Ser. 2, Vol. 11, p. 242.

order p^4. This group must therefore* be of type (1, 1, 1, 1) or (2, 1, 1). I have shewn in my previous paper that the group of p-isomorphisms of a group of order p^4 and type (1, 1, 1, 1) has three independent generators. The same is true† of the group of p-isomorphisms of a group of order p^4 and type (2, 1, 1).

Now it has been shewn, § 2, that the number of independent generators of B/A cannot exceed the number for B.

The assumption that the group G/H exists thus leads to a contradiction; and there is therefore no group of prime power order for which the orders of G/G_1, G_1/G_2, G_2/G_3 and G_3 are p^2, p^3, p^4, p^i ($i \geqslant 1$). This justifies the statement at the end of § 3 that, except when n is 1 or 2, the order of a group of prime power order, whose n-th derived group is not identity, is greater than $p^{\frac{1}{2}(n+1)(n+2)}$.

8. If A is a "maximum" sub-group of G, order p^a, it is not necessarily a self-conjugate sub-group of any group H, order $p^\beta (\beta > a)$, which contains G self-conjugately. Assume that for groups G', order $p^{a'}$ ($a' < a$), it is always possible to choose a "maximum" sub-group, which is self-conjugate in any assigned group of prime power order within which G' is itself self-conjugate. Then if G' is a sub-group, of order p^{a-1}, of G, which is self-conjugate in H (there is always such a sub-group), we may assume that G' has a "maximum" sub-group A' which is self-conjugate in H. No operation of G' not contained in A' is permutable with every operation of A'. If there is no operation of G which is permutable with every operation of A', then A' is a "maximum" sub-group of G.

If P, belonging to G and not to G', is permutable with every operation of A', then $\{P, A'\}$ is Abelian, and so also is $\{S^{-1}PS, A'\}$, where S is any operation of H. Now

$$S^{-1}PS = Pg',$$

where g' belongs to G'; and if both P and Pg' are permutable with every operation of A', so also is g'. It follows that g', belonging to G' and being permutable with every operation of A', must belong to A'. Hence $\{P, A'\}$, or A, is self-conjugate in H. Moreover, A must be a "maximum" sub-group of G; for, if it were not, A' would not be such a sub-group of G'.

Hence, if the assumption made is true for groups of order less than p^a, it is true for groups of order p^a.

THEOREM.—If G is a group of prime power order and H a greater group of prime power order containing G self-conjugately, then G has

* Miller, "Isomorphisms of a Group whose Order is a Power of a Prime," *Transactions of the American Mathematical Society*, Vol. 12, p. 396.

† See note at end of paper.

at least one maximum Abelian self-conjugate sub-group which is self-conjugate in H.

It follows directly that if H is a group of prime power order and G any self-conjugate sub-group of H, then H must contain at least one "maximum" sub-group B, such that the common sub-group of B and G is a "maximum" sub-group of G.

Note to p. **11.**

Let P_1 of order p^2 and $P_2, P_3, \ldots, P_n$ of orders p be independent generators of an Abelian group of order p^{n+1} and type (2, 1, 1, ..., 1).

Denote the isomorphisms (in which the unwritten operations are not altered).

$$\begin{pmatrix} P_2 \\ P_2P_3 \end{pmatrix}, \quad \begin{pmatrix} P_3 \\ P_3P_4 \end{pmatrix}, \quad \ldots, \quad \begin{pmatrix} P_{n-1} \\ P_{n-1}P_n \end{pmatrix}, \quad \begin{pmatrix} P_n \\ P_nP_1^p \end{pmatrix},$$

by $\qquad I_{2,3}, \quad I_{3,4}, \quad \ldots, \quad I_{n-1,n}, \quad I_{n,n+1};$

and the isomorphisms

$$\begin{pmatrix} P_1 \\ P_1^{1+p} \end{pmatrix}, \quad \begin{pmatrix} P_1 \\ P_1P_2 \end{pmatrix}, \quad \ldots, \quad \begin{pmatrix} P_1 \\ P_1P_n \end{pmatrix},$$

by $\qquad I_1, \quad I_2, \quad \ldots, \quad I_n.$

The group of p-isomorphisms of the Abelian group is generated by these, and it is easily verified that no one of the isomorphisms

$$I_n, \quad I_{2,3}, \quad I_{3,4}, \quad \ldots, \quad I_{n,n+1}$$

belongs to the derived group. The group, therefore, has n independent generators. For the case considered on the preceding page n is 3, and this justifies the statement there made.

ON THE USUAL CONVERGENCE OF A CLASS OF TRIGONOMETRICAL SERIES

By W. H. YOUNG.

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1. In a recent communication to the Royal Society I have briefly indicated how, by the use of the generalised concept of integration with respect to a function of bounded variation, it is possible to show that many of the properties of the Fourier series of summable functions are possessed by the derived series of Fourier series of functions of bounded variation. In the present paper I propose to occupy myself with two of the more striking of these properties, which have only quite recently been stated in their most general form. That they should be true not only of Fourier series, but also of the derived series of the Fourier series of functions of bounded variation, seems to me of considerable interest in view of the fact that the class of trigonometrical series constituted by such derived series is a much larger one than that of the Fourier series themselves.

Lebesgue was the first to show that, when summed $(C1)$, *i.e.*, in the Cesàro manner, index unity, a Fourier series converges almost everywhere to the function with which it is associated. Since the publication of his papers more than one equivalent method of defining what may be called fractional Cesàro summability $(C\delta)$ has been given, and it has been shown that in various theorems the more precise statement obtained by substituting for integral summability a fractional summability greater than the next lower integer is true. The latest of these extensions is due to G. H. Hardy, who has shown that this is the case with the theorem of Lebesgue above referred to.*

I propose here to show that we may in this extension substitute the

* For a similar instance reference may be made to Lebesgue's theorem that a Fourier series converges $(C2)$ to its associated function at every point where $f(x)$ is the differential coefficient of its integral. I have shown that the integer 2 may be replaced by any quantity greater than unity. See "On the Convergence of a Fourier Series and its Allied Series," 1911, *Proc. London Math. Soc.*, Ser. 2, Vol. 1, Op. 268. A still more general form of the statement is given below, *infra*, § 9.

trigonometrical series I am here considering for Fourier series. We thus obtain the striking result that the derived series of the Fourier series of a function of bounded variation converges $(C\delta)$ almost everywhere to a function which is equal to the differential coefficient of the function of bounded variation, wherever this exists.

The convergence almost everywhere of a Fourier series when summed in the ordinary manner has been considered by various writers. Recently I showed in particular that the use of the convergence factor

$$1/(\log n)^{1+k} \quad (0 < k)$$

transformed every Fourier series and its allied series into Fourier series which converge almost everywhere. This result, in so far as it relates to Fourier series themselves, has been completed by G. H. Hardy, who has pointed out that we may in the above statement then make k zero. I propose here to show that these results are equally true when for the Fourier series and its allied series we substitute the derived series of the Fourier series of a function of bounded variation and its allied series.

As regards the method employed, little more has been needed than to make the necessary modifications in certain results due to Lebesgue and to myself so as to render them applicable to integration with respect to a function of bounded variation. When this has been done nothing remains but to follow somewhat closely G. H. Hardy in his further arrangement of the reasoning. It seems scarcely possible to improve on his treatment. We can, however, deduce from the known results certain others almost as general as those stated, by remarking, what I have already pointed out, that the derived series of the Fourier series of a function of bounded variation, and its allied series, are both changed into Fourier series when in the one case the typical coefficients of the Fourier series of an even function and in the other case those of an odd function are employed as convergence factors. The carrying out of this process may be left as an exercise for the reader.

One remark may be made in conclusion. The convergence everywhere of a Fourier series, as distinguished from its usual convergence, equally whether it be considered *per* se or as regards the effect produced on it by convergence factors, depends on the class of summable functions to which its associated function belongs, while that of the usual convergence does not, so far as our knowledge goes, depend on this at all. This difference seems to be further illustrated by the results here presented.

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* For Hardy's result see a forthcoming paper by him in the *Proc. London Math. Soc.*

2. We begin by proving a theorem in the integration of bounded sequences. As we prove it *ad hoc* I shall not attempt to give to it all the generality which is possible. Even in the simple form here given, it constitutes, it will be noticed, a considerable advance on the corresponding theorem in ordinary integration, of which I have given the statement and proof in a paper on " The Application of Expansions to Definite Integrals." It should also be remarked that both that proof and that of Lebesgue's in a kindred matter, on which it had been modelled, are abridged to a proof of a few lines only when the new concept (§ 1) is employed. Lebesgue has had occasion to introduce the new concept. As, however, his method involves precisely the same considerations as those he employs in the proof here referred to, there is no gain in simplicity, or at most only a gain in simplicity of arrangement. When, however, the method of monotone sequences, to which I have repeatedly called attention, is employed, we virtually prove the theorems of the more general theory simultaneously with those of the ordinary theory. A word or two of explanation alone is necessary to render the reasoning in the simpler case applicable generally.

THEOREM.—*If $S_n(x)$ converges boundedly to $F(x)$, and $g(x)$ is a function of bounded variation in the finite or infinite interval of integration,*

$$\operatorname*{Lt}_{n\to\infty}\int_a^b S_n(x)\,dg(x) = \int_a^b F(x)\,dg(x).$$

It is evidently only necessary to prove the theorem when $g(x)$ is monotone increasing.

Since we may always integrate bounded sequences term-by-term with respect to a monotone increasing function, the result is at once seen to be true, provided the interval of integration is finite. Assuming this, let $b\to\infty$. Then, by the definition of the integral from a to ∞, we get

$$\int_a^\infty F(x)\,dg(x) = \operatorname*{Lt}_{b\to\infty}\operatorname*{Lt}_{n\to\infty}\int_a^b S_n(x)\,dg(x)$$

$$= \operatorname*{Lt}_{b\to\infty}\operatorname*{Lt}_{n\to\infty}\left[\int_a^\infty S_n(x)\,dg(x) - \int_b^\infty S_n(x)\,dg(x)\right],$$

since the integrals which here appear certainly exist, being numerically less than FG, where F is any number greater than $S_n(x)$ for all values of n and x, and G is the increment of $g(x)$ over the infinite interval.

Now, as the lower limit b in the last integral moves off to infinity, the corresponding increment of $g(x)$ approaches zero, since the total increment of $g(x)$ in the infinite interval (a, ∞) is finite. Hence, using the First

Theorem of the Mean, the repeated limit of the second of the two terms in the bracket on the right in the above equality is zero. That equality therefore becomes the equality to be proved, since the first term in the bracket is independent of b. Thus the theorem is true.

3. The function $t^{-1-k}C_{1+k}(t)$, where $C_{1+k}(t)$ is the generalisation of the sine and cosine functions which I have elsewhere defined* is a function of bounded variation in the whole infinite interval, k being any positive quantity. Hence, if $f(t)$ is a function of bounded variation, or any other summable function whose Fourier series converges boundedly, say

$$f(t) \sim \Sigma(-b_m \cos mt + a_m \sin mt)/m,$$

so that $\tfrac{1}{2}[f(x+t)-f(x-t)] \sim \Sigma \sin mt(a_m \cos mx + b_m \sin mx)/m$

$$\sim \Sigma A_m \sin mt/m,$$

we may integrate term-by-term with respect to

$$nt^{-1-k}C_{1+k}(nt) = g(nt), \text{ say}.$$

Now, $f(x)$ being a function of bounded variation, we may integrate the left-hand side by parts, and get

$$\int_0^\infty \tfrac{1}{2}[f(x+t)-f(x-t)]\,dg(nt)$$
$$= \tfrac{1}{2}\Big[g(nt)[f(x+t)-f(x-t)]\Big]_0^\infty - \int_0^\infty g(nt)\,d\,[f(x+t)-f(x-t)],$$

where the quantity in square brackets is zero, since $g(t)$ vanishes at infinity, while $[f(x+t)-f(x-t)]$ is bounded at infinity, and zero at the origin. Similarly,

$$\int_0^\infty \sin mt\,dg(nt) = -\int_0^\infty g(nt)\,m\cos mt\,dt.$$

Hence, using a known property of the function $g(t)$, we get, changing the variable t to t/n,

$$\frac{2\Gamma(1+k)}{\pi}\,n\int_0^\infty g(t)\,d\,\{\tfrac{1}{2}[f(x+t/n)-f(x-t/n)]\} = \sum_{m=1}^{m=n} A_m(1-m/n)^k.$$

* "On Infinite Integrals Involving a Generalisation of the Sine and Cosine Functions," 1912, *Quarterly Journal of Pure and Applied Mathematics*, pp. 161–177.

$$C_p(x) = \frac{x^p}{\Gamma(p+1)}\left(1 - \frac{x^2}{(p+1)(p+2)} + \frac{x^4}{(p+1)(p+2)(p+3)(p+4)} - \dots\right).$$

Hence

$$\frac{2\Gamma(1+k)}{\pi} n \int_0^\infty g(t)\, d\left\{\tfrac{1}{2}\left[f(x+t/n)-f(x-t/n)\right]-tf'(x)/n\right\}$$
$$= \sum_{m=1}^{m=n} A_m(1-m/n)^k - f'(x).$$

4. We now pass to the generalisation of the important theorem due to Lebesgue, on which he bases the proof of his original theorem and which has been employed by G. H. Hardy in his interesting extension.

As I believe no account exists in English of this theorem, I propose to prove the generalised form of it necessary for my purpose directly, instead of attempting to deduce it from Lebesgue's result. It will be seen that I follow Lebesgue's own reasoning very closely, merely making the changes which correspond to the substitution of the more general for the more special kind of integration. We have first, however, to prove a property of the differential coefficient of the total variation of a function of bounded variation.

5. THEOREM.—*If $f(x)$ is a function of bounded variation, and $V(x)$ is the function representing its total variation, then, except at a set of content zero,*

$$\frac{dV}{dx} = \left|\frac{df}{dx}\right|.$$

Also, if $P(x)$ and $-N(x)$ denote the positive and negative variations of $f(x)$, $dP/dx = df/dx$, $dN/dx = 0$ at a point where $df/dx \geqslant 0$, and $dP/dx = 0$, $dN/dx = -df/dx$, at a point where $df/dx \leqslant 0$, excepting in both cases a set of content zero.

Since a monotone increasing function is the sum of the integral of its derivates plus a monotone increasing function, having zero for differential coefficient except at a set of content zero,* we may write

$$f(x) = P(x) - N(x) = P_0(x) - N_0(x) + I_0(x) - J_0(x),$$

where P_0 and N_0 are the integrals of the positive functions $P'(x)$ and $N'(x)$, and I_0 and J_0 have zero for differential coefficient, except at a set of content zero.

Let
$$g(x) = P_0(x) - N_0(x);$$

then I assert that $P_0(x)$ is the positive variation of $g(x)$, and $-N_0(x)$ is the

* See my paper on "Functions of Bounded Variation," 1910, *Quarterly Journal of Pure and Applied Mathematics*, Vol. XLII, pp. 54–85.

corresponding negative variation. For $N_0(x)$ is a positive monotone increasing function, and, when added to $g(x)$ it changes $g(x)$ into a positive monotone increasing function, while the absolute value of the negative variation is the least positive monotone increasing function which has this property. Thus we may write for the negative variation of $g(x)$,

$$-N_0(x)+N_1(x),$$

where $N_1(x)$ is a nowhere negative function. Now, as we saw,

$$f(x)=g(x)+I_0(x)-J_0(x),$$

where I_0 and J_0 are monotone increasing functions. Thus the addition of the positive monotone increasing function

$$N_0(x)-N_1(x)+J_0(x)$$

to $f(x)$, changes $f(x)$ into a monotone increasing function. Therefore

$$N(x) \leqslant N_0(x)-N_1(x)+J_0(x) \leqslant N(x)-N_1(x).$$

This shows that $N_1(x)$ is identically zero, so that $-N_0(x)$ is, as was asserted, the negative variation of $g(x)$.

Hence, denoting by S_+ the set of points in (a, x) at which $g'(x)$ is positive,

$$\int_{S_+} g'(x)\,dx = P_0(x).$$

In other words, $P_0(x)$ is the integral of the function which is equal to $g'(x)$ wherever it is positive, and is zero elsewhere. Hence, by Lebesgue's theorem, except at a set of content zero G_+,

$$\frac{dP_0}{dx} = g'(x),$$

wherever it is positive, and is elsewhere zero.

Differentiating the identity

$$P(x) = P_0(x)+I_0(x),$$

at a point not belonging to the exceptional set of content zero, consisting of G_+ and the set of content zero at which $I_0(x)$ has not zero for differential coefficient, we see that except at a set of content zero,

$$P'(x) = g'(x),$$

wherever $g'(x)$ is positive, and is elsewhere zero.

But, differentiating the identity

$$f(x)=g(x)+I_0(x)-J_0(x),$$

we have, except at the set of content zero where $I_0(x)$ and $J_0(x)$ have not zero differential coefficients,

$$f'(x) = g'(x).$$

Thus, except at a set of content zero,

$$P'(x) = f'(x), \tag{1}$$

wherever $f(x)$ is positive, and is elsewhere zero.

Similarly, except at a set of content zero,

$$-N'(x) = f'(x), \tag{2}$$

wherever negative, and is elsewhere zero.

From (1) and (2) the whole theorem follows, since

$$V(x) = P(x) + N(x).$$

6. We have still to show that, if $f(x)$ is a function of bounded variation,

$$\left| \int_{x_0}^{x} |d[f(x)-qx]| - \int_{x_0}^{x} |d[f(x)-px]| \right| \leqslant |p-q|\,|x-x_0|. \tag{1}$$

In fact, since the modulus of a sum is less than the sum of the moduli,

$$|\Delta[f(x)-qx]| \leqslant |\Delta[f(x)-px]| + |\Delta(qx-px)|;$$

therefore $\quad |\Delta[f(x)-qx]| - |\Delta[f(x)-px]| \leqslant |p-q|\,|\Delta x|.$

Since this inequality holds for each of a finite number of intervals into which we divide the interval (x, x_0), we have, summing over the intervals,

$$\Sigma|\Delta[f(x)-qx]| - \Sigma|\Delta[f(x)-px]| \leqslant \Sigma|p-q|\,|\Delta x|.$$

But if the intervals are sufficiently small the two summations on the left differ by as little as we please from the corresponding integrals. Therefore

$$\int_{x_0}^{x} |d[f(x)-qx]| - \int_{x_0}^{x} |d[f(x)-px]| \leqslant |p-q|\,|x-x_0|.$$

Interchanging p and q we get a second inequality, and combining the two inequalities (1) follows.

7. We now come to the generalised form of Lebesgue's theorem.*

THEOREM.—*If $f(x)$ is a function of bounded variation, and x_0 does not*

* H. Lebesgue, *Leçons sur les séries trigonométriques*, 1906, p. 13.

belong to a certain set of content zero, $|f'(x_0)-q|$ *is the differential coefficient of* $\int_{x_0}^{x} |d[f(x)-qx]|$ *at the point* $x=x_0$, *whatever value be imputed to* q. *In particular,* $\int_{x_0}^{x} |d[f(x)-xf'(x_0)]|$ *has zero for differential coefficient at* $x=x_0$, *provided* x_0 *does not belong to the set of content zero, above referred to.*

By § 5, since $f(x)$ and therefore $f(x)-qx$ is a function of bounded variation

$$\int_a^x |d[f(x)-qx]|,$$

which is the total variation of $f(x)-qx$, has, except at a set of content zero, say S_q, depending on $f(x)$ and on q,

$$|f'(x)-q|$$

for differential coefficient. The present theorem goes farther; it asserts that this will be the case at $x=x_0$, provided x_0 does not belong to a certain set of content zero, depending on $f(x)$, but not on q.

To prove this, let $r_1, r_2, \ldots$ be the rational points in order. We shall then for each integer i have an exceptional set of content zero S_{r_i}.

We have proved in the preceding article the auxiliary inequality

$$\left| \int_{x_0}^{x} |d[f(x)-qx]| - \int_{x_0}^{x} |d[f(x)-r_i x]| \right| \leqslant |q-r_i|\,|x-x_0|. \tag{1}$$

Now, provided x_0 does not belong to the set of content zero consisting of all the sets S_{r_i}, and provided $|x-x_0| < h_e$, where h_e is a quantity depending on e, we have

$$\frac{1}{x-x_0}\int_{x_0}^{x} |d[f(x)-r_i(x)| = \left[\frac{d}{dx}\int_a^x |d[f(x)-r_i x]|\right]_{x=x_0} + \theta e$$

$$= |f'(x_0)-r_i| + \theta e \quad (-1 \leqslant \theta \leqslant 1).$$

Hence, by (1),

$$\left| \frac{1}{x-x_0}\int_{x_0}^{x} |d[f(x)-qx]| - |f'(x_0)-r_i| + \theta e \right| \leqslant |q-r_i|$$

provided $$x_0-h_e \leqslant x \leqslant x_0+h_e.$$

Let i describe such a sequence of integers that r_i has q as unique limit. The corresponding values of θ will have one or more limits, and we may suppose the integers so chosen that there is only one limit, which lies, of

course, between -1 and 1 inclusive. We thus get in the limit for such a value of θ,

$$\left|\frac{1}{x-x_0}\int_{x_0}^{x} |d[f(x)-qx]| - f'(x_0) - q + \theta e\right| = 0.$$

Now let $e \to 0$, and in consequence $x \to x_0$. The last equation shows that

$$\frac{1}{x-x_0}\int_{x_0}^{x} |d[f(x)-qx)|$$

has an unique limit as $x \to x_0$, namely,

$$f'(x_0) - q.$$

This proves the theorem, since q is any quantity, and may, in particular, be $f'(x_0)$.

8. Hence, except for a set of values of x of content zero,

$$\underset{t\to 0}{\mathrm{Lt}}\, t^{-1}\int_0^t |d[f(x+t)-(x+t)f'(x)]| = \underset{X\to x}{\mathrm{Lt}}\, \frac{1}{X-x}\int_x^X |d[f(X)-Xf'(x)]|$$
$$= 0.$$

Also

$$\underset{t\to 0}{\mathrm{Lt}}\, t^{-1}\int_0^t |d[-f(x-t)-(t-x)f'(x)]| = \underset{X\to x}{\mathrm{Lt}}\, \frac{1}{X-x}\int_x^X |d[-f(X)+Xf'(x)]|$$
$$= 0.$$

But $$t^{-1}\int_0^t |du+dv| \leqslant t^{-1}\int_0^t |du| + t^{-1}\int_0^t |dv|;$$

and therefore $\underset{t\to 0}{\mathrm{Lt}}\, \tfrac{1}{2}t^{-1}\int_0^t |d[f(x+t)-f(x-t)-2tf'(x)]| = 0,$

or, say $$\underset{t\to 0}{\mathrm{Lt}}\, \Phi(t)/t = 0,$$

except for a set of values of x of content zero.

9. We can now prove the first of our main results.

THEOREM.—*The derived series of the Fourier series of a function of bounded variation converges almost everywhere to the differential coefficient of the function.*

We proved, in § 3, that

$$\frac{2\Gamma(1+k)}{\pi}\int_0^\infty t^{-1-k}C_{1+k}(t)\,n\,d\phi(t/n) = \sum_{m=1}^{m=n} A_m(1-m/n)^k - f'(x),$$

where
$$2\phi(t) = f(x+t) - f(x-t) - 2tf'(x).$$

Also, by § 7,
$$\frac{1}{u}\int_0^u |d\phi(u)| \to 0,$$

when
$$u \to 0,$$

except for a set of values of x of content zero.

Consider a value of x not belonging to this exceptional set, and, corresponding to the intervals of integration (0, 1), (1, n), and (n, ∞), let us write our integral on the left as the sum of three integrals $J_1+J_2+J_3$.

Then, since $t^{-1-k}C_{1+k}(t)$ is bounded in (0, 1),

$$|J_1| \leqslant \text{const.}\int_0^1 |n\,d\phi(t/n)| \leqslant \text{const.}\, n\int_0^{1/n} |d\phi(t)| \to 0,$$

when
$$n \to \infty.$$

In J_3 we consider separately the three integrals of which it is the sum, corresponding to the three terms whose sum is $\phi(t)$. The first and second of these are of the same type; the first is numerically

$$< \text{const.}\int_n^\infty t^{-1-k}\, n\, |df(x+t/n)|$$
$$< \text{const.}\; n^{-k}\int_1^\infty t^{-1-k}|df(x+t)| < \text{const.}\; n^{-k}\int_1^\infty |df(x+t)|;$$

since $f(x)$ has bounded variation in the whole infinite interval, this term is of the order of n^{-k}; the same is true of the second term of J_3. The third term, being numerically less than

$$\text{const.}\, f'(x)\int_n^\infty t^{-1-k}\,dt,$$

also approaches zero as $n \to \infty$. Thus

$$J_3 \to 0$$

when
$$n \to \infty.$$

As regards J_2, it is numerically less than a constant multiple of

$$\int_1^n t^{-1-k}|n\,d\phi(t/n)| = n^{-k}\int_{1/n}^1 t^{-1-k}|d\phi(t)|.$$

Writing, as in § 8, $\Phi(t)$ for the total variation of $\phi(t)$, namely,

$$\Phi(t) = \int_0^t |d\phi(t)|,$$

we get, integrating by parts,

$$|J_2| < \text{const.} \left\{ n^{-k}\Phi(1) - n^{-k} n^{1+k}\Phi(1/n) - n^{-k}\int_{1/n}^1 t^{-k-2}\,\Phi(t)\,dt \right\}.$$

Of these three terms the first obviously vanishes with $1/n$; the second is $n\Phi(1/n)$, which also vanishes. The third is of the form

$$n^{-k}\int_{1/n}^{\epsilon} t^{-2-k}\,\Phi(t)\,dt + n^{-k}\int_{\epsilon}^{1} t^{-2-k}\,\Phi(t)\,dt,$$

where ϵ is as small as we please. The second term vanishes with n^{-k}, and the first term is of the order

$$n^{-k}\int_{1/n}^{\epsilon} t^{-2-k}\,o(t)\,dt = n^{-k}\int_{1/n}^{\epsilon} o(t^{-1-k})\,dt = o(n^{-k}n^{k}) = o(1) \to 0.$$

Thus $$J_3 \to 0$$

when $$n \to \infty.$$

Thus, since J_1, J_2, and J_3 all approach zero when $n \to \infty$, the same is true of their sum, which is the left-hand side of the first equation of the present article. Hence, except for a set of values of x of content zero,

$$\sum_{m=1}^{m=n} (a_n \cos nx + b_n \sin nx)(1 - m/n)^k \to f'(x),$$

when $$n \to \infty,$$

$f(x)$ being a function of bounded variation in the infinite interval, and $\Sigma(a_n \cos nx + b_n \sin nx)$ being the derived series of its Fourier series.

10. The result just obtained constitutes a double extension of Lebesgue's result that, if at the point $u = 0$,

$$\int_0^u |f(x+u) + f(x-u) - 2f(x)|\,du$$

has a differential coefficient which is zero, then the Fourier series of $f(x)$ converges $(C1)$ to $f(x)$, and that this happens accordingly almost everywhere in the interval of periodicity of x. In fact, the Fourier series

of a summable function $f(x)$ is replaced by the derived series of a function of bounded variation, while the index unity is replaced by any smaller fractional index. The only change necessary is the substitution of

$$\int_0^u |d[f(x+u)-f(x-u)-2uf'(x)]|$$

for the above integral, where $f(x)$ is the function of bounded variation, and the convergence at the points x in question is to $f'(x)$.

We may similarly give a still greater extension of another result due to Lebesgue. In this extended form it is as follows :—

At a point at which $[f(x+t)-f(x-t)]/2t$ *has an unique limit as* $t \to 0$, *the derived series of the Fourier series of the function* $f(x)$ *converges when summed in any Cesàro manner, having an index integral or fractional greater than unity. Moreover, the sum to which it converges is the unique limit in question.*

I have already given this extension for the case in which the function of bounded variation is an integral. On reference, however, to the proof in § 4, p. 262, of my paper on "The Convergence of a Fourier Series and its Allied Series," it will be seen that, with scarcely any change in the argument used, it applies when $f(x)$ is a function of bounded variation, or again any continuous function whatever. This is of importance, because it may very well happen that a continuous function has at a particular point a differential coefficient or that, more generally, the limit employed in the theorem exists, though it is only in the case of a function of bounded variation, not necessarily continuous, that this must be the case almost everywhere.

That on the previous occasion I stated this theorem in a less extended form is due to the fact that I had Fourier series alone in my mind, and it is only when a function is an integral that its derived series is a Fourier series.

11. We proceed now to occupy ourselves with the second of the two main results of the present paper. We have first, however, to obtain convenient expressions for the sum of the first n terms of the derived series of the Fourier series and allied series of a function of bounded variation. These expressions involve integrals with respect to the function of bounded variation. It will be noted that the coefficients of such series are themselves expressible simply in the form of integrals of this type.

We then pass to the determination of the order of infinity almost everywhere of these partial summations. This is found to be the same as that already obtained for the Fourier series of a summable function.*

12. Let $A_n = a_n \cos nx + b_n \sin nx$ be the typical term of a trigonometrical series, which is the derived series of the Fourier series of a function of bounded variation $f(x)$. Then

$$\psi(t) = \tfrac{1}{2}[f(x+t)-f(x-t)] \sim n^{-1} A_n \sin nt. \tag{1}$$

Therefore, since

$$\operatorname{cosec} \tfrac{1}{2}t \sin (n+\tfrac{1}{2})\, t = \tfrac{1}{2} + \cos t + \cos 2t + \ldots + \cos nt,$$

we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \operatorname{cosec} \tfrac{1}{2}t \sin(n+\tfrac{1}{2})\, t\, d\psi(t) = \frac{1}{2\pi}\int_{-\delta}^{\pi} d\psi(t) + \frac{1}{\pi}\sum_{m=1}^{n}\int_{-\pi}^{\pi} \cos mt\, d\psi(t). \tag{2}$$

Integrating by parts,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos mt\, d\psi(t) = \frac{1}{\pi}\Big[\psi(t) \cos mt\Big]_{-\pi}^{\pi} + \frac{m}{\pi}\int_{-\pi}^{\pi} \sin mt\, \psi(t)\, dt = A_m,$$

by (1).

Hence, by (2),

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \operatorname{cosec} \tfrac{1}{2}t \sin (n+\tfrac{1}{2})\, t\, d\psi(t) = \tfrac{1}{2}A_0 + A_1 + \ldots + A_n = s_n. \tag{3}$$

13. From the last result we have

$$s_n = O\int_0^c t^{-1} \sin nt\, d\psi(t).$$

It will be convenient for our purposes to replace $\phi(t)$ in this relation by the function $\phi(t)$, where

$$2\phi(t) = f(x+t) - f(x-t) - 2tf'(x) = 2\phi(t) - 2tf'(x).$$

This is allowable except for a set of values of x of content zero, since the

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* It should be noticed that other results in the paper just cited are similarly capable of more general statement.

term $2tf'(x)$ does not affect the order at the points where $f'(x)$ exists and is finite. Thus, we may write

$$s_n = O\int_0^c t^{-1}\sin nt\,d\phi(t).$$

14. Again, if we write

$$b_n\cos nx - a_n\sin nx = B_n,$$

we have

$$\tfrac{1}{2}f(x+t)+f(x-t)-\Sigma n^{-1}B_n\cos nt.$$

Let us write $\chi(t) = \tfrac{1}{2}[f(x+t)+f(x-t)]$.

Then

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\sin mt\,d\chi(t) = -\Big[\chi(t)\sin mt\Big]_{-\pi}^{\pi} - \frac{m}{\pi}\int_{-\pi}^{\pi}\chi(t)\cos mt = B_m.$$

Now $\tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}t\{-(1-\cos\tfrac{1}{2}t)+[1-\cos(n+\tfrac{1}{2})t]\} = \sin t+\ldots+\sin nt.$

Therefore

$$\sum_{m=1}^{n} B_m = \frac{1}{\pi}\sum_{m=1}^{n}\int_{-\pi}^{\pi}\sin mt\,d\chi(t)$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}t\,\{-(1-\cos\tfrac{1}{2}t)+[1+\cos(n+\tfrac{1}{2})t]\}\,d\chi(t).$$

15. Denoting by σ_n the sum of $B_1+B_2+\ldots+B_n$, the last result shows that

$$\sigma_n = O\int_{-\pi}^{\pi} t^{-1}(1-\cos nt)\,d\chi(t).$$

Now, if we introduce a new term into $\chi(t)$ and write

$$\theta(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2xf'(x)\},$$

we shall again not alter the order, provided x does not belong to the set of content zero at which $f'(x)$ does not exist and have a finite value. Thus,

$$\sigma_n = O\int_{-\pi}^{\pi} t^{-1}(1-\cos nt)\,d\theta(t)$$

almost everywhere.

We also remark that, as in § 8, we can show $\dfrac{1}{t}\displaystyle\int |\,d\theta(t)\,| \to 0$ as $t\to 0$, except for a set of values of x of content zero.

15. If s_n is the typical partial summation of the derived series of the Fourier series of a function $f(x)$ of bounded variation, we have, except for a set of values of x of content zero,

$$s_n = O\int_0^c t^{-1}\sin nt\, d\phi(t),$$

where, as before,

$$2\phi(t) = f(x+t)-f(x-t)-2tf'(x).$$

Dividing the interval $(0, c)$ at the point $1/n$ we have, therefore,

$$s_n = O\, n\int_0^{1/n} d\phi(t)+O\int_{1/n}^c t^{-1}\, d\phi(t).$$

The first term vanishes as $n\to\infty$, except for a set of values of x of content zero. The second term

$$= O(c^{-1}\Phi(c)-n\Phi(1/n)+\int_{1/n}^{c} t^{-2}\Phi(t)\,dt$$

$$= O(1)+o(1)+\int_{1/n}^{c} o(1/t)\,dt$$

$$= O(1)+o(1)+o(\log n) = o(\log n).$$

Thus, $$s_n = o(\log n).$$

16. Denoting the generic term $(a_n\cos nx+b_n\sin nx)$ of the derived series of the Fourier series of the function $f(x)$ of bounded variation by A_n, we know that $\Sigma A_n/\log n$ is a Fourier series, since $\Sigma\cos nx/\log n$ is a Fourier series.

Now

$$\sum_{n=2}^{N} nA_n/\log n = \sum_{n=2}^{N-1} s_n\Delta(n/\log n)-2s_1/\log 2+s_N N/\log N.$$

The last term is plainly $= o(N)$. The first term is

$$= \Sigma s_n O(1/\log n) = \Sigma o(\log n)\, O(1/\log n) = \Sigma o(1) = o(N).$$

Therefore $$\Sigma A_n/\log n$$

converges almost everywhere, if it converges $(C1)$, which is the case, since it is a Fourier series.

In other words, we have the theorem :

If $\Sigma(a_n\cos nx+b_n\sin nx)$ is the derived series of the Fourier series of a function $f(x)$ of bounded variation, the series

$$\Sigma(a_n\cos nx+b_n\sin nx)/\log n$$

is a Fourier series which converges almost everywhere.

We may also easily write down the function of which it is the Fourier series. Denoting, in fact, by $g(x)$ the even function which has $1/\log n$ for its typical Fourier constant, the sum is expressible as a constant multiple of the integral of $g(t)$ with respect to $[f(x+t)-f(x-t)]$ between the limits of integration $-\pi$ and π.

17. If, instead of employing the result of § 13, we employ that of § 15, we obtain in precisely the same way the corresponding result for the allied series. Since, however, $\Sigma \sin nx/\log n$ is not a Fourier series, we have to replace the factor $1/\log n$ by $(\log n)^{-1-k}$, $(0 < k)$, or by the typical Fourier constant of any odd function whose Fourier series belongs to the logarithmic scale. With this modification the enunciation of § 16 then applies.

THE APPLICATION OF THE CALCULUS OF FINITE DIFFERENCES TO CERTAIN TRIGONOMETRICAL SERIES

By A. E. JOLLIFFE *and* S. T. SHOVELTON.

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1. The object of this paper is to obtain the well-known expansions of $\cos m\theta$, $\dfrac{\sin m\theta}{\cos\theta}$, $\sin m\theta$, and $\dfrac{\cos m\theta}{\cos\theta}$, in series of ascending powers of $\sin\theta$ by the use of two central-difference formulæ. The remainders are investigated and interesting proofs obtained from them of the expansions of $\sin x$ and $\cos x$ in factors. The method is a reversal of that employed by Dr. W. F. Sheppard,* in which the central-difference formulæ are obtained from the expansions of $\sinh n\phi$ and $\cosh n\phi$ in power of $\sinh\phi$.

The two central-difference formulæ required are

$$u_x = u_0 + \sum_{r=1}^{r=m}\left\{\frac{(x+r-1)^{(2r-1)}}{(2r-1)!}\Delta^{2r-1}u_{-(r-1)} + \frac{(x+r-1)^{(2r)}}{(2r)!}\Delta^{2r}u_{-r}\right\} + R_{2m}$$

where
$$R_{2m} = \frac{(x+m)^{(2m+1)}}{(2m)!}\left[\Delta^{2m}\frac{u_x-u_y}{x-y}\right]_{y=-m},$$

and Δ in R_{2m} operates only on y, (1)

and

$$u_x = u_{-1} + \frac{x+1}{2}\Delta u_{-1} + \frac{x^2-1^2}{2^2.2!}\Delta^2 u_{-3} + \frac{(x^2-1^2)(x+3)}{2^3.3!}\Delta^3 u_{-3} + \dots$$
$$+ \frac{(x^2-1^2)(x^2-3^2)\dots[x^2-(2m-1)^2](x+2m+1)}{2^{2m+1}(2m+1)!}\Delta^{2m+1}u_{-(2m+1)} + R_{2m+1},$$

where

$$R_{2m+1} = \frac{(x^2-1^2)(x^2-3^2)\dots[x^2-(2m+1)^2]}{2^{2m+1}(2m+1)!}\left[\Delta^{2m+1}\frac{u_x-u_y}{x-y}\right]_{y=-(2m+1)},$$

* W. F. Sheppard, "Central-Difference Formulæ," *Proc. London Math. Soc.*, Old Series, Vol. XXXI, pp. 453 and 469.

and the differences refer to intervals of 2 instead of unity, and in R_{2m+1} Δ operates only on y. (2)

These series may be obtained by methods similar to those employed in a former paper,* but they can be proved more readily by starting with the remainder. In the first place we have

$$\frac{1}{(x-m)(x-m+1)\ldots(x+m)}$$

$$= \frac{1}{(2m)!}\left\{\frac{1}{x+m} - \frac{^{2m}C_1}{x+m-1} + \ldots - \frac{^{2m}C_1}{x-m+1} + \frac{1}{x-m}\right\}.$$

Therefore $$1 = \frac{(x+m)^{(2m+1)}}{(2m)!}\left[\Delta_y^{2m}\frac{1}{x-y}\right]_{y=-m},$$

where the symbol Δ_y is used to indicate that Δ operates only on y. (A)

Again, the expression

$$\frac{(x+m)^{(2m+1)}}{(2m)!}\left\{\frac{u_{-m}}{x+m} - {}^{2m}C_1\frac{u_{-(m-1)}}{x+m-1} + \ldots + \frac{u_m}{x-m}\right\}$$

is an algebraic function of x of degree $2m$ which takes the values

$$u_{-m},\ u_{-(m-1)},\ \ldots,\ u_m,$$

when x has the values $-m,\ -(m-1),\ \ldots,\ m.$

Denoting this expression by $\phi(x)$ we can express it in the form

$$\phi(x) = a_0 + a_1\frac{x}{1!} + a_2\frac{x(x-1)}{2!} + a_3\frac{(x+1)x(x-1)}{3!} + a_4\frac{(x+1)\ldots(x-2)}{4!}$$

$$+ \ldots + a_{2m}\frac{(x+m-1)^{(2m)}}{(2m)!}.$$

Then
$$\Delta\phi(x) = a_1 + a_2 x + \ldots,$$
$$\Delta^2\phi(x) = a_2 + a_3(x+1) + \ldots,$$
$$\Delta^3\phi(x) = a_3 + a_4(x+1) + \ldots,\ \text{\&c.}$$

Therefore

$$a_0 = \phi(0),\quad a_1 = \Delta\phi(0),\quad a_2 = \Delta^2\phi(-1),\quad a_3 = \Delta^3\phi(-1),\quad \text{\&c.}$$

Hence $a_0 = u_0$, $a_1 = \Delta u_0$, $a_2 = \Delta^2 u_{-1}$, $a_3 = \Delta^3 u_{-1}$, &c.

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* S. T. Shovelton, "A Formula in Finite Differences, &c.," *Messenger of Mathematics*, New Series, 448, August, 1908.

Therefore

$$u_0+\sum_{r=1}^{r=m}\left\{\frac{(x+r-1)^{(2r-1)}}{(2r-1)!}\Delta^{2r-1}u_{-(r-1)}+\frac{(x+r-1)^{(2r)}}{(2r)!}\Delta^{2r}u_{-r}\right\}$$

$$=\phi(x)=\frac{(x+m)^{(2m+1)}}{(2m)!}\left[\Delta_y^{2m}\frac{u_y}{x-y}\right]_{y=-m}. \quad \text{(B)}$$

By combining A and B we have the formula stated in (1). Since this is an identity, it will hold for all values of x, real or complex.

The second formula may be proved in a similar manner, or may be deduced from (1) by changing x into $\frac{1}{2}(x+1)$ and y into $\frac{1}{2}(y+1)$. We then have

$$u_{\frac{1}{2}(x+1)}=u_0+\frac{x+1}{2}\Delta u_0+\frac{x^2-1^2}{2^2.2!}\Delta^2 u_{-1}+\frac{(x^2-1^2)(x+3)}{2^3.3!}\Delta^3 u_{-1}+\dots$$

$$+\frac{(x^2-1^2)(x^2-3^2)\dots[x^2-(2m-1)^2](x+2m+1)}{2^{2m+1}(2m+1)!}\Delta^{2m+1}u_{-m}+R_{2m+1},$$

where

$$R_{2m+1}=\frac{(x^2-1^2)\dots[x^2-(2m+1)^2]}{2^{2m+2}(2m+1)!}\left[\Delta_y^{2m+1}\frac{u_{\frac{1}{2}(x+1)}-u_{\frac{1}{2}(y+1)}}{\frac{1}{2}(x-y)}\right]_{y=-(2m+1)},$$

Δ_y now referring to an interval of 2 instead of unity, and the remainder being taken after one additional term in the series in order to preserve the symmetry of its form.

Writing $v_x=u_{\frac{1}{2}(x+1)}$, so that $v_{-1}=u_0$, $v_{-3}=u_{-1}$, &c., and the differences refer to intervals of 2 instead of unity in the series as well as in R_{2m+1}, we have

$$v_x=v_{-1}+\frac{x+1}{2}\Delta v_{-1}+\frac{x^2-1^2}{2^2.2!}\Delta^2 v_{-3}+\dots$$

$$+\frac{(x^2-1^2)(x^2-3^2)\dots[x^2-(2m-1)^2](x+2m+1)}{2^{2m+1}(2m+1)!}\Delta^{2m+1}v_{-(2m+1)}+R_{2m+1},$$

where

$$R_{2m+1}=\frac{(x^2-1^2)(x^2-3^2)\dots[x^2-(2m+1)^2]}{2^{2m+1}(2m+1)!}\left[\Delta_y^{2m+1}\frac{v_x-v_y}{x-y}\right]_{y=-(2m+1)}.$$

Before proceeding to particular cases it is advantageous to reduce the formulæ (1) and (2) for the special cases of u_x an even function and u_x an odd function of x. To save otherwise necessary explanations the functions will be written as u or v according as Δ refers to intervals of one or two.

If u_x is an odd function of x,

$$\Delta^{2r} u_{-r} = u_{-r} - {}^{2r}C_1 u_{-(r-1)} \dots - {}^{2r}C_1 u_{r-1} + u_r = 0.$$

Hence (1) reduces to

$$u_x = x\Delta u_0 + \frac{x(x^2-1^2)}{3!}\Delta^3 u_{-1} + \dots$$
$$+ \frac{x(x^2-1^2)\dots[x^2-(m-1)^2]}{(2m-1)!}\Delta^{2m-1} u_{-(m-1)} + R_{2m}. \quad (3)$$

If u_x is an even function of x,

$$\begin{aligned}
\Delta^{2r-1} u_{-(r-1)} &= u_r - {}^{2r-1}C_1 u_{r-1} + \dots + {}^{2r-1}C_1 u_{-(r-2)} - u_{-(r-1)} \\
&= u_{-r} - {}^{2r-1}C_1 u_{-(r-1)} + \dots + {}^{2r-1}C_1 u_{(r-2)} - u_{(r-1)} \\
&= \tfrac{1}{2}\{u_r - (1 + {}^{2r-1}C_1)\, u_{r-1} + ({}^{2r-1}C_1 + {}^{2r-1}C_2)\, u_{r-2} - \dots \\
&\qquad - (1 + {}^{2r-1}C_1) u_{-(r-1)} + u_{-r}\} \\
&= \tfrac{1}{2}\{u_r - {}^{2r}C_1 u_{r-1} - \dots - {}^{2r}C_1 u_{-(r-1)} + u_{-r}\} = \tfrac{1}{2}\Delta^{2r} u_{-r}.
\end{aligned}$$

Therefore

$$\frac{(x+r-1)^{(2r-1)}}{(2r-1)!}\Delta^{2r-1} u_{-(r-1)} + \frac{(x+r-1)^{(2r)}}{(2r)!}\Delta^{2r} u_{-r}$$
$$= \frac{(x+r-1)^{(2r-1)}}{(2r)!}\{x-r+r\}\,\Delta^{2r} u_{-r}$$
$$= \frac{x^2(x^2-1^2)\dots[x^2-(r-1)^2]}{(2r)!}\Delta^{2r} u_{-r}.$$

Hence for an even function of x (1) reduces to

$$u_x = u_0 + \frac{x^2}{2!}\Delta^2 u_{-1} + \frac{x^2(x^2-1^2)}{4!}\Delta^4 u_{-2} + \dots$$
$$+ \frac{x^2(x^2-1^2)\dots[x^2-(m-1)^2]}{(2m)!}\Delta^{2m} u_{-m} + R_{2m}. \quad (4)$$

If v_x is an odd function of x,

$$\begin{aligned}
\Delta^{2r} v_{-(2r+1)} &= v_{2r-1} - {}^{2r}C_1 v_{2r-3} + \dots - {}^{2r}C_1 v_{-(2r-1)} + v_{-(2r+1)} \\
&= -v_{-(2r-1)} + {}^{2r}C_1 v_{-(2r-3)} - \dots + {}^{2r}C_1 v_{2r-1} - v_{2r+1} \\
&= -\tfrac{1}{2}\{v_{2r+1} - (1 + {}^{2r}C_1)\, v_{2r-1} + \dots + (1 + {}^{2r}C_1)\, v_{-(2r-1)} - v_{-(2r+1)}\} \\
&= -\tfrac{1}{2}\Delta^{2r+1} v_{-(2r+1)}.
\end{aligned}$$

Therefore

$$\frac{(x^2-1^2)(x^2-3^2)\ldots[x^2-(2r-1)^2]}{2^{2r}(2r)!}\left\{\Delta^{2r}v_{-(2r+1)}+\frac{(x+2r+1)}{2(2r+1)}\Delta^{2r+1}v_{-(2r+1)}\right\}$$
$$=\frac{x(x^2-1^2)\ldots[x^2-(2r-1)^2]}{2^{2r+1}(2r+1)!}\Delta^{2r+1}v_{-(2r+1)}.$$

Hence for an odd function (2) reduces to

$$v_x=\frac{x}{2}\Delta v_{-1}+\frac{x(x^2-1^2)}{2^3.3!}\Delta^3 v_{-3}+\ldots$$
$$+\frac{x(x^2-1^2)\ldots[x^2-(2m-1)^2]}{2^{2m+1}(2m+1)!}\Delta^{2m+1}v_{-(2m+1)}+R_{2m+1}. \quad (5)$$

If v_x is an even function of x,

$$\Delta^{2r+1}v_{-(2r+1)}=v_{2r+1}-{}^{2r+1}C_1 v_{2r-1}+\ldots+{}^{2r+1}C_1 v_{-(2r-1)}-v_{-(2r+1)}=0,$$

and in this case (2) reduces to

$$v_x=v_{-1}+\frac{x^2-1^2}{2^2.2!}\Delta^2 v_{-3}+\ldots$$
$$+\frac{(x^2-1^2)(x^2-3^2)\ldots[x^2-(2m-1)^2]}{2^{2m+1}(2m+1)!}\Delta^{2m}v_{-(2m+1)}+R_{2m+1}. \quad (6)$$

The difference-factors in R_{2m} and R_{2m+1} may clearly be written

$$\Delta_y^{2m}\int_0^1 u'_{x+(y-x)z}\,dz \quad \text{and} \quad \Delta_y^{2m+1}\int_0^1 v'_{x+(y-x)z}\,dz,$$

where y is to be made equal to $-m$ in the first, and $-(2m+1)$ in the second after differencing, provided that u_x and v_x are finite and continuous functions of x throughout any range considered.

2. Taking first the formula given in (4) and writing $u_x=\cos\theta x$, we have

$$\Delta^{2r}u_{-r}=\cos r\theta-{}^{2r}C_1\cos(r-1)\theta+\ldots-{}^{2r}C_1\cos(r-1)\theta+\cos r\theta$$
$$=(e^{\frac{1}{2}i\theta}-e^{-\frac{1}{2}i\theta})^{2r}=(-1)^r(2\sin\tfrac{1}{2}\theta)^{2r}.$$

Therefore

$$\cos\theta x=1-\frac{x^2}{2!}2^2\sin^2\tfrac{1}{2}\theta+\frac{x^2(x^2-1^2)}{4!}2^4\sin^4\tfrac{1}{2}\theta-\ldots$$
$$+(-1)^m\frac{x^2(x^2-1^2)\ldots[x^2-(m-1)^2]}{(2m)!}2^{2m}\sin^{2m}\tfrac{1}{2}\theta+R_{2m},$$

where $$R_{2m} = \frac{x(x^2-1^2)\dots(x^2-m^2)}{(2m)!}\Delta_y^{2m}\int_0^1 u'_{x+(y-x)z}\,dz.$$

Now $$\int_0^1 u'_{x+(y-x)z}\,dz = -\theta\int_0^1 \sin[x+(y-x)z]\,\theta\,dz$$

$$= -\int_0^\theta \sin[\theta x+(y-x)\phi]\,d\phi,$$

by writing $$\theta z = \phi.$$

Also $$\Delta_y \sin[\theta x+(y-x)\phi] = 2\sin\tfrac{1}{2}\phi\cos[\theta x+(y+\tfrac{1}{2}-x)\phi],$$

$$\Delta_y^2 \sin[\theta x+(y-x)\phi] = -(2\sin\tfrac{1}{2}\phi)^2\sin[\theta x+(y+1-x)\phi].$$

Therefore

$$\Delta_y^{2m}\sin[\theta x+(y-x)\phi] = (-1)^m(2\sin\tfrac{1}{2}\phi)^{2m}\sin[\theta x+(y+m-x)\phi].$$

Since y is to be put equal to $-m$ this becomes

$$(-1)^m(2\sin\tfrac{1}{2}\phi)^{2m}\sin x(\theta-\phi).$$

Therefore

$$R_{2m} = (-1)^{m+1}\frac{x(x^2-1^2)\dots(x^2-m^2)}{(2m)!}\int_0^\theta (2\sin\tfrac{1}{2}\phi)^{2m}\sin x(\theta-\phi)\,d\phi.$$

If now $\pi \geqslant \theta > 0$, then $\dfrac{\sin x(\theta-\phi)}{\cos\frac{1}{2}\phi}$ is finite throughout the range of integration, and $(2\sin\frac{1}{2}\phi)^{2m}\cos\frac{1}{2}\phi$ is positive, and therefore, if A is the greatest value of $\left|\dfrac{\sin x(\theta-\phi)}{\cos\frac{1}{2}\phi}\right|$,

$$|R_{2m}| \leqslant A\left|\frac{x(x^2-1^2)\dots(x^2-m^2)}{(2m)!}\right|\int_0^\theta (2\sin\tfrac{1}{2}\phi)^{2m}\cos\tfrac{1}{2}\phi\,d\phi$$

$$\leqslant A\,\frac{|x(x^2-1^2)\dots(x^2-m^2)|}{(2m)!}\,\frac{2^{2m+1}(\sin\frac{1}{2}\theta)^{2m+1}}{2m+1}$$

$$\leqslant A\left|\frac{2x\left(1-\dfrac{x^2}{1^2}\right)\left(1-\dfrac{x^2}{2^2}\right)\dots\left(1-\dfrac{x^2}{m^2}\right)}{\left(1+\dfrac{1}{2}\right)\left(1+\dfrac{1}{4}\right)\dots\left(1+\dfrac{1}{2m}\right)}\right|(\sin\tfrac{1}{2}\theta)^{2m+1}, \qquad (7)$$

the limit of which is zero since the denominator diverges to infinity, and the numerator is convergent.

Therefore $$\operatorname*{Lt}_{m\to\infty} |R_{2m}| = 0,$$

for $$\pi \geqslant \theta > 0.$$

It will be seen that if θ be changed to $-\theta$, R_{2m} is unchanged, and we therefore have

$$\cos\theta x = 1 - \frac{x^2}{2!}\, 2^2 \sin^2 \tfrac{1}{2}\theta + \frac{x^2(x^2-1^2)}{4!}\, 2^4 \sin^4 \tfrac{1}{2}\theta - \dots,$$

for $$-\pi \leqslant \theta \leqslant \pi.$$

If θ be changed to 2ϕ and x to $\frac{1}{2}m$, we have the expansion in the more familiar form

$$\cos m\phi = 1 - \frac{m^2}{2!} \sin^2\phi + \frac{m^2(m^2-2^2)}{4!} \sin^4\phi - \dots,$$

for $$-\tfrac{1}{2}\pi \leqslant \phi \leqslant \tfrac{1}{2}\pi. \tag{8}$$

It is clear from (7) that the series for $\cos\theta x$ is uniformly convergent for all values of x and for all values of θ between $-\pi$ and $+\pi$. The discussion of the uniform convergency for the case of $\theta = \pi$ is deferred until the elementary results arising from (3), (5) and (6) have been obtained.

3. If we put $u_x = \sin\theta x$ in (3) we find that

$$\Delta^{2r-1} u_{-(r-1)} = (-1)^{r-1}\, 2^{2r-1} \cos\tfrac{1}{2}\theta\, (\sin\tfrac{1}{2}\theta)^{2r-1}.$$

Proceeding, as in § 2,*

$$R_{2m} = (-1)^m \frac{x(x^2-1^2)\dots(x^2-m^2)}{(2m)!} \int_0^\theta (2\sin\tfrac{1}{2}\phi)^{2m} \cos x(\theta-\phi)\, d\phi,$$

and, if $0 < \theta < \pi$,

$$|R_{2m}| < B \left| \frac{x(x^2-1^2)\dots(x^2-m^2)}{(2m)!} \right| \frac{2^{2m+1}(\sin\frac{1}{2}\theta)^{2m+1}}{2m+1},$$

where B is the greatest value of $\left| \dfrac{\cos x(\theta-\phi)}{\cos\frac{1}{2}\phi} \right|$.

Hence, as before, R_{2m} converges uniformly to zero for all values of x,

* Mr. Bromwich points out to the authors that this result is at once obtained by differentiating, with respect to θ, the series for $\cos\theta x$ obtained in § 2.

and for all values of θ such that $-\pi < \theta < \pi$. Therefore, writing 2ϕ for θ and $\frac{1}{2}m$ for x, we have

$$\frac{\sin m\phi}{\cos\phi} = m\sin\phi - \frac{m(m^2-2^2)}{3!}\sin^3\phi + \frac{m(m^2-2^2)(m^2-4^2)}{5!} - \dots,$$

for $$-\tfrac{1}{2}\pi < \phi < \tfrac{1}{2}\pi. \tag{9}$$

4. If $\theta = \pi$, the limit of R_{2m} in § 3 is not zero. The integral in R_{2m} is

$$\int_0^\pi (2\sin\tfrac{1}{2}\phi)^{2m}\cos x(\pi-\phi)\,d\phi = \int_0^\pi (2\cos\tfrac{1}{2}\phi)^{2m}\cos x\phi\,d\phi$$

$$= \int_0^\pi (2\cos\tfrac{1}{2}\phi)^{2m}\left(1-2\sin^2\tfrac{1}{2}\phi\,\frac{\sin^2\frac{1}{2}x\phi}{\sin^2\frac{1}{2}\phi}\right)d\phi.$$

Now $$\left|\int_0^\pi (2\cos\tfrac{1}{2}\phi)^{2m}\sin^2\tfrac{1}{2}\phi\,\frac{\sin^2\frac{1}{2}x\phi}{\sin^2\frac{1}{2}\phi}\,d\phi\right|$$

$$\leqslant \int_0^\pi (2\cos\tfrac{1}{2}\phi)^{2m}\sin^2\tfrac{1}{2}\phi\left|\frac{\sin^2\frac{1}{2}x\phi}{\sin^2\frac{1}{2}\phi}\right|d\phi$$

$$\leqslant A\int_0^\pi (2\cos\tfrac{1}{2}\phi)^{2m}\sin^2\tfrac{1}{2}\phi\,d\phi,$$

where A is the greatest value of $\left|\dfrac{\sin^2\frac{1}{2}x\phi}{\sin^2\frac{1}{2}\phi}\right|$ between the values 0 and π of ϕ and is necessarily finite. Also

$$\int_0^\pi (2\cos\tfrac{1}{2}\phi)^{2m}\,d\phi = 2^{2m}\,\pi\,\frac{(2m-1)(2m-3)\dots 1}{2m(2m-2)\dots 2},$$

and $$\int_0^\pi (2\cos\tfrac{1}{2}\phi)^{2m}\sin^2\tfrac{1}{2}\phi\,d\phi = 2^{2m}\,\pi\,\frac{(2m-1)(2m-3)\dots 1}{2m(2m-2)\dots 2}\,\frac{1}{2m+2};$$

Therefore

$$R_{2m} = \sin\pi x = \pi x\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\dots\left(1-\frac{x^2}{m^2}\right)\left(1-\frac{C}{m+1}\right),$$

where $|C| \leqslant A$, defined above. Therefore

$$\sin\pi x = \pi x\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\dots,$$

for all values of x, real or complex.

Writing θ for πx we have the expansion of $\sin\theta$ in factors in the usual form.

5. If we put $v_x = \sin\theta x$ in (5), we have

$$\therefore \quad \Delta^{2r+1} v_{-(2r+1)} = (-1)^r (2\sin\theta)^{2r+1}.$$

The difference-factor in R_{2m+1} may be shown to reduce to

$$\left[\Delta_y^{2m+1} \int_0^\theta \cos\left[\theta x + (y-x)\,\phi\right] d\phi\right]_{y=-(2m+1)}.$$

Remembering that differences now refer to intervals of 2 in y this becomes

$$(-1)^{m+1} \int_0^\theta (2\sin\phi)^{2m+1} \sin x\,(\theta-\phi)\,d\phi.$$

Therefore

$$R_{2m+1} = (-1)^{m+1} \frac{(x^2-1^2)(x^2-3^2)\dots\left[x^2-(2m+1)^2\right]}{(2m+1)!} \times \int_0^\theta (\sin\phi)^{2m+1} \sin x\,(\theta-\phi)\,d\phi.$$

It will readily be seen, as in § 2, that

$$\operatorname*{Lt}_{m\to\infty} R_{2m+1} = 0,$$

for

$$-\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi,$$

and that R_{2m+1} converges uniformly to zero for all values of x, and for all values of θ between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. The discussion of the uniform convergency at the extremities of the range is given later.

We therefore have

$$\sin\theta x = x\sin\theta - \frac{x\,(x^2-1^2)}{3!}\sin^3\theta + \dots,$$

for

$$-\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi. \qquad (10)$$

6. If we put $v_x = \cos\theta x$ in (6), we find that

$$\Delta^{2r} v_{-(2r+1)} = (-1)^r (2\sin\theta)^{2r} \cos\theta.$$

Proceeding as in § 5, or by differentiation of the series for $\sin\theta x$,

$$R_{2m+1} = (-1)^{m+1} \frac{(x^2-1^2)\dots\left[x^2-(2m+1)^2\right]}{(2m+1)!} \int_0^\theta (\sin\phi)^{2m+1} \cos x\,(\theta-\phi)\,d\phi$$

As in § 3 it is seen that R_{2m+1} converges uniformly to zero for all

values of x and for all values of θ such that $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$. Hence

$$\frac{\cos\theta x}{\cos\theta} = 1 - \frac{x^2-1^2}{2!}\sin^2\theta + \frac{(x^2-1^2)(x^2-3^2)}{4!}\sin^4\theta \ldots,$$

for $$-\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi. \tag{11}$$

7. If $\theta = \frac{1}{2}\pi$, the integral in R_{2m+1} in § 6 is

$$\int_0^{\frac{1}{2}\pi} (\sin\phi)^{2m+1} \cos x(\tfrac{1}{2}\pi - \phi)\, d\phi = \int_0^{\frac{1}{2}\pi} (\cos\phi)^{2m+1} \cos x\phi\, d\phi$$

$$= \int_0^{\frac{1}{2}\pi} (\cos\phi)^{2m+1} \left(1 - 2\sin^2\phi \frac{\sin^2\frac{1}{2}x\phi}{\sin^2\phi}\right) d\phi$$

$$= \int_0^{\frac{1}{2}\pi} (\cos\phi)^{2m+1} (1 - 2D\sin^2\phi)\, d\phi$$

$\left(\text{where } |D| \text{ is } \not> \text{ the greatest value of } \left|\frac{\sin^2\frac{1}{2}x\phi}{\sin^2\phi}\right| \text{ between the values 0 and } \frac{1}{2}\pi \text{ of } \phi\right)$

$$= \frac{2m(2m-2)\ldots 2}{(2m+1)\ldots\ldots 3}\left(1 - \frac{2D}{2m+3}\right).$$

Therefore

$$\cos\frac{x\pi}{2} = \left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{3^2}\right)\cdots\left(1-\frac{x^2}{(2m+1)^2}\right)$$

$$\times \frac{1^2.3^2.5^2\ldots(2m+1)^2}{(2m+1)!}\,\frac{2m\ldots\ldots\ldots 2}{(2m+1)\ldots 3}\left(1-\frac{2D}{2m+3}\right)$$

$$= \left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{3^2}\right)\cdots\left(1-\frac{x^2}{(2m+1)^2}\right)\left(1-\frac{2D}{2m+3}\right).$$

Hence $$\cos\frac{\pi x}{2} = \left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{3^2}\right)\ldots,$$

for all values of x, real or complex. Writing θ for $\frac{1}{2}\pi x$, we have the expansion of $\cos\theta$ in the usual form.

8. To investigate the remainder in the series for $\cos\theta x$ in § 2 more closely we require the asymptotic value of $\frac{(m!)^2}{(2m)!}$, which can be obtained from Stirling's theorem, but in a more elementary fashion from the following considerations.

If $u_n = \int_0^{\frac{1}{2}\pi} \sin^n\theta\, d\theta$, then $u_{n-1} > u_n > u_{n+1}$, and $u_n u_{n+1} = \frac{1}{2}\pi/(n+1)$, and $u_n u_{n-1} = \frac{1}{2}\pi/n$; therefore $\frac{1}{2}\pi/n > u_n^2 > \frac{1}{2}\pi/(n+1)$.

Hence u_n lies between $\sqrt{(\frac{1}{2}\pi/n)}$ and $\sqrt{\{\frac{1}{2}\pi/(n+1)\}}$. Now

$$\frac{(2m)!}{(m!)^2} = \frac{2^{2m}(2m-1)(2m-3)\dots 3}{(2m)(2m-2)\dots\dots\dots 2} = 2^{2m}\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\sin^{2m}\theta\, d\theta,$$

which lies between $2^{2m}\sqrt{\left(\frac{2}{\pi}\right)}\frac{1}{\sqrt{(2m)}}$ and $2^{2m}\sqrt{\left(\frac{2}{\pi}\right)}\frac{1}{\sqrt{(2m+1)}}$. Therefore

$$|R_{2m}| \leqslant A\,\frac{(m!)^2}{(2m)!}\left|x\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\dots\left(1-\frac{x^2}{m^2}\right)\right|\frac{2^{2m+1}(\sin\frac{1}{2}\theta)^{2m+1}}{2m+1}$$

$$\leqslant A\sqrt{\left(\frac{2\pi}{2m+1}\right)}(\sin\tfrac{1}{2}\theta)^{2m+1}\left|x\left(1-\frac{x^2}{1^2}\right)\dots\left(1-\frac{x^2}{m^2}\right)\right|.$$

We can therefore choose m such that for all values of x whatever and for all values of θ in the range $(-\pi, \pi)$, $|R_{2m}| < \epsilon$, an arbitrarily assigned positive quantity, so that the convergence is uniform both with respect to x and θ for the values specified. The series can therefore be integrated with respect to x between any limits, and with respect to θ anywhere in the range $(-\pi, \pi)$.

If we differentiate with respect to θ, the differential coefficient of R_{2m} is

$$(-1)^{m+1}\frac{x^2(x^2-1^2)\dots(x^2-m^2)}{(2m)!}\int_0^\theta (2\sin\tfrac{1}{2}\phi)^{2m}\cos x(\theta-\phi)\,d\phi.$$

If $\pi > \theta > 0$,

$$\left|\int_0^\theta (\sin\tfrac{1}{2}\phi)^{2m}\cos x(\theta-\phi)\,d\phi\right| = \left|\int_0^\theta(\sin\tfrac{1}{2}\phi)^{2m}\cos\tfrac{1}{2}\phi\,\frac{\cos x(\theta-\phi)}{\cos\frac{1}{2}\phi}\,d\phi\right|$$

$$< \frac{2A}{2m+1}(\sin\tfrac{1}{2}\theta)^{2m+1},$$

where A is the greatest value of $\left|\frac{\cos x(\theta-\phi)}{\cos\frac{1}{2}\phi}\right|$ in the range of integration, and this value is finite if θ is anywhere in the range 0 to π, π being excluded. If θ be changed to $-\theta$, R_{2m} is unchanged, and its differential coefficient changes sign so that if $\pi > \theta > -\pi$,

$$\left|\int_0^\theta(\sin\tfrac{1}{2}\phi)^{2m}\cos x(\theta-\phi)\,d\phi\right| < \frac{2A}{2m+1}(\sin\tfrac{1}{2}\theta)^{2m+1}.$$

We can then show as above that $d/d\theta\,(R_{2m})$ tends uniformly to zero in any interval between $-\pi$ and $+\pi$, which excludes both $-\pi$ and π, so that we can differentiate term by term with respect to θ in such an interval.

The differential coefficient of R_{2m} with respect to x is

$$(-1)^{m+1}\frac{x(x^2-1^2)\dots(x^2-m^2)}{(2m)!}\int_0^\theta (2\sin\tfrac{1}{2}\phi)^{2m}(\theta-\phi)\cos x(\theta-\phi)\,d\phi$$

$$+(-1)^{m+1}\frac{x(x^2-1^2)\dots(x^2-m^2)}{(2m)!}\left(\frac{1}{x}+\frac{2x}{x^2-1^2}+\dots+\frac{2x}{x^2-m^2}\right)$$
$$\times\int_0^\theta (2\sin\tfrac{1}{2}\phi)^{2m}\sin x(\theta-\phi)\,d\phi.$$

Since $\dfrac{(\theta-\phi)\cos x(\theta-\phi)}{\cos\frac{1}{2}\phi}$ and $\dfrac{\sin x(\theta-\phi)}{\cos\frac{1}{2}\phi}$ are both finite throughout the range of integration if θ is anywhere in the interval $(-\pi, \pi)$ both end points now included, we see that $d/dx\,(R_{2m})$ tends uniformly to zero for values of x in any interval, and for values of θ in the interval $(-\pi, \pi)$. We can therefore differentiate term by term with respect to x for all values of x, so long as θ is in the interval $(-\pi, \pi)$, the end points included.

9. If $\theta=\pi+a$, where $\pi>a>0$, the integral in R_{2m} in the series for $\cos\theta x$, in § 2, is

$$\int_0^{\pi+a}(2\sin\tfrac{1}{2}\phi)^{2m}\sin x(\pi+a-\phi)\,d\phi,$$

which is equal to

$$\sin ax\int_0^{\pi+a}(2\sin\tfrac{1}{2}\phi)^{2m}\cos x(\pi-\phi)\,d\phi+\cos ax\int_0^{\pi+a}(2\sin\tfrac{1}{2}\phi)^{2m}\sin x(\pi-\phi)\,d\phi$$

$$=\sin ax\left[2\int_0^{\pi}(2\cos\tfrac{1}{2}\theta)^{2m}\cos x\theta\,d\theta-\int_a^{\pi}(2\cos\tfrac{1}{2}\theta)^{2m}\cos x\theta\,d\theta\right]$$
$$+\cos ax\int_0^{\pi-a}(2\sin\tfrac{1}{2}\phi)^{2m}\sin x(\pi-\phi)\,d\phi.$$

Hence

$$R_{2m}=(-1)^{m+1}\frac{x(x^2-1^2)\dots(x^2-m^2)}{(2m)!}\left\{(A\cos ax-B\sin ax)\frac{(2\cos\frac{1}{2}a)^{2m+1}}{2m+1}\right.$$
$$\left.+2\sin ax\,.\,2^{2m}\,.\,\pi\frac{(2m-1)\dots1}{2m\,\dots\dots\,2}\left(1-\frac{c}{m+1}\right)\right\},$$

where A, B and C are finite.

Therefore R_{2m} tends to the form

$$-2\sin ax\,.\,\pi x\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\dots.$$

But when $\theta = \pi + a$, the series

$$1 - \frac{x^2}{2!}(2\sin\tfrac{1}{2}\theta)^2 + \dots$$

has the same value as when $\theta = \pi - a$, and is therefore equal to

$$\cos(\pi - a)\,x.$$

Therefore

$$R_{2m} = \cos(\pi + a)\,x - \cos(\pi - a)\,x = -2\sin\pi x \sin ax.$$

Equating the two values of R_{2m} we have a second proof of the infinite product for $\sin\pi x$.

10. Considerations similar to those in § 8 will establish the fact that the series for $\sin\theta x$ in § 5 (10) is uniformly convergent for all values of x, and for all values of θ in the range $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$. It can also be shown that the series can be differentiated with respect to θ for all values of x for values of θ in the interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ the end points excluded, and with respect to x for all values of x for values of θ in the same interval with the end points included.

The consideration of the case $\theta = \frac{1}{2}\pi + a$ will lead to the ordinary expression of $\cos\frac{1}{2}\pi x$ in factors.

11. It is interesting to notice that all the series can be obtained without the theory of differences by taking the remainders in the integral form and finding the reduction formulæ. We have

$$\int_0^\theta (2\sin\tfrac{1}{2}\phi)^{2m}\sin x(\theta-\phi)\,d\phi$$

$$= \left[(2\sin\tfrac{1}{2}\phi)^{2m}\frac{\cos x(\theta-\phi)}{x}\right]_0^\theta - \int_0^\theta 2m(2\sin\tfrac{1}{2}\phi)^{2m-1}\cos\tfrac{1}{2}\phi\,\frac{\cos x(\theta-\phi)}{x}\,d\phi$$

$$= \frac{1}{x}(2\sin\tfrac{1}{2}\theta)^{2m} + \frac{2m}{x^2}\left[(2\sin\tfrac{1}{2}\phi)^{2m-1}\cos\tfrac{1}{2}\phi\sin x(\theta-\phi)\right]_0^\theta$$

$$-\frac{2m}{x^2}\int_0^\theta\left\{(2m-1)(2\sin\tfrac{1}{2}\phi)^{2m-2}\cos^2\tfrac{1}{2}\phi - \tfrac{1}{4}(2\sin\tfrac{1}{2}\phi)^{2m}\right\}\sin x(\theta-\phi)\,d\phi$$

$$= \frac{1}{x}(2\sin\tfrac{1}{2}\theta)^{2m} - \frac{2m(2m-1)}{x^2}\int_0^\theta(2\sin\tfrac{1}{2}\phi)^{2m-2}\sin x(\theta-\phi)\,d\phi$$

$$+\frac{(2m)^2}{x^2}\,\tfrac{1}{4}\int_0^\theta(2\sin\tfrac{1}{2}\phi)^{2m}\sin x(\theta-\phi)\,d\phi.$$

Therefore

$$\int_0^\theta (2\sin\tfrac{1}{2}\phi)^{2m}\sin x(\theta-\phi)\,d\phi$$

$$=\frac{x}{(x^2-m^2)}(2\sin\tfrac{1}{2}\theta)^{2m}-\frac{2m(2m-1)}{(x^2-m^2)}\int_0^\theta(2\sin\tfrac{1}{2}\phi)^{2m-2}\sin x(\theta-\phi)\,d\phi,$$

and

$$(-1)^{m+1}\frac{x(x^2-1^2)\ldots(x^2-m^2)}{(2m)!}\int_0^\theta(2\sin\tfrac{1}{2}\phi)^{2m}\sin x(\theta-\phi)\,d\phi$$

$$+(-1)^m\frac{x^2(x^2-1^2)\ldots[x^2-(m-1)^2]}{(2m)!}(2\sin\tfrac{1}{2}\theta)^{2m}$$

$$=(-1)^m\frac{x(x^2-1^2)\ldots[x^2-(m-1)^2]}{(2m-2)!}\int_0^\theta(2\sin\tfrac{1}{2}\phi)^{2m-2}\sin x(\theta-\phi)\,d\phi,$$

which leads to the expansion of $\cos\theta x$ in powers of $\sin\tfrac{1}{2}\theta$.

The other series may be obtained in a similar manner by taking the corresponding remainders.

EINIGE UNGLEICHUNGEN FÜR ZWEIMAL DIFFERENTIIERBARE FUNKTIONEN

Von EDMUND LANDAU in Göttingen.

[Received May 26th, 1913.—Read June 12th, 1913.]

DIE folgenden Betrachtungen knüpfen an den Ausgangspunkt einer grösseren Reihe von Sätzen der Differentialrechnung an, welche das zweite Kapitel der Abhandlung ausmachen: "Contributions to the Arithmetic Theory of Series," by G. H. Hardy and J. E. Littlewood.* Jenen Ausgangspunkt (pp. 416, 417) bildet der schon früher von Herrn Littlewood† ("The Converse of Abel's Theorem on Power Series," pp. 437, 438) gefundene

SATZ.—*Es sei $f(x)$ eine für $x > x_0$ definierte reelle Funktion; $f''(x)$ sei ebenda vorhanden, stetig und beschränkt. Für $x \to \infty$ sei $f(x) \to s$; dann ist $f'(x) \to 0$.*

Dass die Stetigkeit von $f''(x)$ nicht voll benutzt wird, ist den Verfassern bewusst. Es lässt sich aber beweisen, dass über $f''(x)$ ausser der blossen Existenz und Beschränktheit gar nichts vorausgesetzt zu werden braucht.‡ Nach dem Paradigma des Beweises von Hilfssatz 1 (p. 266) meiner Abhandlung "Über einen Satz des Herrn Littlewood"§ verläuft die Begründung so: Nach Voraussetzung ist $|f''(x)| < c$ für $x > x_0$ Für jedes $\epsilon > 0$ und $x > x_0$ ist nach dem Taylorschen Satz

$$f(x+\epsilon)-f(x) = \epsilon f'(x)+\tfrac{1}{2}\epsilon^2 f''(\xi) \quad (x < \xi < x+\epsilon),$$

* *Proc. London Math. Soc.*, Ser. 2, Vol. 11 (1912–1913), pp. 411–478.

† Ebenda, Ser. 2, Vol. 9 (1910–1911), pp. 434–448.

‡ Hierin ist auch ein Satz von Herrn C. N. Moore ["On the Introduction of Convergence Factors into Summable Series and Summable Integrals," *Trans. American Math. Soc.*, Vol. 8 (1907), pp. 299–330 (Lemma 5, p. 316)] enthalten, der ausser meinen Voraussetzungen die weitere $f''(x) \to 0$ macht.

§ *Rendiconti del Circolo Matematico di Palermo*, Bd. 35 (1913), S. 265–276.

$$|f'(x)| \leqslant \frac{|f(x+\epsilon)-f(x)|}{\epsilon} + \tfrac{1}{2}\epsilon\,|f''(\xi)|$$

$$< \frac{|f(x+\epsilon)-f(x)|}{\epsilon} + \tfrac{1}{2}\epsilon c.$$

Wegen $f(x) \to s$ ist also für $x > x_1 = x_1(\epsilon)$

$$|f'(x)| < \epsilon + \tfrac{1}{2}\epsilon\, c = \epsilon\,(1+\tfrac{1}{2}c),$$

d. h.

$$f'(x) \to 0.$$

Hinter dem so verschärften Littlewoodschen Satze stecken nun folgende weiteren Sätze, die ihn enthalten* und sich dadurch auszeichnen, dass gewisse in ihnen auftretenden Konstanten bestmögliche Werte haben. $f(x)$ bezeichnet eine reelle Funktion.

SATZ 1.—*Wenn in einem Intervall der Länge* $\geqslant 2$

$$|f(x)| \leqslant 1, \qquad |f''(x)| \leqslant 1$$

ist,† *so ist ebenda*

$$|f'(x)| \leqslant 2.$$

SATZ 2.—*Die Konstante* 2 *in der Behauptung des Satzes* 1 *lässt sich durch keine kleinere Zahl ersetzen.*

SATZ 3.—*Die Konstante* 2 *in der Voraussetzung des Satzes* 1 *lässt sich durch keine kleinere Zahl ersetzen.*

SATZ 4.—*Wenn*

$$\limsup_{x=\infty} |f(x)| \leqslant 1,$$

$$\limsup_{x=\infty} |f''(x)| \leqslant 1$$

ist, so ist

$$\limsup_{x=\infty} |f'(x)| \leqslant \sqrt{2}.$$

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* In Satz 1 ist der obige Satz enthalten. Denn, wenn $a > 0$, $b > 0$ ist und in Satz 1

$$x = y\sqrt{(b/a)}, \quad f(x) = f[y\sqrt{(b/a)}] = [g(y)]/a$$

gesetzt wird, so lehrt Satz 1: Wenn in einem Intervall der Länge $\geqslant 2\sqrt{(a/b)}$

$$|g(y)| \leqslant a, \quad |g''(y)| \leqslant b$$

ist, so ist ebenda

$$|g'(y)| \leqslant 2\sqrt{(ab)}.$$

Im Falle $g(y) = F(y) - s \to 0$, $|g''(y)| = |F''(y)| < c$ (für $y > y_0$) ist nun bei festem $\delta > 0$ für alle $y > y_1(\delta)$

$$|g(y)| \leqslant \delta, \quad |g''(y)| \leqslant c,$$

also

$$|g'(y)| \leqslant 2\sqrt{\delta}\,\sqrt{c},$$

so dass $g'(y) \to 0$ ist.

† Es ist gleichgiltig, ob man in den Endpunkten $f'(x)$ und $f''(x)$ einseitig nach innen oder zweiseitig meint.

SATZ 5.—*Die Konstante* $\sqrt{2}$ *des Satzes* 4 *lässt sich durch keine kleinere Zahl ersetzen.*

BEWEIS VON SATZ 1. — Ohne Beschränkung der Allgemeinheit sei $0 \leqslant x \leqslant 2$ das Intervall (d. h. seine Länge $= 2$ und seine Lage rechts an 0 anliegend). Dann ist

$$f(x)-f(0) = xf'(x)-\tfrac{1}{2}x^2 f''(\xi_1) \quad (0 \leqslant \xi_1 \leqslant x \leqslant 2),$$

$$f(2)-f(x) = (2-x)f'(x)+\tfrac{1}{2}(2-x)^2 f''(\xi_2) \quad (0 \leqslant x \leqslant \xi_2 \leqslant 2),$$

$$f(2)-f(0) = 2f'(x)-\tfrac{1}{2}x^2 f''(\xi_1)+\tfrac{1}{2}(2-x)^2 f''(\xi_2),$$

$$2|f'(x)| \leqslant 1+1+\tfrac{1}{2}x^2 . 1+\tfrac{1}{2}(2-x)^2 . 1 = 4-x(2-x) \leqslant 4,$$

$$|f'(x)| \leqslant 2.$$

BEWEIS VON SATZ 2.—Die Funktion $f(x) = \frac{1}{2}x^2-1$ genügt im Intervall 0 bis 2 den Bedingungen

$$|f(x)| \leqslant 1, \quad |f''(x)| = |1| \leqslant 1,$$

und es ist
$$|f'(2)| = 2,$$

also für kein $\delta > 0$ im Intervall beständig $|f'(x)| \leqslant 2-\delta$.

BEWEIS VON SATZ 3.—Es sei ein positives $\rho < 2$ gegeben. Die Funktion

$$f(x) = \tfrac{1}{2}x^2-\left(\frac{2}{\rho}+\frac{\rho}{2}\right)x+1$$

ist im Intervall $0 \leqslant x \leqslant \rho$ absolut $\leqslant 1$, nämlich 1 für $x = 0$, -1 für $x = \rho$ und dazwischen abnehmend, da

$$f'(x) = x-\left(\frac{2}{\rho}+\frac{\rho}{2}\right) < \rho-2 < 0$$

ist. Es ist ferner
$$|f''(x)| = 1 \leqslant 1,$$

aber
$$|f'(0)| = \frac{2}{\rho}+\frac{\rho}{2} > 2.$$

BEWEIS VON SATZ 4.—Wenn in einem Intervall der Länge $\geqslant 2\sqrt{2}$ sowohl $|g(x)| \leqslant 1$ als auch $|g''(x)| \leqslant 1$ ist,* so ist in seinem Mittelpunkte

$$|g'(x)| \leqslant \sqrt{2},$$

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* Es ist gleichgiltig, ob in den Endpunkten $g'(x)$ und $g''(x)$ zweiseitig oder nur einseitig gemeint sind.

wie aus
$$g(x+\sqrt{2})-g(x) = \sqrt{2}\,.\,g'(x)+\tfrac{1}{2}.2g''(\xi_1),$$
$$g(x)-g(x-\sqrt{2}) = \sqrt{2}\,.\,g'(x)-\tfrac{1}{2}.2g''(\xi_2),$$
$$g(x+\sqrt{2})-g(x-\sqrt{2}) = 2\sqrt{2}.g'(x)+g''(\xi_1)-g''(\xi_2),$$
$$2\sqrt{2}\,|g'(x)| \leqslant 1+1+1+1 = 4$$
folgt. Bei festem $\epsilon > 0$ ist nun,
$$g(x) = f(x)/(1+\epsilon)$$
gesetzt, von einer Stelle $x_1(\epsilon)$ an
$$|g(x)| \leqslant 1, \qquad |g''(x)| = \left|\frac{f''(x)}{1+\epsilon}\right| \leqslant 1,$$
also von der Stelle
$$x_1(\epsilon)+\sqrt{2} = x_2(\epsilon)$$
an
$$|g'(x)| = \left|\frac{f'(x)}{1+\epsilon}\right| \leqslant \sqrt{2}.$$
Daher ist
$$\limsup_{x=\infty} |f'(x)| \leqslant (1+\epsilon)\sqrt{2}$$
für jedes $\epsilon > 0$, also
$$\limsup_{x=\infty} |f'(x)| \leqslant \sqrt{2}.$$

BEWEIS VON SATZ 5.—Es sei ϵ gegeben und $0 < \epsilon < \sqrt{2}$. Ich werde eine bestimmte für alle reellen x definierte, zweimal differentiierbare Funktion der Periode $4\sqrt{2}$ betrachten, die ungerade ist und der Funktionalgleichung
$$f(2\sqrt{2}-x) = f(x)$$
genügt. Eine solche Funktion ist eindeutig charakterisiert, wenn ich sie auf der Strecke $0 \leqslant x \leqslant \sqrt{2}$ angebe und dabei $f'(x)$ und $f''(x)$ existieren (in 0 nach rechts, in $\sqrt{2}$ nach links), ferner
$$f(0) = 0, \quad f'_-(\sqrt{2}) = 0, \quad f''_+(0) = 0$$
ist. Denn die auf Grund der Funktionalgleichungen
$$f(x+4\sqrt{2}) = f(x), \quad f(-x) = -f(x), \quad f(2\sqrt{2}-x) = f(x)$$
überall weiter definierte Funktion $f(x)$ ist alsdann auch in den Punkten zweimal differentiierbar, die Multipla von $\sqrt{2}$ sind.

Ich setze nun für $0 \leqslant x \leqslant \sqrt{2}$
$$f(x) = \int_0^x dt \int_t^{\sqrt{2}} h(u)\,du,$$

wo $h(x)$ folgende Bedeutung hat:

$$h(x)=\begin{cases} x/\epsilon & \text{für } 0\leqslant x\leqslant \epsilon, \\ 1 & \text{für } \epsilon\leqslant x\leqslant \sqrt{2}. \end{cases}$$

$h(x)$ ist stetig, $f(x)$ also für $0\leqslant x\leqslant\sqrt{2}$ zweimal differentiierbar (in 0 nach rechts, in $\sqrt{2}$ nach links) und

$$\int_x^{\sqrt{2}} h(u)\,du=\begin{cases} f'(x) & \text{für } 0<x<\sqrt{2}, \\ f'_+(x) & \text{für } x=0, \\ f'_-(x) & \text{für } x=\sqrt{2}; \end{cases}$$

$$-h(x)=\begin{cases} f''(x) & \text{für } 0<x<\sqrt{2}, \\ f''_+(x) & \text{für } x=0, \\ f''_-(x) & \text{für } x=\sqrt{2}. \end{cases}$$

Wegen $$f(0)=0,\quad f'_-(\sqrt{2})=0,\quad f''_+(0)=0$$

ist die durch die obigen drei Funktionalgleichungen überall definierte Funktion $f(x)$ überall zweimal differentiierbar. Offenbar ist überall

$$|f(x)|\leqslant f(\sqrt{2})=\int_0^{\sqrt{2}}dt\int_t^{\sqrt{2}}h(u)\,du\leqslant\int_0^{\sqrt{2}}dt\int_t^{\sqrt{2}}du=1,$$

ferner, da $$|f''(x)|=|h(x)|\leqslant 1\quad \text{für}\quad 0\leqslant x\leqslant\sqrt{2}$$

ist, überall $$|f''(x)|\leqslant 1;$$

also ist $$\limsup_{x=\infty}|f(x)|\leqslant 1,\quad \limsup_{x=\infty}|f''(x)|\leqslant 1.$$

Andererseits ist im Punkte 0, also für alle durch $2\sqrt{2}$ teilbaren x,

$$|f'(x)|=\int_0^{\sqrt{2}}h(u)\,du\geqslant\int_\epsilon^{\sqrt{2}}du=\sqrt{2}-\epsilon,$$

folglich $$\limsup_{x=\infty}|f'(x)|\geqslant\sqrt{2}-\epsilon,$$

womit Satz 5 bewiesen ist.

Die Sätze 1 und 4 lassen sich noch dahin verschärfen, dass auf die Existenz von $f''(x)$ verzichtet wird und dafür nur die entsprechende Beschränktheitsannahme dem Differenzenquotienten von $f'(x)$ auferlegt wird. Es gelten nämlich die beiden folgenden Sätze, die offenbar die Sätze 1 resp. 4 enthalten.

SATZ 6.—*Wenn in einem Intervall der Länge* $\geqslant 2$

$$|f(x)| \leqslant 1$$

und

$$\frac{|f'(y)-f'(x)|}{y-x} \leqslant 1 \quad (x<y)$$

ist, so ist ebenda

$$|f'(x)| \leqslant 2.$$

BEWEIS.—Das Intervall sei $(0 \dots 2)$. Dann ist [da $f'(t)$ nach Voraussetzung für $0 \leqslant t \leqslant 2$ stetig ist]

$$|f(x)-f(0)-xf'(x)| = \left|\int_0^x [f'(t)-f'(x)]\,dt\right|$$

$$\leqslant \int_0^x |t-x|\,dt = \int_0^x u\,du = \tfrac{1}{2}x^2,$$

$$|f(2)-f(x)-(2-x)f'(x)| = \left|\int_x^2 [f'(t)-f'(x)]\,dt\right|$$

$$\leqslant \int_x^2 (t-x)\,dt = \int_0^{2-x} u\,du = \tfrac{1}{2}(2-x)^2,$$

etc. wie beim Beweis des Satzes 1.

SATZ 7.—*Wenn*

$$\limsup_{x=\infty} |f(x)| \leqslant 1$$

und

$$\limsup_{x=\infty} \left| \textit{obere Grenze von } \frac{|f'(y)-f'(x)|}{y-x} \textit{ für } y>x \right| \leqslant 1$$

ist, so ist

$$\limsup_{x=\infty} |f'(x)| \leqslant \sqrt{2}.$$

BEWEIS.—Wenn in einem Intervall der Länge $\geqslant 2\sqrt{2}$ sowohl

$$|g(x)| \leqslant 1$$

als auch

$$\frac{|g'(y)-g'(x)|}{y-x} \leqslant 1 \quad (x<y)$$

ist, ist im Mittelpunkte

$$|g'(x)| \leqslant \sqrt{2},$$

wie aus

$$|g(x+\sqrt{2})-g(x)-\sqrt{2}\,g'(x)|$$

$$= \left|\int_x^{x+\sqrt{2}} [g'(t)-g'(x)]\,dt\right| \leqslant \int_x^{x+\sqrt{2}} (t-x)\,dt = 1,$$

$$|\, g(x)-g(x-\sqrt{2})-\sqrt{2}\, g'(x)\,|$$
$$= \left| \int_{x-\sqrt{2}}^{x} [g'(t)-g'(x)]\, dt \right| \leqslant \int_{x-\sqrt{2}}^{x} |\, t-x\,|\, dt = 1$$

folgt. Es werde für $\epsilon > 0$

$$g(x) = f(x)/(1+\epsilon)$$

gesetzt. Dann gibt es ein $x_1(\epsilon)$ derart, dass für $x \geqslant x_1(\epsilon)$

$$|\, g(x)\,| \leqslant 1$$

und für $y > x \geqslant x_1(\epsilon)$

$$\frac{|\, g'(y)-g'(x)\,|}{y-x} \leqslant 1$$

ist. Für $x \geqslant x_1(\epsilon)+\sqrt{2} = x_2(\epsilon)$ ist also

$$|\, g'(x)\,| \leqslant \sqrt{2},$$

etc. wie beim Beweis von Satz 4.

THE ELECTROMAGNETIC FORCE ON A MOVING CHARGE IN RELATION TO THE ENERGY OF THE FIELD

By J. LARMOR.

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THE basis of the modern theory of the electric field consists in the conjugate circuital relations of Maxwell *; these relate to the electric and magnetic forces in the æther, the medium that establishes connexion between material bodies at a distance apart, through the agency of the electrons which they contain. It is a direct consequence of these equations of the field that an electron e, considered simply as a mobile pole of the electric force, continually generates, while in motion, a field both electric and magnetic, which spreads out from it with the velocity of radiation †: that in particular, while it is moving with uniform velocity v, it thus establishes and convects with it a steady magnetic field, arranged in circular lines of force around its line of motion, and with distribution of intensity following the Amperean law $evr^{-2}\sin\theta$, where e is in electromagnetic units.

In order to complete the foundation on which the theory of ordinary electric currents rests, it is usual, in purely formal discussions which do not attempt to probe into the dynamical connexions of phenomena, to add another principle—namely, that an electron e, moving with velocity v, in a magnetic field H, is subject to electro-magnetic force $e[vH]$, where $[vH]$ represents in magnitude and direction the vector product of v and H.

This principle, which determines all the mechanical phenomena of electric currents, and so dominates electrical technology, must however be, on the face of it, in intimate relation to the derived principle described above, which gives the magnetic field belonging to a moving electron. It ought, therefore, to be deducible as a consequence of the circuital relations

* Cf. J. Clerk Maxwell, *Phil. Trans.*, 1868: *Collected Papers*, Vol. II, pp. 137–143.

† Cf. for example *Phil. Mag.*, December, 1897: *Æther and Matter*, pp. 221–234.

and the value of the energy; and it would not, in a completed logical exposition, persist under the guise of an independent relation.

Its complete demonstration, by means of the formulation of the activity in the electric field under the general dynamical principle of Least Action, is a somewhat complex piece of mathematical analysis, involving abstruse points of interpretation.* It is desirable, however, that a connexion so fundamental should be amenable to elucidation in simple and direct physical terms, and that it should be viewed from various angles. The brief discussion which follows is a contribution in this direction, deducing the law of the mechanical force from consideration of a special simple system of moving charges.

We have to deal with electrons moving in a magnetic field; and the fundamental dynamical quantity is the energy concerned in the motion. If we take the field to be uniform, and unlimited in extent, the mutual part of the energy of a single electron and the field is found to involve divergent integrals. Thus, if we considered one electron by itself, we would have to introduce the origin of the field, which is, of course, in actuality not unlimited in extent, and so have to discuss the interaction between the existing magnets and the electron. This, however, would carry us back into the complexities of the general discussion. But all electric charge arises, in fact, from electric separation: and we may, therefore, rather inquire whether the infinities can be evaded by dealing with a pair of conjugate electrons, positive and negative, moving together in the uniform magnetic field.

When we proceed on this track, mathematical simplicity will be gained if, instead of two point charges, we consider the positive and negative charges uniformly spread and insulated on the faces of a flat condenser, having a thin uniform dielectric space of thickness t, and in motion with velocity v through a uniform magnetic field H. We can calculate the forces acting on this double sheet of electric charge from the principle that each element of charge δE constitutes an Amperean element of current $\delta E . v$, and is therefore acted on by a force $\delta E\,[vH]$ as above postulated. We resolve all such forces into their components transverse to the condenser and along its plane: the former components cancel as regards opposite conjugate elements of charge $+\delta E$ and $-\delta E$: the latter sum up in all to a torque or couple, in the plane containing the pole of $[vH]$ and n the normal to the condenser, and of moment equal to $Et\,[vH]\sin(vH . n)$, where $(vH . n)$ represents the angle between these directions.

* Cf. *Æther and Matter*, pp. 82 *seq.*: or *Camb. Phil. Trans.*, Stokes Commemoration volume, 1899.

Now let us examine whether the same result as is here deduced by postulating the law of magnetic force on a moving charge, can be obtained directly, in this simple case, from the energy in the field. If so, it will follow, since it remains true for a condenser of any form of contour and any thickness of dielectric, that the forcive on a pair of conjugate elements of charge $+\delta E$ and $-\delta E$ forms an elementary torque, the same as if each charge were acted on as would be the equivalent Amperean current-element $[\delta E . v]$. The essential dependence between these electromagnetic forcives sustained by moving charges and the electromotive properties of the field will then have been confirmed in a simple way. And if, further, we can obtain a like confirmation for the component, at right angles to its plane, of the mechanical force on one of the charges by itself, we shall have deduced the complete law of mechanical force from the relations of the field.

Now the moving charges of the condenser will produce a magnetic field, here to be determined, which is superposed on the uniform field H. The Amperean element $\delta E . v$ can be replaced by its components, $\delta E . v \cos\theta$ normal to the condenser and $\delta E . v \sin\theta$ in its plane, where θ is the angle vn. In the steady motion of the system the magnetic field of the former is annulled by that arising from the opposite negative element $-\delta E$. The magnetic fields of the latter components aggregate, for all the elements of the system, to a field of uniform intensity $4\pi\sigma v \sin\theta$ in a direction along the dielectric and at right angles to v, where σ is the surface density of charge; while they give *no field whatever* outside the plates of the condenser.

Generally, if a magnetic field $(\alpha_0, \beta_0, \gamma_0)$ is superposed on an existing field (α, β, γ), the kinetic energy $(8\pi)^{-1}\int(\alpha^2+\beta^2+\gamma^2)\,d\tau$ is increased by

$$\frac{1}{4\pi}\int(\alpha\alpha_0+\beta\beta_0+\gamma\gamma_0)\,d\tau+\frac{1}{8\pi}\int(\alpha_0^2+\beta_0^2+\gamma_0^2)\,d\tau.$$

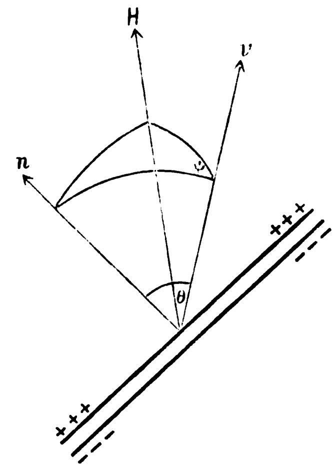
In the present case the superposed field exists only between the plates, and there

$$\alpha\alpha_0+\beta\beta_0+\gamma\gamma_0 = H . 4\pi\sigma v \sin\theta \cos(H . vn),$$

$$\alpha_0^2+\beta_0^2+\gamma_0^2 = (4\pi\sigma v \sin\theta)^2.$$

Thus the mutual or interacting term on the magnetic energy is, integrated over the volume of dielectric,

$$H . Evt \sin\theta \cos(H . vn).$$



Now on the spherical projection in the diagram

$$\cos(H.vn) = \sin(Hv)\sin\psi,$$

and $\quad \sin\theta\sin\psi = \cos\chi,$

where ψ is $(vH.vn)$, and χ is $(n.vH)$, viz., the inclination of n, the normal to the condenser, to the fixed direction of the pole of the vector $[vH]$.

The mutual term in the magnetic energy is therefore

$$H.Evt\sin(Hv)\cos\chi.$$

On varying χ, viz., the angle $(Hv.n)$, by rotation of the condenser, this local store of energy is altered; and if this is done not too fast, it might be thought that there would be no gain or loss of energy by radiation or otherwise. Then the increment of energy would represent work that must be put into the system consisting of the charged condenser and its surroundings, by an extraneous torque assisting the variation of χ: the reacting forcive exerted by the charged condenser itself on external bodies is the opposed torque, whose moment in the direction of χ increasing would therefore be

$$H.Evt\sin(Hv)\sin\chi,$$

measured in the direction of χ increasing. This is the same as the value $Et[vH]\cos(vH.n)$ already obtained: but it has the opposite sign,* so the hypothesis of conservation of the energy is not borne out.

[The momentum along v is here T/v, which is $H.Et\sin(nH)\cos(v.nH)$, showing that the resultant momentum in the system, of which this is a component, is $H.Et\sin(nH)$ directed towards the pole of (nH). This momentum arises from the interaction of the uniform magnetic field H with the uniform electric field of the charged condenser. It is thus to be regarded as uniformly distributed throughout the dielectric space, and so of density equal to the vector product of magnetic force and æthereal displacement—in accordance with the well known formula: there are in addition the intrinsic momenta of the individual electrons of the charge. This derivation of it is exact when v is diminished indefinitely, *i.e.*, in all

* I am indebted to Dr. Bromwich for pointing out this mistake in the original draft of the paper, and its source in the kinetic character of the energy.

ordinary systems: the modification required for exceedingly rapid convection does not now concern us.

What is the source of the energy that is expended by the system when the convected condenser does work on external bodies? As the working force exerted by the system is $+\partial T/\partial \phi$, it appears at first sight to gain in the end energy of the same amount $+\delta T$ as it has expended in work, which would involve a supply $2\delta T$ from outside. This could only come from some of the energy of the extraneous field being transmitted by radiation across the intervening region, during the time that the condenser is undergoing change of orientation. But this change can be as slow as we please; and, as the rate of receipt of radiation must be proportional to the square of the rate of change, while the duration of receipt is inversely as that rate, the total receipt can be diminished indefinitely. Thus the conservation of energy cannot be secured in this way; though conservation of momentum can thus be effected.

We therefore revert to the general theory of the kinetics of a system, made up of two parts, which exert interaction expressed by mixed terms in the energy, say of type

$$T = \ldots + L v_1 v_2 + \ldots,$$

where L depends on the coordinates of position, of type ϕ, of the two parts of the system. Let us suppose that the second part is maintained steady in its coordinates and velocities: it will however suffice if its field of influence is maintained steady throughout the region in which the other part is situated, which covers the case of the magnets producing the extraneous field of our present system. The first part of the system then exerts on external bodies a force $-v_2 . dL/dt$ in the direction which works on v_1. But it also exerts in the direction which works on ϕ a force $v_1 v_2 \partial L/\partial \phi$. The total rate of working of this part on external bodies is thus

$$-v_2 \frac{dL}{dt} v_1 + \Sigma v_1 v_2 \frac{\partial L}{\partial \phi} \dot{\phi},$$

which is

$$-v_1 v_2 \frac{\partial L}{\partial t}.$$

Thus when L does not contain the time explicitly, *i.e.*, when the structure of the first part of the system is not changing with the time, and the field arising from the second part is steady, there is no loss of energy from the first part of the system to external bodies, the work expended through one

coordinate being recovered through the others. Interesting hydrodynamic examples can readily be stated.

In the present case, when the condenser system acquires energy through work gained in rotating itself, this gain is merely a transfer from the energy of convection: for that convection is opposed by a force, extremely small, which appears as the rate of loss of intrinsic momentum of the portion of the system under consideration.]

The torque on the condenser vanishes when its plane is parallel to H and v. Under the continued free operation of this forcive the condenser ought, in fact, finally to assume the position in which it cannot increase the local magnetic energy further, *i.e.*, the position in which it is maximum, which is that in which the field due to the moving charges is in the same direction as the extraneous field and as great as possible. This agrees with the result above stated.

This calculation of the torque from the energy of the field confirms, but does not prove, the expression for the force on a moving electron. We can now go further. We may displace one face of the moving condenser away from the other. Variation of the energy with regard to t should thus give the component, in the direction of t, of the force acting on the moving charge δE. The value thus obtained is easily seen to agree with the Amperean formula. As then the torque for the pair $+\delta E$, $-\delta E$ agrees, and also one component of the force on a single charge, it is easy to see that the formula for the mechanical force on a charge moving in a magnetic field is derivable completely from the electromotive relations of the field and its energy.

The intrinsic electrokinetic energy of the moving charged condenser is the remaining term, viz., $(8\pi)^{-1}(4\pi\sigma v \sin\theta)^2$ per unit volume of the dielectric, or in all $2\pi E\sigma v^2 t \sin^2\theta$, where θ is the inclination of the normal to the plane of the condenser to the direction of its motion. Thus the condenser sustains a torque $2\pi E\sigma v^2 t \sin 2\theta$ tending to set it transverse to that direction. With these small quantities we are, however, in the debatable domain of the principle of relativity. It can be shown* that by means of such second order forcives the energy of the absolute motion of the Earth through æther could be drawn upon for terrestrial work, unless motion of bodies through the æther is accompanied by the FitzGerald-

* Cf. Note appended in *Scientific Writings of G. F. FitzGerald* (Dublin, 1902), pp. 566–8.

Lorentz shrinkage, which would annul, up to the second order inclusive, both this and all other methods of testing the presence of absolute motion through the æther by operations within the system.

The forces above considered arise from the motional, or electromagnetic, energy of the system. Are there also forces of electrostatic origin? The electrostatic field of a uniformly convected system is not altered, up to the first order (cf. *Æther and Matter*, p. 154), the electric force of the field being defined relative to the moving charges; thus the electric potentials exist as before, and there are no new first-order forces of this type.

[Reference has just been made to annulments in a convected system by adjustment up to the second order. We have, however, obtained in this discussion a force of the first order: thus, as the impressed magnetic field may be itself due to magnets forming part of the convected system, a contradiction is suggested. The explanation is to remember that a moving magnet produces, on Amperean principles, a field of æthereal force, measurable as force exerted on stationary ions: this gives a force equal and opposite at each point to the force experienced by an ion on account of its motion with the system, so that, on the whole, a convected ion is undisturbed. Thus, if the magnets producing the impressed field are moving with the system, then, even though that field be absolutely uniform, the state of motion of its source is revealed by the presence of an æthereal force which would disturb a fixed ion, but for a moving ion convected with the magnets would just annul the other force on it arising from its motion. Further elucidation as to how this comes about would be interesting, but not now relevant.]

ON AN ASSUMPTION CONTAINED IN EUCLID'S PROOFS OF CERTAIN PROPOSITIONS IN HIS TWELFTH BOOK

By J. ROSE-INNES.

[Received March 11th, 1913.—Read April 24th, 1913.]

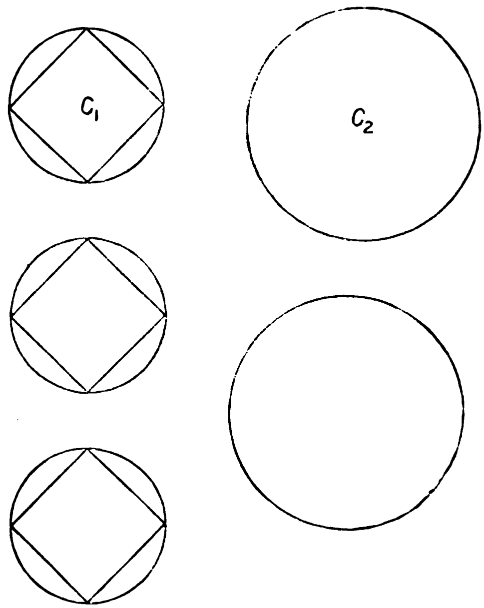
IT is well known that Euclid in his proof of V, 18 made the assumption that, given any three quantities—the first two being of the same kind—there always exists a fourth proportional to them. Simson, who objected to the proof as it stands, devised another free from the supposed blemish. The assumption reappears in Euclid in five propositions of his Twelfth Book, viz., XII, 2, 5, 11, 12, 18, and Simson did not give any proof of these theorems removing the difficulty; nor has he put forward any grounds which entitle him to condone the want of rigour in the Twelfth Book while condemning it in the Fifth. As far as I know, none of the subsequent commentators has cleared up the matter. It seems interesting, therefore, to enquire to what extent the above assumption is really necessary, and to examine how Euclid could have managed to do without it, had he entertained any objection to it, while adhering in other respects to his own methods and his own definitions. In other words, we might wish to know whether it is possible to make for the five theorems of the Twelfth Book the same kind of amendment that Simson effected in V, 18.

After a careful examination as to the best way of removing the difficulty, I am led to believe that in the case of XII, 2 we can proceed most readily by making use of Stolz's theorem. This theorem tells us that in order to satisfy ourselves that four quantities, say A, B, C, D are in proportion, we need only show that $pA \gtrless qB$, according as $pC \gtrless qD$; where pA, pC are any equimultiples of A and C, and qB, qD are any independent equimultiples of B and D. The remaining test contemplated by Euclid as to $pA = qB$, when $pC = qD$, need not be applied. (See Stolz, *Vorlesungen über allgemeine Arithmetik*, Pt. 1, p. 87, published in 1885.)

The proof of XII, 2 on these lines will now be given.

To show that any two circles have the same ratio as the squares on their diameters.

Let C_1 and C_2 be the two circles; take n_1C_1, n_2C_2 multiples of C_1 and C_2 respectively, and first suppose that $n_1C_1 > n_2C_2$.



Inscribe a square in each of the circles equal to C_1; then just as in Euclid we may show that each square is greater than the half of the circle in which it is inscribed; and that on substituting for each square an inscribed regular polygon of double the number of sides we shall be removing from the segments left over from the squares more than their half. (See Euclid's proof; Heath, Vol. III, pp. 371, 372.)

Thus, by starting with the inscribed squares and continually substituting inscribed regular polygons of double the number of sides, we shall at length leave some segments of the circles which will be together less than the excess by which n_1C_1 exceeds n_2C_2 (X, 1).

Let P_1 denote the inscribed polygon when this stage has been reached: then

$$n_1C_1 - n_1P_1 < n_1C_1 - n_2C_2,$$

so that

$$n_1P_1 > n_2C_2.$$

Let P_2 denote a polygon similar to P_1 but inscribed in C_2; then

$$n_2C_2 > n_2P_2.$$

Hence

$$n_1P_1 > n_2P_2.$$

But similar polygons inscribed in circles are to one another as the squares on the diameters (XII, 1). Therefore

$$n_1 S_1 > n_2 S_2,$$

where S_1, S_2 are the squares on the diameters of C_1, C_2 respectively (V, Def. 5).

Secondly, if $n_1 C_1 < n_2 C_2$, we may show in a similar manner that

$$n_1 S_1 < n_2 S_2.$$

By Stolz's theorem these two cases are sufficient to prove that C_1, C_2, S_1, S_2 are proportional.

An objection might be raised to the above proof, viz., that Stolz's theorem does not actually occur in Euclid, and that consequently we have no right to use the theorem in a proof conducted professedly on Euclidean lines. To this we might urge in reply that though Stolz's theorem is not actually included in Euclid's Fifth Book, it can easily be proved by Euclidean methods (see Hill's *Euclid*, 2nd ed., p. 29); and we may therefore imagine that it has been proved and added to the Fifth Book just as Simson, for example, added his propositions A—K.

The remaining propositions referred to above, viz., XII, 5, 11, 12, 18, can all be proved by Stolz's theorem, without making use of Euclid's assumption, on much the same lines as XII, 2 in the proof just given. It seems worth mentioning that for three of the propositions, XII, 5, 11, 12, we can easily devise a proof by ordinary geometrical manipulation, which shall avoid Euclid's assumption, and yet be independent of Stolz's theorem. For XII, 12, though not for XII, 5 and 11, the alternative proof is decidedly the simpler.

THE DIOPHANTINE EQUATION $y^2-k=x^3$.

By L. J. MORDELL.

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1. This equation was brought into prominence by Fermat,* who had proposed as a problem to the English mathematicians, to shew that there was only one integral solution of the equation $y^2+2=x^3$. Concerning this he † says: "Peut on trouver en nombres entiers un carré autre que 25, qui, augmenté de 2, fasse un cube? A la première vue cela paraît d'une recherche difficile, en fractions une infinité de nombres se déduisent de la méthode de Bachet; mais la doctrine des nombres entiers, qui est assurément très-belle et très-subtile, n'a été cultivée ni par Bachet, ni par aucun autre dont les écrits venus jusqu'à moi." He did not publish his method, which is not known at present.

We shall consider the equation from three points of view. Firstly, we shall find general formulæ for k, for which there are no solutions (we consider integral values only of the unknowns throughout our paper); secondly, we shall apply ideal numbers; and, finally, we shall make use of the arithmetical theory of the binary cubic.

In a series of notes and papers published by Lebesgue,‡ Gerono,§ Jonquières,|| Realis,¶ and Pepin,** †† various values and formulæ have been given for k for which our equation is insoluble. These results can be considerably extended. Moreover, the same method supplies us with a very useful tentative method for solving such equations.‡‡

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* Ball, *Mathematical Recreations*, p. 40.

† Brassinne's *Précis*, p. 122, or Fermat's *Diophantus*, Bk. VI, Prop. 19, p. 320.

‡ *Nouvelles Annales de Mathématiques*, 1st series, Vol. 9, 1850; 2nd series, Vol. 8, 1869.

§ *Ibid.*, 2nd series, Vol. 8, 1869; Vol. 9, 1870; Vol. 10, 1871; Vol. 16, 1877.

|| *Ibid.*, Vol. 17, 1878.

¶ *Ibid.*, Vol. 22, 1883.

** Liouville, *Journal de Math. pures et appliquées*, 3rd series, Vol. 1, 1875.

†† *Annales de la Société Scientifique de Bruxelles*, 1882, Pt. 2.

‡‡ Cf. Cunningham, *Educational Times Reprints*, Vol. 13, Question 15697, and Vol. 14, Question 16408.

2. A few preliminary considerations are necessary. It is well known that if p is any odd prime factor of x^2-ky^2, then $x^2 \equiv ky^2 \bmod p$; so that, if p is prime to ky, $(k/p)=1$. Thus, by the use of the law of quadratic reciprocity and the supplementary laws, we find that p is congruent to certain residues to mod k or mod $4k$. And any prime q such that $(k/q)=-1$ cannot be a divisor of x^2-ky^2 unless it divides both x and y. Hence any odd number t such that $(k/t)=-1$ cannot be a divisor of x^2-ky^2 unless all the prime factors q of t, for which $(k/q)=-1$, divide both x and y.

Consider now the equation

$$y^2-klb^2 = x^3-k^3a^3$$
$$= (x-ka)\,N, \tag{1}$$

where k has no square factors and is prime to bl. Moreover, $N=x^2+kax+k^2a^2$ is essentially positive, and a will be hereafter so chosen that N is odd.

Then $$y^2 \equiv klb^2 \bmod N,$$

and so $(kl/N)=1$ if N is prime to klb^2. Now N is prime to k if x is so, and from (1) we see that this is the case, since k is prime to bl. As to N being prime to $l\,(\neq \pm 1)$, this is best postponed to the stage when particular values of l are considered; but this will always be the case.

If now N is prime to b, then, since $(N/k)=1$, we find

$$(l/N) = (k/N)(N/k)$$
$$= (-1)^{\frac{1}{4}(k-1)(N-1)}, \tag{2}$$

if k is positive, and it is also true if k is negative. If, therefore, values of a and b can be chosen such that for given k and l, (2) is untrue, it will follow that (1) is insoluble. In particular, when $l=1$, (1) is insoluble, if $N \equiv 3 \bmod 4$ and $k \equiv 3 \bmod 4$. If, further, $k=-1$, (2) is replaced by $(-1/N)=1$; and here again (1) is insoluble if $N \equiv 3 \bmod 4$.

Now suppose (2) is untrue, *i.e.* $(kl/N)=-1$, then from (1), since $y^2-klb^2 \equiv 0 \bmod N$, it follows that b and N have a common prime factor q such that $(kl/q)=-1$. Putting $y=qy_1$, $b=qb_1$, we have

$$y_1^2-klb_1^2 = (x-ka)\,N/q^2. \tag{3}$$

But, since $N=x^2+kax+k^2a^2 \equiv 0 \bmod q$,

$$(x-ka)^2+3kax \equiv 0 \bmod q.$$

Hence $x-ka$ is prime to q if $3kax$ is so. Now a is prime to q if a and b have no common factors of the type q; 3 is prime to q if $b \not\equiv 0 \bmod 3$ when $(kl/3) = -1$; k is prime to q, since b and k are prime to each other. Moreover, x is prime to q, since ka is so. Hence $x-ka$ is prime to q, and hence $N \equiv 0 \bmod q^2$. But, putting $N = M_1 q^2$, we see that $(kl/M_1) = -1$.

Continuing this process, we can remove all the factors of b typified by q, and finally arrive at an equation of the form

$$Y^2 - klB^2 = (x-ka)M,$$

where $(kl/M) = -1$; but B and M have no common factors q such that $(kl/q) = -1$. Hence the equation is impossible. But $(kl/N) = (kl/M)$. Hence, if (2) is untrue the equation (1) is insoluble if a and b have no common prime factors q such that $(kl/q) = -1$, and $b \not\equiv 0 \bmod 3$ if $(kl/3) = -1$. We may note that the equation (1) is still insoluble even if a and b possess common factors of the type q, provided the indices of these factors satisfy certain conditions which are easily found in any particular case.

Let us now consider (1) when $l = 1$, so that our equation becomes

$$y^2 - kb^2 = x^3 - k^3a^3. \tag{3a}$$

We suppose $k \equiv 3 \bmod 4$, free from square factors and prime to b. Also a and b have no common prime factors q for which $(k/q) = -1$, and $b \not\equiv 0 \bmod 3$ if $(k/3) = -1$. Hence (3a) is insoluble if a and b are such that $N \equiv 3 \bmod 4$.

We now solve the congruence

$$x^2 + kax + k^2a^2 \equiv 3 \bmod 4 \text{ (or since } k \equiv 3 \bmod 4),$$

$$x^2 - \ ax + \ a^2 \equiv 3 \bmod 4.$$

$$\left.\begin{array}{lll} \text{Hence, if } & a \equiv 1, \text{ then} & x \equiv -1, 2 \\ \quad ,, & a \equiv 2, \quad ,, & x \equiv -1, 1 \\ \quad ,, & a \equiv 3, \quad ,, & x \equiv \ \ \ 1, 2 \end{array}\right\}. \tag{4}$$

If we now take any one of these values of a mod 4 and can find values of b such that x satisfying (3) must be congruent to one or both of the corresponding residues mod 4 given in (4), then (3) is insoluble for these values of a and b.

But $$y^2 \equiv x^3 + a^3 - b^2 \bmod 4,$$

and if we take $a \equiv -1 \bmod 4$ and $b \equiv 0 \bmod 2$, then $x \equiv 1 \bmod 4$. Hence we have the first insoluble equation

$$y^2 = x^3 - k^3(4a-1)^3 + 4kb^2.$$

Take $a \equiv 2 \bmod 4$ and $b \equiv 1 \bmod 2$, then $x \equiv 1 \bmod 4$, and hence the insoluble equation

$$y^2 = x^3 - k^3(4a+2)^3 + k(2b+1)^2.$$

Take $a \equiv 2 \bmod 4$ and $b \equiv 0 \bmod 2$, then $x \equiv 1 \bmod 4$ or even; so if we can find values of a and b satisfying these congruences, and also such that x cannot be even,* then (3) is insoluble.

Now when $A \equiv 4 \bmod 8$, and

$$y^2 = x^3 + A$$

admits of even values for x, we have $y \equiv 2 \bmod 4$, whence

$$A \equiv 12 \bmod 16 \text{ or } \equiv 4 \bmod 32,$$

i.e.

$$A \not\equiv -12 \bmod 32.$$

Hence the values of a and b needed are given by

$$-k^3a^3 + kb^2 \equiv -12 \bmod 32,$$

from which we obtain

$$a \equiv -k-3 \bmod 8 \text{ and } b \equiv 2 \bmod 4,$$

and hence the insoluble equation

$$y^2 = x^3 - k^3(5+8c-k)^3 + 4k(2b+1)^2.$$

Finally, taking $a \equiv 1 \bmod 4$ and $b \equiv 0 \bmod 2$, then $x \equiv 3 \bmod 4$ or even. The equation (3) becomes, writing $4a+1$ for a and $2b$ for b,

$$y^2 = x^3 - k^3(4a+1)^3 + 4kb^2 = x^3 + A,$$

say, and x cannot be even if $A \equiv 5 \bmod 8$. This gives us

$$4a+1 \equiv 4b^2 + 3k \bmod 8,$$

and we have the insoluble equation

$$y^2 = x^3 - k^3(4b^2+3k+8c)^3 + 4kb^2,$$

* We shall have frequent occasion to use equations of the form $y^2 = x^3 + k$, which do not admit of even values for x. This is the case when $k \equiv 5 \bmod 8$, $k \equiv -12 \bmod 32$ and also $k \equiv -16$ or $32 \bmod 64$.

which is equivalent to the two equations

$$y^2 = x^3 - k^3(3k+8c)^3 + 16kb^2,$$

$$y^2 = x^3 - k^3(4+3k+8c)^3 + 4k(2b+1)^2.$$

In particular, by taking $k = -1$, we find some of the known insoluble equations $y^2 = x^3 + A^3 - B^2$, say, and our conditions become $B \not\equiv 0 \bmod 3$, and A and B have no common prime factor congruent to 3 mod 4. We thus find the insoluble equations $y^2 + k = x^3$ where $k = 9, 3, 12, 43, 91, 99$. and $-k = 95, 47, 39, 11, 67, 53, 13, 20$.

3. Taking now $l = 2$ in equation (1), and following the same procedure as before, we find the insoluble equations

$$y^2 = x^3 - k^3a^3 + 2kb^2,$$

when (1) $a \equiv 2, 4 \bmod 8$, $b \equiv 1 \bmod 2$,

(2) $a \equiv 4 \bmod 8$, $b \equiv 4 \bmod 8$,

(3) $a \equiv 2 + 4(-1)^{\frac{1}{2}(k-1)} \bmod 16$, $b \equiv 2 \bmod 4$,

and k is odd, free from square factors and prime to b. Also $b \not\equiv 0 \bmod 3$, when $(k/3) = 1$, and a and b have no common prime factor q for which $(2k/q) = -1$. As particular cases, when $k = \pm 1$, we find the insoluble equations $y^2 + k = x^3$, where $k = 62, 98$, and $-k = 62, 46, 32, 66, 90, 96$.

When $l = 3$, we find the insoluble equations

$$y^2 = x^3 - k^3a^3 + 3kb^2,$$

when (1) $a \equiv 1 \bmod 4$, $b \equiv 0 \bmod 2$,

(2) $a \equiv 2 \bmod 4$, $b \equiv 1 \bmod 2$,

(3) $a \equiv 2(-1)^{\frac{1}{4}(k+3)} \bmod 8$, $b \equiv 2 \bmod 4$,

(4) $a \equiv 5k - 2(-1)^{\frac{1}{2}b} \bmod 8$, $b \equiv 0 \bmod 2$,

where $k \equiv 1 \bmod 4$, free from square factors and prime to $3b$. Also a and b have no common odd prime factor to q such that $(3k/q) = -1$, and b is prime to 3. When $k = 1$, we find the insoluble equations $y^2 - k = x^3$, $k = 75, 84$.

Finally, when $l = 6$, we find the insoluble equations

$$y^2 = x^3 - k^3a^3 + 6kb^2.$$

when (1) $a \equiv -2, 4 \mod 8$, $b \equiv 1 \mod 2$,

(2) $a \equiv 4 \mod 8$, $b \equiv 4 \mod 8$,

(3) $a \equiv 6+4(-1)^{\frac{1}{2}(k-1)} \mod 16$, $b \equiv 2 \mod 4$,

where k is an odd number possessing no square factors, and k and b are prime to each other and to 3. Also a and b have no common prime factor q, such that $(6k/q) = -1$.

4. These results can be immediately extended to equations of higher degrees. Thus we have the insoluble equation

$$y^2 = x^{2n+1}-k^{2n+1}a^{2n+1}+kb^2,$$

where $a \equiv 2 \mod 4$ and $b \equiv 1 \mod 2$, and k satisfies the conditions of equation (3a). Further $2n+1$, a possible common factor of $x-ka$, and $(x^{2n+1}-k^{2n+1}a^{2n+1})/(x-ka)$, plays the same part in the remaining conditions that 3 does.

Another insoluble equation is

$$y^2 = x^{4n+3}-k^{4n+3}a^{4n+3}+kb^2,$$

where $a \equiv -1 \mod 4$, $b \equiv 0 \mod 2$, and with similar conditions to those above. The proof of these equations is very simple. We give no more of them as it is simply rewriting our previous results with slight changes. We may, however, notice the insoluble equations

$$y^2 = x^{4n+3}+k,$$

when (1), $k = -1-4b^2$ and b and $4n+3$ have no common prime factors congruent to 3 mod 4; (2) $k = -1+12b^2$, b is prime to 3 and b and $4n+3$ have no common prime factors congruent to ± 5 mod 12; and finally when $k = 1+12(2b+1)^2$ and $2b+1$ is prime to 3, and $2b+1$ and $4n+3$ have no common prime factors congruent to ± 5 mod 12.

5. The preceding impossible equations which we have given are very simple. We can obtain more complicated ones as follows. Suppose

$$y^2 = x^3+8k+5.$$

Here $x \equiv -1 \mod 4$, *i.e.*, $x \equiv -1, 3 \mod 8$.

If this equation can be written in the two ways

$$y^2-2b^2 = (x+a)(x^2-ax+a^2),\quad \text{where } a \equiv 3 \mod 8,\quad b \equiv 1 \mod 2,$$

and

$$y^2-8d^2 = (x+c)(x^2-cx+c^2),\quad ,,\quad c \equiv -3 \mod 8,\ d \equiv 1 \mod 2,$$

where our usual conditions are satisfied by a, b and c, d, we see that $x \equiv -1 \bmod 8$ is excluded by the first form of the equation, while $x \equiv 3 \bmod 8$ is excluded by the second form. Hence the equation is insoluble.

To find values for k, we have

$$b^2 - 4d^2 = \tfrac{1}{2}(c^3 - a^3).$$

We can easily find an indefinite number of solutions of this equation. We easily find $a \equiv c \bmod 2$, and take any values for a and c consistent with this condition. We then split $\tfrac{1}{2}(c^3-a^3)$ into two factors p and q say, and take

$$b + 2d = p,$$

$$b - 2d = q.$$

In particular we may take

$$2p = c - a \quad \text{and} \quad q = c^2 + ac + a^2,$$

or, again,
$$2p = c^3 - a^3 \quad \text{and} \quad q = 1.$$

Thus we find insoluble equations, provided that the value we find for b, viz., $\tfrac{1}{2}(p+q)$ and the assumed value of a; and likewise d and c satisfy our usual conditions.

As a particular case of this equation consider

$$y^2 = x^3 + 45.$$

We note that x is prime to 3, and throwing the equation in the two forms

$$y^2 - 18 = (x+3)(x^2 - 3x + 9),$$

$$y^2 - 72 = (x-3)(x^2 + 3x + 9),$$

we see that when $x \equiv -1 \bmod 8$, $x^2 - 3x + 9 \equiv 5 \bmod 8$, and prime to 3. This excludes $x \equiv -1 \bmod 8$. Similarly, when $x \equiv 3 \bmod 8$, from the second form of the equation. Hence it is insoluble.

By similar methods we can shew $y^2 = x^3 + k$ is insoluble for

$$k = -24,\ 29,\ -36,\ 51,\ 59,\ 85,\ \pm 88,\ -92,\ 93.$$

Many of these can also be proved by the methods introduced in the latter part of the paper.

6. We can now give a tentative method for finding solutions of

$$y^2 = x^3 + k.$$

We find $x \equiv p, q, r, \ldots \bmod 8$. If the equation can be written in any of the forms

$$y^2 = x^3 + a^3 - b^2 \quad \text{or} \quad y^2 = x^3 + a^3 \pm 2b^2,$$

we may be able to exclude $x \equiv p, q, \ldots \bmod 8$, and are left with $x \equiv s, \ldots \bmod 8$.

We now carry out the same process with other moduli, say 3, 5, 7, &c., giving us $x \equiv t, \ldots \bmod 3$, &c. We then gather our results together and find $x \equiv A, B, \ldots \bmod N$, where the density of the values of x is considerably diminished. We now test the values of x as follows. We take another modulus M say, find the residues of $x \bmod M$ and reject those for which $x^3 + k$ is a non-quadratic residue of M. We may also find additional information, if the equation can be written in the form

$$y^2 \pm 2, 3, 5, \ldots Mb^2 = x^3 + a^3.$$

Finally, if the necessary conditions are satisfied for many moduli M, we test the value of x by actual substitution. When $k = -31$, we find that there are no solutions with $x < 10^9$.

7. We shall now pass on to other methods, and give the first direct method for determining in many cases, sufficient conditions for the insolubility of our equation. We shall also shew the existence of new classes of equations of our type admitting a limited number only of solutions.

It would be interesting to know, if the method given below was that used by Fermat for his equation $y^2 + 2 = x^3$. He knew, it is thought, that all factors of numbers of the form $a^2 + 2b^2$ are of the same form, but further proof is required before one can say that the complete solution of his equation is given by

$$x = a^2 + 2b^2, \quad y + \sqrt{(-2)} = [a + b\sqrt{(-2)}]^3.$$

As a matter of fact, Euler* himself has fallen into error on this point, with a similar equation. It is very curious that the application of ideal numbers has been overlooked, especially as Pepin's† paper is chiefly concerned with the equation

$$y^2 + kz^2 = x^m,$$

and Dirichlet‡ had considered similar equations.

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* *Algebra*, Pt. 2, Chap. 12.

† Liouville, 1875.

‡ *Collected Works*, "De Quelques Equations du cinquième degré," Vol. 1, p. 31.

8. We suppose k negative, free from square factors, and congruent to 2, 3, mod 4, so that x is prime to $2k$. Further let h, the number of classes of ideal numbers of determinant k, which is here the number of properly primitive classes of determinant k, be not divisible by 3. Then the equation has no solutions, unless $-k = 3a^2 \pm 1$, when it has but one.

Since $y^2-k = x^3$, writing $\theta = \sqrt{k}$, we have

$$(y+\theta)(y-\theta) = x^3.$$

But any common factor of $y+\theta$ and $y-\theta$ is a factor of 2θ, and since x is prime to 2θ, it follows that any prime ideal factor of x cannot be a factor both of $y+\theta$ and $y-\theta$; consequently its cube must be a factor either of $y+\theta$ or of $y-\theta$. And since the only units in the domain of θ are ± 1, and $\pm i$ when $k=-1$, and as -1 and $\pm i$ can be absorbed in T_1^3, we obtain

$$y+\theta = T_1^3,$$

$$y-\theta = T_2^3,$$

where T_1 and T_2 are ideal numbers in the domain of θ. But since $h \not\equiv 0$ mod 3, T_1 and T_2 are primary numbers, and we can write

$$T_1 = a+b\theta, \quad T_2 = a-b\theta,$$

and so $$y+\theta = (a+b\theta)^3, \quad \text{and} \quad x = a^2-kb^2.$$

Therefore $$1 = b(3a^2+kb^2),$$

whence $$b = \mp 1,$$

and $$-k = 3a^2 \pm 1,$$

also $$x = 4a^2 \pm 1.$$

As illustrations,

for $k = -2$, the only solution is $x = 3$, $y = 5$,

,, $k = -13$, ,, ,, $x = 17$, $y = 70$,

,, $k = -74$, ,, ,, $x = 99$, $y = 985$,

while there are none for $-k = 5$, 6, 10, 14, 17, 21, 22, 30, 33, 34, 37, 41, 42, 46, 57, 58, 65, 66, 69, 70, 73, 77, 78, 82, 85, 86, 90, 93, 94, 97.

Similar results hold when $k \equiv 5$ mod 8, $k \neq -3$, for which h is equal to the number of improperly primitive classes of determinant k (or $\frac{1}{3}$ the number of properly primitive classes of determinant k). Also when $k \equiv 1$ mod 8, for which h is equal to the number of properly primitive

classes of determinant k. But now there is great difficulty in dealing with the case of x being even, though our method applies to the odd values of x.

We have $$y+\theta = (a+b\phi)^3,$$
where $$2\phi = -1+\sqrt{k},$$
and $$x = a^2-ab+\tfrac{1}{4}(1-k)\,b^2.$$
Therefore $$8 = b\,[3(2a-b)^2+kb^2].$$
Hence either $b = \mp 1$ and $-k = 3(2a\pm 1)^2 \pm 8$,

or $b = \mp 2$ and $-k = 3(a\pm 1)^2 \pm 1$.

Thus the equation is insoluble for
$$-k = 43,\ 51,\ 91.$$
Also $-k = 11$ gives only $x = 3$, $y = 4$, and $x = 15$, $y = 58$,

$-k = 19$ „ $x = 7$, $y = 18$,

$-k = 35$ „ $x = 11$, $y = 36$,

$-k = 67$ „ $x = 23$, $y = 110$.

These results can be extended to equations of the form
$$y^2-kf^2 = x^3,$$
where f is such that x is prime to $2kf$, and k satisfies the previous conditions. Thus when $k \equiv 2, 3 \bmod 4$, we must have
$$f = b(3a^2+kb^2),$$
$$x = a^2-kb^2,$$
and when $k \equiv 1, 5 \bmod 8$, $k \neq -3$,
$$8f = b\,[3\,(2a-b)^2+kb^2],$$
$$x = a^2-ab+\tfrac{1}{4}(1-k)\,b^2,$$
and the equation either has no solutions or only a limited number.

In particular, when $f = 4$, and $k \equiv 2, 3 \bmod 4$, x is prime to $2k$, whence $4 = b(3a^2+kb^2)$, which is easily seen to require $b = \mp 1$ and $-k = 3a^2 \pm 4$. Thus we have no solutions of $y^2+16k = x^3$ for $k = 1, 2, 5, 6$.

Or, again, when $k \equiv 5 \bmod 8$, and $f = 2$, x is prime to $2k$, hence
$$16 = b\,[3\,(2a-b)^2+kb^2],$$

hence, if $b = \mp 1$, $\quad -k = 3(2a \pm 1)^2 \pm 16$,

$b = \mp 2$, $\quad -k = 3(a \pm 1)^2 \pm 2$,

$b = -4$, $\quad -4k = 3(a+2)^2+1$.

Thus, for
$$y^2+4k = x^3,$$

$k = 11$ gives only $x = 5$, $y = 9$,

$k = 19$ „ $x = 5$, $y = 7$, and $x = 101$, $y = 1015$.

Other illustrations are given by
$$y^2+52 = x^3 \quad \text{and} \quad y^2+68 = x^3.$$

For, if x is even, $x \equiv 2 \bmod 4$, and this value is excluded by putting the equations under the forms
$$y^2+25 = (x-3)(x^2+3x+9) \quad \text{and} \quad y^2+4 = (x-4)(x^2+4x+16),$$
and noting that
$$x-3 \equiv 3 \bmod 4,$$
and that
$$x^2+4x+16 \equiv 12 \bmod 16.$$
Further, h for 13 or 17, $\not\equiv 0 \bmod 3$, and $2 = b(a^2+kb^2)$ for $-k = 13, 17$ has no solutions, and hence the equations have none.

9. The simplicity of these results is due to the fact that the only units are ± 1, except for the determinants -1, where no inconvenience is caused, and -3 which was excluded from the discussion. We have, however, interesting results when the units must be taken into account.

Thus, let $y^2-kf^2 = x^3$, where $k \equiv 2, 3 \bmod 4$, free from square factors and positive, $h \not\equiv 0 \bmod 3$, and f is such that x is prime to $2kf$ (*e.g.*, if $f = 1$). Also let the unit for which U* has its least non-zero value be given by $T^2-kU^2 = 1$. We easily find
$$x = a^2-kb^2 \quad \text{and} \quad y+f\sqrt{k} = (a+b\sqrt{k})^3,$$
which will give none, or a finite number of values for x; or
$$y+f\sqrt{k} = (T+U\sqrt{k})(a+b\sqrt{k})^3.$$
This is fairly obvious if $T+U\sqrt{k}$ is the fundamental unit ϵ. If it is not,

* We refer to this as the first solution, and similarly for the equation $T^2-kU^2 = 4$. Solutions for the first equation may be found in the tables at end of Vol. 1 of Legendre's *Théorie des Nombres*. For $T^2-kU^2 = 4$, see Cayley, *Crelle*, Vol. 53, p. 371.

putting $\epsilon\epsilon_1 = -1$, we have

$$y+f\sqrt{k} = \epsilon\,(a+b\sqrt{k})^3.$$

Also $\epsilon = \epsilon^3\epsilon_1^2$, and ϵ^3 can be absorbed in $(a+b\sqrt{k})^3$.

We now consider different cases arising from the residues of kf^2 mod 9. Thus, let $kf^2 \equiv 4, 7 \bmod 9$. This gives $x \equiv 0 \bmod 3$, and $k \equiv 1 \bmod 3$.

Hence $$x \equiv a^2-b^2 \equiv 0 \bmod 3,$$

$$f = U(a^3+3kab^2)+T(3a^2b+kb^3).$$

But since $$T^2-kU^2 = 1,$$

and $k \equiv 1 \bmod 3$; $U \equiv 0 \bmod 3$ and $T^2 \equiv 1 \bmod 9$.

Hence
$$\begin{aligned} f &\equiv Tkb^3 \bmod 3 \\ &\equiv Tkb \quad ,, \end{aligned}$$

Thus $$b \equiv Tf \quad ,,$$

and $$a^2 \equiv 1 \quad ,,$$

Hence $$f \equiv Ua^3+3T^2a^2f+T^4f^3k \bmod 9,$$

or $$f \equiv Ua^3+3f+f^3k \quad ,,$$

or $$0 \equiv Ua^3+f(2+kf^2) \quad ,,$$

Hence our equation is insoluble if

$$kf^2 \equiv 4 \bmod 9, \quad \text{and} \quad U \equiv 0 \bmod 9,$$

$$kf^2 \equiv 7 \bmod 9, \quad \text{and} \quad U \equiv \pm 3 \bmod 9.$$

And in particular the equations

$$y^2-k = x^3, \qquad k = 7,\ 34,\ 58,\ 70.$$

If, however, $k \equiv 1, 5 \bmod 8$, we find

$$y+f\sqrt{k} = \left[T_1+\frac{U_1}{2}(1+\sqrt{k})\right]\left[a_1+\frac{b_1}{2}(1+\sqrt{k})\right]^3,$$

$$x = a_1^2+a_1b_1+\tfrac{1}{4}(1-k)b_1^2,$$

$$1 = T_1^2+T_1U_1+\tfrac{1}{4}(1-k)U_1^2,$$

and also the same equation with $T_1 = 1$, $U_1 = 0$. We dispose of this by saying, it can give only a finite number of values for x.

Putting
$$a = 2a_1 + b_1,$$
$$b = \quad b_1,$$
$$-T = 2T_1 + U_1,$$
$$-U = \quad U_1,$$
we find
$$2f \equiv U(a^3 + 3kab^2) + T(3a^2b + kb^3) \mod 9,$$
$$x \equiv a^2 - b^2 \equiv 0 \mod 3,$$
$$T^2 - kU^2 = 4.$$
Hence $U \equiv 0 \mod 3, \quad T^2 \equiv 4 \mod 9, \quad b \equiv -Tf \mod 3,$

and
$$0 \equiv Ua^3 - f(5 + 7kf^2) \mod 9.$$
Hence our equation is insoluble if
$$kf^2 \equiv 4 \mod 9 \quad \text{and} \quad U \equiv 0 \mod 9,$$
$$kf^2 \equiv 7 \mod 9 \quad \text{and} \quad U \equiv \pm 3 \mod 9.$$
It will be noticed that these results are of the same form as the previous ones. We thus find the particular equations
$$y^2 - k = x^3, \quad k = 61, 85.$$
For
$$kf^2 \equiv -4, -7 \mod 9, \quad k \equiv 2, 3 \mod 4,$$
and
$$T^2 - kU^2 = 1.$$
Then
$$T^2 + U^2 \equiv 1 \mod 3,$$
and either
$$T \text{ or } U \equiv 0 \mod 3.$$
Carrying out the same process as before, we find if $T \equiv 0 \mod 3$, our equation is insoluble if
$$kf^2 \equiv -4 \mod 9 \quad \text{and} \quad T \equiv \pm 3 \mod 9,$$
$$kf^2 \equiv -7 \mod 9 \quad \text{and} \quad T \equiv 0 \mod 9.$$
We find exactly the same results when $U \equiv 0 \mod 3$, if we replace T by U in the above conditions. We also find the same results when
$$k \equiv 1 \mod 4,$$
but of course T and U are now given by
$$T^2 - kU^2 = 4.$$

We now find the particular insoluble equations

$$y^2-k = x^3, \quad k = 14, 23, 59, 83, 86.$$

For $kf^2 \equiv \pm 3 \bmod 9$, *i.e.*, $f^2 \equiv 1 \bmod 3$, and $k \equiv \pm 3 \bmod 9$, and for $k \equiv 2, 3 \bmod 4$, we have, as before,

$$T^2-kU^2 = 1,$$

$$f \equiv U(a^3+3kab^2)+T(3a^2b+kb^3) \bmod 9.$$

Thus our equation is impossible if $U \equiv 0 \bmod 3$. If this be not the case, and we take $k \equiv -3 \bmod 9$, we find

$$a \equiv fU \bmod 3.$$

Thus $$f \equiv f^3U^4+3bf^2U^2T-3Tb^3 \bmod 9,$$

or $$f(1-f^2U^4) \equiv 3bT(U^2-b^2) \bmod 9.$$

And since $b(U^2-b^2) \equiv 0 \bmod 3$ as $U \not\equiv 0 \bmod 3$, our equation is insoluble if $f^2U^4 \not\equiv 1 \bmod 9$, *i.e.*, $f \not\equiv \pm U \bmod 9$.

When $k \equiv 1 \bmod 4$, we find by putting $2f$ for f, the conditions $U \equiv 0 \bmod 3$ or $2f \not\equiv \pm U \bmod 9$, where $T^2-kU^2 = 4$.

In particular we have $y^2-k = x^3$ impossible for

$$k = 6, 21, 42, 69, 78, 87, 93.$$

10. Another illustration is given by $y^2-60 = x^3$. The even values of x satisfy $x \equiv 2 \bmod 4$, and this is excluded by putting the equation in the form

$$y^2+4 = (x+4)(x^2-4x+16),$$

and noticing that the last factor is congruent to 12 mod 16. Hence x is prime to 30, and h for determinant 15 is 4. Also $f = 2$, $U = \pm 1$, and $fU^2 \not\equiv \pm 1 \bmod 9$. Hence the impossibility of the equation.

And finally consider $y^2-27 = x^3$. Firstly, let x be not divisible by 3. Then we have

$$y+3\sqrt{3} = (T+U\sqrt{3})(a+b\sqrt{3})^3 \quad \begin{Bmatrix} T = \pm 2, & U = \pm 1 \\ T = \pm 1, & U = 0 \end{Bmatrix},$$

$$x = a^2-3b^2.$$

Thus $$3 = U(a^3+9ab^2)+T(3a^2b+3b^3).$$

For $T = \pm 2$, this gives $a \equiv 0 \bmod 3$, contrary to our supposition For $U = 0$, there are no solutions.

Secondly, put $x = 3\xi$, $y = 9\eta$, then

$$3\eta^2 - 1 = \xi^3.$$

Thus
$$\sqrt{3}\,\eta + 1 = (T + U\sqrt{3})(a + b\sqrt{3})^3,$$

or
$$1 = T(a^3 + 9ab^2) + 3U(3a^2b + 3b^3).$$

For $T = \pm 2$, this gives $1 \equiv \pm 2a^3 \bmod 9$, which is absurd.
For $T = \pm 1$, this gives $x = -3$.

11. Other results may be found by considering congruences to mod 7. Thus for

$$y^2 - kf^2 = x^3, \quad k \equiv 2, 3 \bmod 4,$$

where k and f satisfy our usual conditions and $kf^2 \equiv 4 \bmod 7$, we easily find

$$x \equiv a^2 - kb^2 \equiv 0 \bmod 7,$$

or
$$a^2f^2 \equiv 4b^2 \bmod 7.$$

Thus
$$a \equiv 2c \quad ,,$$
$$b \equiv \pm fc \quad ,,$$

Also
$$f = U(a^3 + 3kab^2) + T(3a^2b + kb^3),$$

and we can shew that this equation is impossible if $U \equiv 0 \bmod 7$. We have then

$$f \equiv T(3a^2b + kb^3) \bmod 7,$$

or
$$f \equiv \pm(12f + kf^3)c^3 \bmod 7,$$

or
$$1 \equiv \pm 2c^3 \bmod 7,$$

which is absurd.

The same result holds when $k \equiv 1 \bmod 4$, but of course U is now given by $T^2 - kU^2 = 4$.

12. We can also prove our results by the theory of the binary cubic,* which also has the advantage of throwing additional light upon our exceptional case, when the class number is divisible by 3.

Thus, let $y^2 - kf^2 = x^3$, where k possesses no square factors, and f is such that x is prime to $2kf$. Then when this equation has solutions, f can be properly represented by a binary cubic of determinant $4k$ (or what comes to the same thing, such cubics exist whose first coefficient is f);

* All that we need is contained in a paper by Arndt in *Crelle*, Vol. 53, p. 309.

and conversely, if such representations of f exist, our equation has solutions.

Taking the latter part first, consider the binary cubic (f, b, c, d) of determinant $4k$. Calling its Hessian (F, G, H), where

$$F = b^2-fc, \quad 2G = bc-fd, \quad H = C^2-bd, \quad k = G^2-FH,$$

we have by equating the coefficients of x^3 in the syzygy of the cubic,

$$(bF-fG)^2-kf^2 = F^3,$$

which proves the second part.

Now suppose we have a solution of our original equation in the form $q^2-kf^2 = F^3$. Hence we can find binary quadratics of determinant k, whose first coefficient is F. Let (F, G, H) be one of these, where we suppose G given by $fG \equiv -q \bmod F$. We shall now shew that we can find a binary cubic (f, b, c, d) of determinant $4k$, with (F, G, H) as its Hessian. It will be found that b, c, d are given by

$$\frac{1}{F}(q+Gf), \quad \frac{1}{F^2}(kf+2qG+fG^2), \quad \frac{1}{F^3}(kq+3kfG+3qG^2+fG^3).$$

It is easily seen that b is an integer. Also

$$\begin{aligned}(kf+2qG+fG^2)(kf-2qG+fG^2) &\equiv f^2(k+G^2)^2-4kf^2G^2 \bmod F^2,\\ &\equiv f^2(G^2-k)^2 \qquad \text{,,}\\ &\equiv 0 \qquad \text{,,}\end{aligned}$$

And these two factors have no factor in common with F since $4qG$ is prime to F; and remembering $fG \equiv -q \bmod F$, we find

$$kf+2qG+fG^2 \equiv 0 \bmod F,$$

and hence mod F^2, so that c is an integer. Similarly, we can shew that d is an integer, which proves the first part of our theorem.

13. When k is negative, and the class number (now and hereafter we mean by this the number of properly primitive classes of binary quadratics of determinant k) is not divisible by 3, the binary cubics are comprised in the class $(0, 1, 0, k)$, and hence we have either none or a limited number of proper representations of f, and hence none or a limited number of solutions of our original equation.

When k is positive, there are three classes of binary cubics corresponding to our given Hessian, and when the class number is not divisible by 3, only one of the classes consists of reducible cubics, while the other

two are improperly equivalent, and it suffices to consider only one of them.

Again, when k is negative and congruent to 2, 3 mod 4, and the index of irregularity for $k \not\equiv 0 \bmod 3$, then besides the principal form, there will be two other subtriplicate binary quadratic forms of determinant k. These two will be improperly equivalent, as also the binary cubics corresponding to them, and it suffices to consider only one of the latter. Now, if q, p is a solution of $q^2-k=p^3$, $(1, 0, -p, 2q)$ is a binary cubic of determinant $4k$, and if p is not represented by the principal quadratic form, which occurs when $-k=3a^2\pm 1$, $p=4a^2\pm 1$, $\pm q=8a^3\pm 3a$, this cubic and the one derived from it by changing the sign of q, constitute our classes of irreducible cubics of determinant $4k$. Hence under our conditions, when $-k\neq 3a^2+1$, all the solutions of $y^2-kf^2=x^3$ are given by $x=pm^2-2qmn+p^2n^2$, where $m^3-3pmn^2+2qn^3=f$. It is easily seen, that whatever be the index of irregularity, there will always be a finite number of expressions of this sort, giving all the values of x. When $-k=3a^2\pm 1$, there are a finite number of others given by the above expressions with $p=4a^2\pm 1$, $\pm q=8a^3\pm 3a$, since the cubic has then the factor $m\mp 2an$.

We may note that in all cases, the values of m and n furnish us with solutions of our equation.

Thus	k	p	q	m	n	x	y
$f=1$	17,	-2,	3,	-23,	26,	5234,	378661
	24,	-2,	4,	-31,	28,	8158,	736844

Now suppose k is negative and equal to $-8n-3$, $n\neq 0$, then it will be found that the classes $(2n+1, \pm 1, 4)$, produce by triplication the principal class, and these three classes will be the only ones to possess this property if the index of irregularity is not divisible by 3, which of course includes the case of regular determinants. The cubics corresponding to these classes are $(n, \mp 1, -2, 0)$ and $(0, 1, 0, k)$, and are all reducible. Hence there will be a finite number of representations of f by the cubics; and we have the interesting result that the equation

$$y^2-kf^2=x^3,$$

where
$$k\equiv -3 \bmod 8,$$

negative, and free from square factors, and f is such that x is prime to $2kf$, and also the index of irregularity for proper binary quadratics of

determinant $+k$, $\not\equiv 0 \bmod 3$, has none or limited number of solutions.* The case $n=0$ is no exception, for then there is only one properly primitive class. In particular when $f=1$, the only solutions are when

$$8n+3 = 3a^2 \pm 1, \quad \text{for which} \quad x = 4a^2 \pm 1,$$

or

$$8n+3 = 3a^2 \pm 8, \quad \text{,,} \quad x = a^2 \pm 2.$$

Instead of $y^2-kf^2 = x^3$, we might have considered $y^2-k = x^3$, where k may have square factors, but is such that x is prime to $2k$. It can be shewn in exactly the same way, that the solution now depends upon the representation of unity, by binary cubics of determinant $4k$.

I have since shewn that the solution of $y^2 = x^3+Ax+B$, depends upon the representation of unity by binary quartics with invariants $g_2, g_3 = -4A, -4B$.

14. At present, there is no general method of determining whether or not a given number can be represented by a given binary cubic; but, as before, we can obtain interesting results by considering congruences to various moduli.

Let (F, G, H) be a subtriplicate binary quadratic of determinant k. This class belongs to the principal genus, and hence we can take

$$F \equiv 1 \equiv G \bmod 3.$$

If (a, b, c, d) is the cubic having this for its Hessian, we find if

$$q^2-kp^2 = F^3,$$

$$a \equiv p, \quad b \equiv p+q, \quad c \equiv (k+1)p+2q, \quad d \equiv (1+3k)p+(3+k)q,$$

to mod 9. We may suppose that F is prime to $2kp$, and incapable of representation by the principal class, as the cubic is then reducible, and we can easily see if it supplies us with values of x. Now let $kf^2 \equiv 1 \bmod 3$ in our equation, then $k \equiv 1 \bmod 3$ and $q^2-p^2 \equiv 1 \bmod 3$, whence

$$p \equiv 0 \bmod 3, \quad q \equiv \pm 1 \bmod 9.$$

Taking the positive sign (the negative one leads to the same results), we have from the representation of f by this cubic

$$pm^3+3m^2n+6mn^2+(4p+3+k)n^3 \equiv f \bmod 9.$$

Taking $f \equiv 1 \bmod 3$ (the value $f \equiv -1 \bmod 3$ leads to the same re-

* For application to irregular determinants, see note in *Messenger of Math.*, Dec. 1912.

sults), we find $n \equiv 1 \bmod 3$, whence

$$pm^3+3m^2+6m+4p+3+k \equiv f \bmod 9,$$

or
$$m^2+\tfrac{1}{3}(p+6)m \equiv \tfrac{1}{3}(f-4p-k-3) \bmod 3,$$

and if $p \equiv 0 \bmod 9$, this congruence is impossible if

$$f-k-3 \equiv 3 \bmod 9,$$

or
$$f^3-kf^2 \equiv -3f^2 \bmod 9,$$

or
$$kf^2 \equiv 4 \bmod 9.$$

This applies to all the values of p, and if it holds for all the subtriplicate binary quadratics, our equation $y^2-kf^2 = x^3$ is impossible.

So, if $p \equiv \pm 3 \bmod 9$, the corresponding congruence is impossible if $kf^2 \equiv 7 \bmod 9$.

Similarly, if $F \equiv -1 \bmod 3$, we find $q \equiv 0 \bmod 3$, and that the above results hold if p is replaced by q.

Proceeding similarly with the various values of kf^2, we have the following scheme for impossible equations:—

$kf^2 \equiv 4 \bmod 9$	and	$p \equiv 0$	mod 9 (or q if $F \equiv -1 \bmod 3$),
$kf^2 \equiv 7$	„	$p \equiv \pm 3$	„
$kf^2 \equiv 2$	„	q or $p \equiv 0$	„
$kf^2 \equiv 5$	„	q or $p \equiv \pm 3$	„
$kf^2 \equiv -3$	„	$p \not\equiv \pm f$	„
$kf^2 \equiv 3$	„	$p \equiv 0 \bmod 3$.	

Also $q^2-kp^2 = F^3$, where $F \equiv 1 \bmod 3$, except in the first two cases where we may take $F \equiv -1 \bmod 3$. Further, F is prime to $2kp$. When k is negative, we can take F to be a prime, and there will be only two values for p, differing only in sign. Thus for $k = -29$ and $q^2+29p^2 = 125$, $q = 3$, and for $k = -38$, $q^2+38p^2 = 343$ and $p = 3$, and hence the two equations $y^2-k = x^3$, $-k = 29, 38$ are impossible, as the principal binary quadratic form of determinant $+k$ gives rise to a reducible cubic giving no values of x, and further there are only two irreducible cubics (which are improperly equivalent) arising from two improperly equivalent quadratics.

15. When k is positive, there are an infinite number of values for p. In particular, when the class number is not divisible by 3, we can take F

equal to 1, and hence $q^2-kp^2=1$. If either q or p in the first solution is divisible by 9, this will be the case with every solution, but if q or $p\equiv\pm3$ mod 9, this will also be the case with all solutions except those given by
$$q_1+p_1\sqrt{k}=(q+p\sqrt{k})^{3n}.$$
The corresponding cubic, however, belongs to the reducible class, and our scheme will be found to agree with the results of § 9.

When $k\equiv5$ mod 8, we have for $kf^2\equiv4$ mod 9, the condition $p\equiv0$ mod 9, where q, p is the first solution of $q^2-kp^2=1$. But since the class number is not divisible by 3, T and U are both odd in the first solution of $T^2-kU^2=4$, and we have the relation
$$q+p\sqrt{k}=\tfrac{1}{8}(T+U\sqrt{k})^3.$$
Hence $8p=U(3T^2+kU^2)$, but $U\equiv0$ mod 3, and hence
$$p\equiv0 \text{ mod } 9,$$
so that our equation $y^2-kf^2=x^3$ is impossible. It is easily seen that a similar result holds when $kf^2\equiv2,\ \pm3$ mod 9.

16. As before we can obtain some results by considering congruences to mod 7. Thus, let (F, G, H) be the Hessian, where we may take $F\equiv1$, 2 or 4 mod 7 and $G\equiv1$ mod 7. Then for the representation of f by the corresponding cubic, we have
$$(Fm+Gn+n\sqrt{k})^3(q+p\sqrt{k})-(Fm+Gn-n\sqrt{k})^3(q-p\sqrt{k})=2F^3f\sqrt{k},$$
where
$$q^2-kp^2=F^3.$$
If $kf^2\equiv4$ mod 7, then since $F^3\equiv1$ mod 7, $kp^2\equiv0$, 1 mod 7, and we can shew that if $p\equiv0$ mod 7, our equation is impossible. For then $q\equiv\pm1$ mod 7, $f\sqrt{k}\equiv\pm2$ mod 7, and hence the above equation can be written $P^3+Q^3\equiv4$ mod 7, which is impossible.

If k is positive, we may take $F=1$, and if in the first solution $p\equiv0$ mod 7, this will be true for all the values of p, and our equation is impossible. In particular if $k\equiv5$ mod 8, then since the class number is supposed not divisible by 3, we have for the first solution p, q,
$$8p=U(3T^2+kU^2),$$
where T, U is the first solution of
$$T^2-kU^2=4.$$
But since $kf^2\equiv4$ mod 7, we find either $U\equiv0$ mod 7, or $U\equiv\pm f$, $T\equiv\pm1$ mod 7, and in both cases $p\equiv0$ mod 7, and the equation is insoluble.

17. We can now draw up a scheme, giving the values of k between ± 100 for which $y^2-k=x^3$ is soluble or not. Thus, for

$-k = 7$, 15, 18, 20, 23, 25, 26, 28, 39, 40, 45, 47, 48, 53, 54, 55, 56, 60, 61, 63, 71, 72, 79, 83, 87, 89, 95, 100,

there are, I believe, an infinite number of solutions. For the other values of $-k$ between 1 and 100 there are none or a finite number of solutions, except when $-k = 31$, 84, and in these cases, whether or not the equations are insoluble I cannot say. When

$k = 1$, 4, 6, 7, 11, 13, 14, 16, 20, 21, 23, 25, 27, 29, 32, 34, 39, 42, 45, 46, 47, 49, 51, 53, 58, 59, 60, 61, 62, 66, 67, 69, 70, 75, 77, 78, 83, 84, 85, 86, 87, 88, 90, 93, 95, 96,

the equations are insoluble or admit only a limited number of solutions. For the remaining values of k, there are an infinite number of solutions, except when $k = 74$, in which case nothing can be said about the equation. When k is a perfect square, we may note that the solution* of $y^2 = x^3+k$ involves that of $x^3+y^3 = 2\sqrt{k}$ and *vice versa*. We have made use of this when $k = 1, 4, \ldots$.

I take this opportunity of acknowledging my great indebtedness to the referees who have suggested many improvements.

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* Due to Lucas, I believe, *Nouvelles Annales*, 1878, p. 425.

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FACTORIAL MOMENTS IN TERMS OF SUMS OR DIFFERENCES

By W. F. SHEPPARD.

[Received and Read May 8th, 1913.]

1. The g-th moment of $u_p, u_{p+1}, \ldots, u_{p+m}$ is (supposing them to be placed at successive intervals 1) of the form

$$(\theta-p)^g u_p+(\theta-p-1)^g u_{p+1}+\ldots+(\theta-p-m)^g u_{p+m} \equiv M_g;$$

and any linear compound of $M_0, M_1, \ldots, M_g$ can be written in the form

$$\sum_{q=0}^{q=m} f_g(q)\, u_{p+q},$$

where $f_g(x)$ is a polynomial in x of degree g. The moments are now often determined by a process of successive additions; and this process leads to expressions of the above form, $f_g(q)$ being a reduced factorial (*i.e.*, a factorial divided by $g!$).

2. The notation used for the reduced factorials is

$$\left.\begin{aligned}(\theta, q) &\equiv \theta(\theta-1)\ldots(\theta-q+1)/q! = [\theta-q+1, q]\\ [\theta, q] &\equiv \theta(\theta+1)\ldots(\theta+q-1)/q! = (\theta+q-1, q)\end{aligned}\right\}, \tag{1}$$

$$\begin{aligned}(\theta, q] &\equiv (\theta-\tfrac{1}{2}q+\tfrac{1}{2})(\theta-\tfrac{1}{2}q+\tfrac{3}{2})\ldots(\theta+\tfrac{1}{2}q-\tfrac{1}{2})/q!\\ &= [\theta-\tfrac{1}{2}q+\tfrac{1}{2}, q] = (\theta+\tfrac{1}{2}q-\tfrac{1}{2}, q),\end{aligned} \tag{2}$$

$$[\theta, q) \equiv \tfrac{1}{2}\{(\theta-\tfrac{1}{2}, q]+(\theta+\tfrac{1}{2}, q]\} = \theta/q\,.\,(\theta, q-1]. \tag{3}$$

There are various relations, depending mainly on

$$(\theta+\tfrac{1}{2}, q]-(\theta-\tfrac{1}{2}, q] = (\theta, q-1]. \tag{4}$$

This latter is identical with an elementary property of binomial coefficients, but is easier to remember.

Also we denote by $L\{u_\theta;\ u_p, u_{p+1}, \ldots, u_{p+m}\}$ the Lagrangian formula

for u_θ in terms of $u_p, u_{p+1}, \ldots, u_{p+m}$; *i.e.*,

$$L\{u_\theta;\ u_p, u_{p+1}, \ldots, u_{p+m}\} \equiv \frac{(\theta-p)(\theta-p-1)\ldots(\theta-p-m)}{m!}$$

$$\times\left\{\frac{1}{\theta-p-m}u_{p+m}-\frac{(m,\,1)}{\theta-p-m+1}u_{p+m-1}+\ldots+(-)^m\frac{(m,\,m)}{\theta-p}u_p\right\}. \quad (5)$$

Similarly, if we find the values of $u_{\theta-\frac{1}{2}}$ and $u_{\theta+\frac{1}{2}}$ by this formula, their mean is denoted by

$$L\{\mu u_\theta;\ u_p, u_{p+1}, \ldots, u_{p+m}\}.$$

3. *Particular formulæ (advancing differences).*—In the ordinary notation Σ is the inverse of Δ, so that

$$\Delta\Sigma u_q = u_q, \quad \Sigma u_q = \ldots + u_{q-3} + u_{q-2} + u_{q-1}. \quad (6)$$

If we are dealing only with $u_0, u_1, \ldots, u_m$, it is sometimes more convenient to use Σ' to denote summation from u_0 to u_q, both included, and Σ'' summation from u_q to u_m, both included; *i.e.*,

$$\Sigma' u_q \equiv u_0+u_1+\ldots+u_q, \quad \Sigma'^2 u_q \equiv \Sigma' u_0+\Sigma' u_1+\ldots+\Sigma' u_q, \ldots, \quad (7)$$

$$\Sigma'' u_q \equiv u_q+u_{q+1}+\ldots+u_m, \quad \Sigma''^2 u_q \equiv \Sigma'' u_q+\Sigma'' u_{q+1}+\ldots+\Sigma'' u_m, \ldots. \quad (8)$$

We have then

$$\Delta\Sigma' u_q = u_{q+1} = E u_q, \quad \Delta\Sigma'' u_q = -u_q. \quad (9)$$

By successive summation it will be found that*

$$\Sigma'^{g+1} u_m = \sum_{q=0}^{q=m}[m-q+1,\, g]\,u_q, \quad (10)$$

$$\Sigma''^{g+1} u_0 = \sum_{q=0}^{q=m}[q+1,\, g]\,u_q, \quad (11)$$

$$\Sigma''^{g+1} u_g = \sum_{q=g}^{q=m}[q-g+1,\, g]\,u_q = \sum_{q=g}^{q=m}(q,\, g)\,u_q. \quad (12)$$

4. We know that

$$u_m = u_0+(m,1)\,\Delta u_0+(m,2)\,\Delta^2 u_0+\ldots+(m,m)\,\Delta^m u_0 = \sum_{s=0}^{s=m}(m,\,s)\,\Delta^s u_0, \quad (13)$$

and that $(0 \leqslant q \leqslant m)$,

$$u_q = u_0+(q,1)\,\Delta u_0+(q,2)\,\Delta^2 u_0+\ldots+(q,m)\,\Delta^m u_0. \quad (14)$$

* Cf. W. P. Elderton, *Frequency-Curves and Correlation*, p. 20.

Putting $q = 0, 1, 2, \ldots, m$ in succession, and adding, and repeating the process, we find that

$$\Sigma'^{g+1} u_m = (m+g+1, g+1) u_0 + (m+g+1, g+2)\Delta u_0 + (m+g+1, g+3)\Delta^2 u_0 + \ldots + (m+g+1, m+g+1)\Delta^m u_0$$

$$= (m+1)(m+2)\ldots(m+g+1) \sum_{s=0}^{s=m} \frac{(m, s)}{(s+1)(s+2)\ldots(s+g+1)} \Delta^s u_0. \tag{15}$$

To obtain the corresponding formula for $\Sigma''^{g+1} u_0$, we have, with the usual notation,

$$1+E+E^2+\ldots+E^m$$

$$= \frac{E^{m+1}-1}{E-1} = \frac{(1+\Delta)^{m+1}-1}{\Delta}$$

$$= (m+1, 1)+(m+1, 2)\Delta+(m+1, 3)\Delta^2+\ldots+(m+1, m+1)\Delta^m$$

$$= (m+1)\left\{\tfrac{1}{1}+\tfrac{1}{2}(m, 1)\Delta+\tfrac{1}{3}(m, 2)\Delta^2+\ldots+\frac{1}{m+1}(m, m)\Delta^m\right\}.$$

Differentiating with regard to Δ (or E), and then replacing m by $m+1$, we have

$$1+2E+3E^2+\ldots+(m+1)E^m$$

$$= (m+1)(m+2)\left\{\tfrac{1}{2}+\tfrac{1}{3}(m, 1)\Delta+\tfrac{1}{4}(m, 2)\Delta^2+\ldots+\frac{1}{m+2}(m, m)\Delta^m\right\}.$$

Repeating the process,* we find that

$$1.2\ldots g+2.3\ldots(g+1)E+3.4\ldots(g+2)E^2+\ldots+(m+1)(m+2)\ldots(m+g)E^m$$

$$= (m+1)(m+2)\ldots(m+g+1)$$

$$\times\left\{\frac{1}{g+1}+\frac{1}{g+2}(m, 1)\Delta+\frac{1}{g+3}(m, 2)\Delta^2+\ldots+\frac{1}{m+g+1}(m, m)\Delta^m\right\}.$$

---

* Another method is by means of the reduction-formula

$$\int_0^\Delta \Delta^g(1+\Delta)^m\, d\Delta = \frac{1}{m+g+1}\Delta^{g+1}(1+\Delta)^m + \frac{m}{m+g+1}\int_0^\Delta \Delta^g(1+\Delta)^{m-1}\, d\Delta,$$

which gives

$$\frac{1}{g+1}\Delta^{g+1}+\frac{(m, 1)}{g+2}\Delta^{g+2}+\ldots+\frac{(m, m)}{m+g+1}\Delta^{m+g+1}$$

$$= \frac{1}{m+g+1}\Delta^{g+1}E^m + \frac{m}{(m+g+1)(m+g)}\Delta^{g+1}E^{m-1}+\ldots+\frac{m(m-1)\ldots 1}{(m+g+1)(m+g)\ldots(g+1)}\Delta^{g+1}.$$

Operating on u_0, and comparing with (11), we see that

$$\Sigma''^{g+1} u_0 = \frac{(m+1)(m+2)\dots(m+g+1)}{1.2\dots g}$$

$$\times \left\{ \frac{1}{g+1} u_0 + \frac{1}{g+2}(m,1)\Delta u_0 + \frac{1}{g+3}(m,2)\Delta^2 u_0 + \dots \right.$$

$$\left. + \frac{1}{m+g+1}(m,m)\Delta^m u_0 \right\}$$

$$= [m+1, g] \sum_{s=0}^{s=m} \frac{m+g+1}{s+g+1}(m,s)\Delta^s u_0. \tag{16}$$

5. The identity of the expressions obtained in (10) and (15) for $\Sigma'^{g+1} u_m$, or in (11) and (16) for $\Sigma''^{g+1} u_0$, may be verified as follows. Let

$$w_x \equiv \frac{1}{x+g+1}, \tag{17}$$

and let E_1 and Δ_1 denote operations on u, and E_2 and Δ_2 operations on w. Then

$$\left.\begin{aligned} E_2^s w_0 &= \frac{1}{s+g+1} \\ \Delta_2^s w_0 &= (-)^s \frac{s!}{(g+1)(g+2)\dots(g+s+1)} \\ E_2^{m-q}\Delta_2^q w_0 &= (-)^q \frac{q!}{(m+g+1)(m+g)\dots(m+g-q+1)} \end{aligned}\right\}. \tag{18}$$

Also
$$(E_2 - E_1\Delta_2)^m = (1-\Delta_1\Delta_2)^m, \tag{19}$$

$$(E_1E_2 - \Delta_2)^m = (1+E_2\Delta_1)^m. \tag{20}$$

Operating with both members of (19) on $u_0 w_0$, we have

$$\frac{1}{m+g+1} u_0 + \frac{m}{(m+g+1)(m+g)} u_1 + \frac{m(m-1)}{(m+g+1)\dots(m+g-1)} u_2 + \dots$$

$$+ \frac{m(m-1)\dots 1}{(m+g+1)\dots(g+1)} u_m$$

$$= \frac{1}{g+1} u_0 + \frac{m}{(g+1)(g+2)} \Delta u_0 + \frac{m(m-1)}{(g+1)(g+2)(g+3)} \Delta^2 u_0 + \dots$$

$$+ \frac{m(m-1)\dots 1}{(g+1)\dots(m+g+1)} \Delta^m u_0, \tag{21}$$

which equates the two expressions for $\Sigma'^{g+1} u_m$; and, operating with both members of (20) on $u_0 w_0$,

$$\frac{1}{m+g+1} u_m + \frac{m}{(m+g+1)(m+g)} u_{m-1} + \frac{m(m-1)}{(m+g+1)\dots(m+g-1)} u_{m-2} + \dots$$

$$+ \frac{m(m-1)\dots 1}{(m+g+1)\dots(g+1)} u_0$$

$$= \frac{1}{g+1} u_0 + \frac{(m, 1)}{g+2} \Delta u_0 + \frac{(m, 2)}{g+3} \Delta^2 u_0 + \dots + \frac{(m, m)}{m+g+1} \Delta^m u_0, \tag{22}$$

which equates the two expressions for $\Sigma''^{g+1} u_0$.

6. From (15) and (10), or from (16) and (11), by taking $g = 0, 1, 2, \dots, j$, we see that any linear compound of $M_0, M_1, \dots, M_j$ is of the form

$$\sum_{s=0}^{s=m} \frac{\phi_j(s)}{(s+1)(s+2)\dots(s+j+1)} (m, s) \Delta^s u_0,$$

where $\phi_j(s)$ is some polynomial in s of degree j; and conversely, if we have an expression $\Phi \equiv a_0 u_0 + a_1 \Delta u_0 + \dots + a_m \Delta^m u_0$, which we know to be of this latter form, j being not greater than m, and if we know the values of $j+1$ of the coefficients $a_0, a_1, \dots, a_m$, we can determine $\phi_j(s)$, and thence by means of (16) we can express Φ in terms of $\Sigma'' u_0, \Sigma''^2 u_0, \dots, \Sigma''^{j+1} u_0$.

7. *Central factorial moments in terms of central differences.*—(i) For obtaining the central-difference formulæ corresponding to (16), the fundamental formulæ are

$$\epsilon^{2n} + \epsilon^{-2n} = 2 \sum_{t=0}^{t=n} [n, 2t)(\epsilon - \epsilon^{-1})^{2t}, \tag{23}$$

$$\epsilon^{2n} - \epsilon^{-2n} = (\epsilon + \epsilon^{-1}) \sum_{t=1}^{t=n} (n, 2t-1](\epsilon - \epsilon^{-1})^{2t-1}, \tag{24}$$

$$\epsilon^{2n-1} + \epsilon^{-(2n-1)} = (\epsilon + \epsilon^{-1}) \sum_{t=1}^{t=n} (n - \tfrac{1}{2}, 2t-2](\epsilon - \epsilon^{-1})^{2t-2}, \tag{25}$$

$$\epsilon^{2n-1} - \epsilon^{-(2n-1)} = 2 \sum_{t=1}^{t=n} [n - \tfrac{1}{2}, 2t-1)(\epsilon - \epsilon^{-1})^{2t-1}, \tag{26}$$

$$1+(\epsilon^2+\epsilon^{-2})+(\epsilon^4+\epsilon^{-4})+\ldots+(\epsilon^{2n}+\epsilon^{-2n})$$

$$= 2\sum_{t=0}^{t=n} [n+\tfrac{1}{2},\ 2t+1)(\epsilon-\epsilon^{-1})^{2t}, \tag{27}$$

$$(\epsilon+\epsilon^{-1})+(\epsilon^3+\epsilon^{-3})+\ldots+\{\epsilon^{2n-1}+\epsilon^{-(2n-1)}\}$$

$$= (\epsilon+\epsilon^{-1})\sum_{t=1}^{t=n} (n,\ 2t-1](\epsilon-\epsilon^{-1})^{2t-2}. \tag{28}$$

It is sufficient to establish any one of these, and the others can then be obtained in rotation. Thus (24) can be deduced from (23) by differentiating and multiplying by ϵ, (28) from (24) by dividing by $\epsilon-\epsilon^{-1}$, (25) from (28) by changing n into $n-1$ and subtracting, (26) from (25) by dividing by ϵ and integrating, (27) from (26) by changing n into $n+1$ and dividing by $\epsilon-\epsilon^{-1}$, and (23) from (27) by changing n into $n-1$ and subtracting.

(ii) From (27) it can be proved, by induction, that

$$\sum_{r=-n}^{r=n} (r,\ 2h]\epsilon^{2r} = (4h+2)(n+\tfrac{1}{2},\ 2h+1]\sum_{t=0}^{t=n} \frac{(n+\frac{1}{2},\ 2t]}{2t+2h+1}(\epsilon-\epsilon^{-1})^{2t}, \tag{29}$$

$$\sum_{r=-n}^{r=n} [r,\ 2h+1)\epsilon^{2r} = (n+\tfrac{1}{2},\ 2h+1](\epsilon+\epsilon^{-1})\sum_{t=1}^{t=n} \frac{2t(n+\frac{1}{2},\ 2t]}{2t+2h+1}(\epsilon-\epsilon^{-1})^{2t-1}, \tag{30}$$

and from (28) it can be proved that

$$\sum_{r=1}^{r=n} (r-\tfrac{1}{2},\ 2h-1]\{\epsilon^{2r-1}-\epsilon^{-(2r-1)}\}$$

$$= 4h(n,\ 2h]\sum_{t=1}^{t=n} \frac{(n,\ 2t-1]}{2t+2h-1}(\epsilon-\epsilon^{-1})^{2t-1}, \tag{31}$$

$$\sum_{r=1}^{r=n} [r-\tfrac{1}{2},\ 2h)\{\epsilon^{2r-1}+\epsilon^{-(2r-1)}\}$$

$$= (n,\ 2h](\epsilon+\epsilon^{-1})\sum_{t=1}^{t=n} \frac{(2t-1)(n,\ 2t-1]}{2t+2h-1}(\epsilon-\epsilon^{-1})^{2t-2}. \tag{32}$$

(iii) Taking ϵ to be the operator which converts u_0 into $u_{\frac{1}{2}}$, so that

$$\mu = \tfrac{1}{2}(\epsilon+\epsilon^{-1}), \qquad \delta = \epsilon-\epsilon^{-1},$$

and operating on u_0 with (23), (24), (27), (29), and (30), and on $u_{\frac{1}{2}}$ with

(25), (26), (28), (31), and (32), we have*

$$\left.\begin{aligned} u_n+u_{-n} &= 2\sum_{t=0}^{t=n}[n,\ 2t)\,\delta^{2t}u_0 \\ u_n-u_{-n} &= 2\sum_{t=1}^{t=n}(n,\ 2t-1]\,\mu\delta^{2t-1}u_0 \end{aligned}\right\}, \tag{33}$$

$$\left.\begin{aligned} u_n+u_{-n+1} &= 2\sum_{t=1}^{t=n}(n-\tfrac{1}{2},\ 2t-2]\,\mu\delta^{2t-2}u_{\frac{1}{2}} \\ u_n-u_{-n+1} &= 2\sum_{t=1}^{t=n}[n-\tfrac{1}{2},\ 2t-1)\,\delta^{2t-1}u_{\frac{1}{2}} \end{aligned}\right\}, \tag{34}$$

$$\sum_{r=-n}^{r=n} u_r = 2\sum_{t=0}^{t=n}[n+\tfrac{1}{2},\ 2t+1)\,\delta^{2t}u_0, \tag{35}$$

$$\sum_{r=-n+1}^{r=n} u_r = 2\sum_{t=1}^{t=n}(n,\ 2t-1]\mu\delta^{2t-2}u_{\frac{1}{2}}, \tag{36}$$

$$\sum_{r=-n}^{r=n}(r,\ 2h]u_r = (4h+2)(n+\tfrac{1}{2},\ 2h+1]\sum_{t=0}^{t=n}\frac{(n+\frac{1}{2},\ 2t]}{2t+2h+1}\,\delta^{2t}u_0, \tag{37}$$

$$\sum_{r=-n}^{r=n}[r,\ 2h+1)\,u_r = 2(n+\tfrac{1}{2},\ 2h+1]\sum_{t=1}^{t=n}\frac{2t\,(n+\frac{1}{2},\ 2t]}{2t+2h+1}\,\mu\delta^{2t-1}u_0, \tag{38}$$

$$\sum_{r=-n+1}^{r=n}(r-\tfrac{1}{2},\ 2h-1]\,u_r = 4h(n,\ 2h]\sum_{t=1}^{t=n}\frac{(n,\ 2t-1]}{2t+2h-1}\,\delta^{2t-1}u_{\frac{1}{2}}, \tag{39}$$

$$\sum_{r=-n+1}^{r=n}[r-\tfrac{1}{2},\ 2h)\,u_r = 2(n,\ 2h]\sum_{t=1}^{t=n}\frac{(2t-1)(n,\ 2t-1]}{2t+2h-1}\,\mu\delta^{2t-2}u_{\frac{1}{2}}. \tag{40}$$

(iv) It follows from (37)–(40) that, if $f_k(x)$ is any polynomial in x of degree k,

$$\sum_{r=-n}^{r=n} f_k(r^2)\,u_r = \sum_{t=0}^{t=n}\frac{\phi_k(t)}{(2t+1)(2t+3)\dots(2t+2k+1)}(n+\tfrac{1}{2},\ 2t]\,\delta^{2t}u_0,$$

$$\sum_{r=-n}^{r=n} rf_k(r^2)\,u_r = \sum_{t=1}^{t=n}\frac{2t\,\phi_k(t)}{(2t+1)(2t+3)\dots(2t+2k+1)}(n+\tfrac{1}{2},\ 2t]\,\mu\delta^{2t-1}u_0,$$

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* The formulæ (37) and (38) have been stated by me in the paper mentioned on p. 97 below; (33) and (34) are familiar, but the method is new.

$$\sum_{r=-n+1}^{r=n} f_k\{(r-\tfrac{1}{2})^2\}\,u_r = \sum_{t=1}^{t=n} \frac{\psi_k(t)}{(2t+1)(2t+3)\dots(2t+2k-1)}\,(n,\ 2t-1]\,\mu\delta^{2t-2}u_{\frac{1}{2}},$$

$$\sum_{r=-n+1}^{r=n} (r-\tfrac{1}{2})\,f_k\{(r-\tfrac{1}{2})^2\}\,u_r$$
$$= \sum_{t=1}^{t=n} \frac{\chi_k(t)}{(2t+1)(2t+3)\dots(2t+2k+1)}\,(n,\ 2t-1]\,\delta^{2t-1}u_{\frac{1}{2}},$$

$$\sum_{r=-n+1}^{r=n} (r-\tfrac{1}{2})^2 f_k\{(r-\tfrac{1}{2})^2\}\,u_r$$
$$= \sum_{t=1}^{t=n} \frac{(2t-1)\,\chi_k(t)}{(2t+1)(2t+3)\dots(2t+2k+1)}\,(n,\ 2t-1]\,\mu\delta^{2t-2}u_{\frac{1}{2}},$$

where $\phi_k(t)$, $\psi_k(t)$, $\chi_k(t)$ are some polynomials in t of degree k.

(v) Conversely, if we have an expression Φ which we know to be of one of the forms in (iv), and if we know the values of $k+1$ of the coefficients (of u_0 or $\mu u_{\frac{1}{2}}$ and the successive differences), we can express Φ in terms of the values, for $h = 0, 1, 2, \dots, k$, of the sum mentioned on the right-hand side of (37), (38), (39), or (40). These sums, however, are not simple to calculate; and moreover the cases in which we require to use central-difference formulæ are usually cases in which the $2n+1$ or $2n$ terms with which we are dealing form part only of a sequence. It is therefore more convenient to be able to express Φ in terms of the successive sums obtained from the sequence as a whole; for this purpose we require certain general formulæ.

8. *General formulæ.*—We know that $L\{u_\theta;\ u_p,\ u_{p+1},\ \dots,\ u_{p+m}\}$ can be expressed in a good many different ways in terms of differences. In particular there are the advancing-difference and the receding-difference formulæ

$$L\{u_\theta;\ u_0,\ u_1,\ \dots,\ u_m\} = u_0+(\theta,\ 1)\,\Delta u_0 + (\theta,\ 2)\,\Delta^2 u_0 + \dots + (\theta,\ m)\,\Delta^m u_0,$$
$$L\{u_\theta;\ u_0,\ u_{-1},\ \dots,u_{-m}\} = u_0+[\theta,1]\,\Delta u_{-1}+[\theta,2]\Delta^2 u_{-2}+\dots+[\theta,m]\Delta^m u_{-m},$$

and the central-difference formulæ

$$L\{u_\theta;\ u_{-n},\ u_{-n+1},\ \dots,\ u_n\}$$
$$= u_0 + (\theta,\ 1]\,\mu\delta u_0 + [\theta,\ 2)\,\delta^2 u_0 + \dots + [\theta,\ 2n)\,\delta^{2n} u_0,$$

$$\overline{L}\{\overline{u_\theta};\ \overline{u_{-n+1}},\ u_{-n+2},\ \dots,\ u_n\}$$
$$= \mu u_{\frac{1}{2}} + [\theta-\tfrac{1}{2},\ 1)\,\delta u_{\frac{1}{2}} + (\theta-\tfrac{1}{2},\ 2]\,\mu\delta^2 u_{\frac{1}{2}} + \dots + [\theta-\tfrac{1}{2},\ 2n-1)\,\delta^{2n-1} u_{\frac{1}{2}}.$$

In the more general formula we start from any u and proceed by any diagonal steps to $\Delta^m u_p$. The formula is most safely expressed in terms of central differences; if the last difference introduced is $\delta^s u_t$, the next term is $(\theta-t,\ s+1]\,\delta^{s+1}u_{t\pm\frac{1}{2}}$, provided that the suffix $t\pm\frac{1}{2}$ is not less than $p+\frac{1}{2}s+\frac{1}{2}$ and not greater than $p+m-\frac{1}{2}s-\frac{1}{2}$.

For μu_θ, we take $u_{\theta-\frac{1}{2}}$ and $u_{\theta+\frac{1}{2}}$ separately. The general formula for $u_{\theta-\frac{1}{2}}$ is as described above, with $(\theta-\frac{1}{2}-t,\ s+1]\,\delta^{s+1}u_{t\pm\frac{1}{2}}$ substituted for $(\theta-t,\ s+1]\,\delta^{s+1}u_{t\pm\frac{1}{2}}$; and the formula for $u_{\theta+\frac{1}{2}}$ is similar, with $\theta+\frac{1}{2}$ substituted for $\theta-\frac{1}{2}$. Hence, taking the mean, we get a similar formula, but with $[\theta-t,\ s+1)$ in place of $(\theta-t,\ s+1]$. In particular,

$$L\{\mu u_\theta;\ u_{-n},\ u_{-n+1},\ \ldots,\ u_n\}$$

$$=u_0+[\theta,\ 1)\,\delta u_{\frac{1}{2}}+[\theta-\tfrac{1}{2},\ 2)\,\delta^2 u_0+[\theta,\ 3)\,\delta^3 u_{\frac{1}{2}}+\ldots+[\theta-\tfrac{1}{2},\ 2n)\,\delta^{2n}u_0 \quad (41)$$

$$=u_0+[\theta,\ 1)\,\delta u_{-\frac{1}{2}}+[\theta+\tfrac{1}{2},\ 2)\,\delta^2 u_0+[\theta,\ 3)\,\delta^3 u_{-\frac{1}{2}}+\ldots+[\theta+\tfrac{1}{2},\ 2n)\,\delta^{2n}u_0, \quad (41a)$$

$$L\{\mu u_{\theta+\frac{1}{2}};\ u_{-n+1},\ u_{-n+2},\ \ldots,\ u_n\}$$

$$=u_0+[\theta+\tfrac{1}{2},\ 1)\,\delta u_{\frac{1}{2}}+[\theta,\ 2)\,\delta^2 u_0+[\theta+\tfrac{1}{2},\ 3)\,\delta^3 u_{\frac{1}{2}}+\ldots+[\theta+\tfrac{1}{2},\ 2n-1)\,\delta^{2n-1}u_{\frac{1}{2}} \quad (42)$$

$$=u_1+[\theta-\tfrac{1}{2},\ 1)\,\delta u_{\frac{1}{2}}+[\theta,\ 2)\,\delta^2 u_1+[\theta-\tfrac{1}{2},\ 3)\,\delta^3 u_{\frac{1}{2}}+\ldots+[\theta-\tfrac{1}{2},\ 2n-1)\,\delta^{2n-1}u_{\frac{1}{2}}. \quad (42a)$$

9. Having constructed $L\{u_\theta;\ u_{p+1},\ u_{p+2},\ \ldots,\ u_{p+g}\}$, we can introduce either u_p by adding

$$(\theta-p-\tfrac{1}{2}g-\tfrac{1}{2},\ g]\,\delta^g u_{p+\frac{1}{2}g}=(\theta-p-1,\ g)\,\Delta^g u_p$$

or u_{p+g+1} by adding

$$(\theta-p-\tfrac{1}{2}g-\tfrac{1}{2},\ g]\,\delta^g u_{p+\frac{1}{2}g+1}=(\theta-p-1,\ g)\,\Delta^g u_{p+1}.$$

Hence

$$L\{u_\theta;\ u_{p+1},\ u_{p+2},\ \ldots,\ u_{p+g+1}\}-L\{u_\theta;\ u_p,\ u_{p+1},\ \ldots,\ u_{p+g}\}$$

$$=(\theta-p-1,\ g)(\Delta^g u_{p+1}-\Delta^g u_p)$$

$$=(\theta-p-1,\ g)\,\Delta^{g+1}u_p \quad (43)$$

$$=(\theta-p-\tfrac{1}{2}g-\tfrac{1}{2},\ g]\,\delta^{g+1}u_{p+\frac{1}{2}g+\frac{1}{2}}. \quad (44)$$

Replacing p by $p+1$, $p+2$, ..., $p+m$, and adding,

$$L\{u_\theta;\ u_{p+m+1},\ u_{p+m+2},\ \ldots,\ u_{p+m+g+1}\} - L\{u_\theta;\ u_p,\ u_{p+1},\ \ldots,\ u_{p+g}\}$$

$$= \sum_{q=p}^{q=p+m} (\theta-q-1,\ g)\,\Delta^{g+1} u_q \tag{45}$$

$$= \sum_{q=p}^{q=p+m} [\theta-q-g,\ g]\,\Delta^{g+1} u_q \tag{46}$$

$$= \sum_{q=p}^{q=p+m} (\theta-q-\tfrac{1}{2}g-\tfrac{1}{2},\ g]\,\delta^{g+1} u_{q+\frac{1}{2}g+\frac{1}{2}} \tag{47}$$

$$= (-)^g \sum_{q=p}^{q=p+m} (g+q-\theta,\ g)\,\Delta^{g+1} u_q \tag{48}$$

$$= (-)^g \sum_{q=p}^{q=p+m} [q-\theta+1,\ g]\,\Delta^{g+1} u_q \tag{49}$$

$$= (-)^g \sum_{q=p}^{q=p+m} (\tfrac{1}{2}g+q-\theta+\tfrac{1}{2},\ g]\,\delta^{g+1} u_{q+\frac{1}{2}g+\frac{1}{2}}. \tag{50}$$

Replacing θ in (47) and (50) by $\theta-\frac{1}{2}$ and by $\theta+\frac{1}{2}$, and taking the mean of the two expressions,

$$L\{\mu u_\theta;\ u_{p+m+1},\ u_{p+m+2},\ \ldots,\ u_{p+m+g+1}\} - L\{\mu u_\theta;\ u_p,\ u_{p+1},\ \ldots,\ u_{p+g}\}$$

$$= \sum_{q=p}^{q=p+m} [\theta-q-\tfrac{1}{2}g-\tfrac{1}{2},\ g)\,\delta^{g+1} u_{q+\frac{1}{2}g+\frac{1}{2}} \tag{51}$$

$$= (-)^g \sum_{q=p}^{q=p+m} [\tfrac{1}{2}g+q-\theta+\tfrac{1}{2},\ g)\,\delta^{g+1} u_{q+\frac{1}{2}g+\frac{1}{2}}. \tag{52}$$

10. For expressing a factorial moment in terms of receding sums (corresponding to advancing differences), we replace u in (45), (46), (48), and (49), by $\Sigma^{g+1} u$, and we alter θ to $\theta+1$ in (45) and (49) and to $\theta+g$ in (46) and (48); denoting $F(P)-F(p)$ by $[F(x)]_{x=p}^{x=P}$, this gives

$$\sum_{q=p}^{q=p+m} (\theta-q,\ g)\,u_q$$

$$= \left[L\{\Sigma^{g+1} u_{\theta+1};\ \Sigma^{g+1} u_x,\ \Sigma^{g+1} u_{x+1},\ \ldots,\ \Sigma^{g+1} u_{x+g}\}\right]_{x=p}^{x=p+m+1}, \tag{53}$$

$$\sum_{q=p}^{q=p+m} (q-\theta,\ g)\,u_q$$

$$= \left[(-)^g L\{\Sigma^{g+1} u_{\theta+g};\ \Sigma^{g+1} u_x,\ \Sigma^{g+1} u_{x+1},\ \ldots,\ \Sigma^{g+1} u_{x+g}\}\right]_{x=p}^{x=p+m+1}, \tag{54}$$

$$\sum_{q=p}^{q=p+m} [\theta-q,\, g]\, u_q$$
$$= [L\{\Sigma^{g+1} u_{\theta+g};\ \Sigma^{g+1} u_x,\ \Sigma^{g+1} u_{x+1},\ \ldots,\ \Sigma^{g+1} u_{x+g}\}]_{x=p}^{x=p+m+1}, \tag{55}$$

$$\sum_{q=p}^{q=p+m} [q-\theta,\, g]\, u_q$$
$$= [(-)^g L\{\Sigma^{g+1} u_{\theta+1};\ \Sigma^{g+1} u_x,\ \Sigma^{g+1} u_{x+1},\ \ldots,\ \Sigma^{g+1} u_{x+g}\}]_{x=p}^{x=p+m+1}. \tag{56}$$

The following are particular cases :—

(i) $\Sigma'^{g+1} u_m$ is $\Sigma^{g+1} u_{m+g+1}$ obtained by taking

$$\Sigma u_0 = 0, \quad \Sigma^2 u_1 = 0, \quad \Sigma^3 u_2 = 0, \quad \ldots, \quad \Sigma^{g+1} u_g = 0,$$

so that $$\Sigma^{g+1} u_0 = 0, \quad \Sigma^{g+1} u_1 = 0, \quad \ldots, \quad \Sigma^{g+1} u_g = 0.$$

Putting $p = 0$, $\theta = m+g$ in (53), or $p = 0$, $\theta = m+1$ in (55), the first term on the right-hand side becomes $\Sigma^{g+1} u_{m+g+1} = \Sigma'^{g+1} u_m$, and the second term is zero; hence

$$\Sigma'^{g+1} u_m = \sum_{q=0}^{q=m} (m+g-q,\, g)\, u_q$$
$$= \sum_{q=0}^{q=m} [m-q+1,\, g]\, u_q,$$

which agrees with (10).

(ii) For $\Sigma'' u_0$, $\Sigma''^2 u_0$, ..., $\Sigma''^{g+1} u_0$, we take

$$\Sigma u_{m+1} = 0, \quad \Sigma^2 u_{m+1} = 0, \quad \ldots, \quad \Sigma^{g+1} u_{m+1} = 0,$$

and change the signs of Σu_0, $\Sigma^3 u_0$, $\Sigma^5 u_0$, ... Putting $p = 0$, $\theta = -g$ in (54), or $p = 0$, $\theta = -1$ in (56), the first term on the right-hand side is zero, and the second term is $(-)^{g+1} \Sigma^{g+1} u_0 = \Sigma''^{g+1} u_0$; and therefore

$$\Sigma''^{g+1} u_0 = \sum_{q=0}^{q=m} (q+g,\, g)\, u_q$$
$$= \sum_{q=0}^{q=m} [q+1,\, g]\, u_q,$$

which agrees with (11).

11. *Central factorial moments in terms of central sums.*—To obtain central-sum formulæ, we replace q by $q-\frac{1}{2}g-\frac{1}{2}$, and p by $p-\frac{1}{2}g-\frac{1}{2}$,

in (47), (50), (51), and (52), and u by $\sigma^{g+1}u$ throughout; and we have

$$\sum_{q=p}^{q=P} (\theta-q,\, g]\, u_q$$

$$= [L\,\{\sigma^{g+1}u_\theta;\ \sigma^{g+1}u_{x-\frac{1}{2}g},\ \sigma^{g+1}u_{x-\frac{1}{2}g+1},\ \ldots,\ \sigma^{g+1}u_{x+\frac{1}{2}g}\}]_{x=p-\frac{1}{2}}^{x=P+\frac{1}{2}}, \qquad (57)$$

$$\sum_{q=p}^{q=P} (q-\theta,\, g]\, u_q$$

$$= [(-)^g L\,\{\sigma^{g+1}u_\theta;\ \sigma^{g+1}u_{x-\frac{1}{2}g},\ \sigma^{g+1}u_{x-\frac{1}{2}g+1},\ \ldots,\ \sigma^{g+1}u_{x+\frac{1}{2}g}\}]_{x=p-\frac{1}{2}}^{x=P+\frac{1}{2}}, \qquad (58)$$

$$\sum_{q=p}^{q=P} [\theta-q,\, g)\, u_q$$

$$= [L\,\{\mu\sigma^{g+1}u_\theta;\ \sigma^{g+1}u_{x-\frac{1}{2}g},\ \sigma^{g+1}u_{x-\frac{1}{2}g+1},\ \ldots,\ \sigma^{g+1}u_{x+\frac{1}{2}g}\}]_{x=p-\frac{1}{2}}^{x=P+\frac{1}{2}}, \qquad (59)$$

$$\sum_{q=p}^{q=P} [q-\theta,\, g)\, u_q$$

$$= [(-)^g L\,\{\mu\sigma^{g+1}u_\theta;\ \sigma^{g+1}u_{x-\frac{1}{2}g},\ \sigma^{g+1}u_{x-\frac{1}{2}g+1},\ \ldots,\ \sigma^{g+1}u_{x+\frac{1}{2}g}\}]_{x=p-\frac{1}{2}}^{x=P+\frac{1}{2}}. \qquad (60)$$

12. Hence we obtain the following special results:—

(i) $g = 2h$.

$$\sum_{r=-n}^{r=n} (r,\, 2h]\, u_r$$

$$= [L\,\{\sigma^{2h+1}u_0;\ \sigma^{2h+1}u_{x-h},\ \sigma^{2h+1}u_{x-h+1},\ \ldots,\ \sigma^{2h+1}u_{x+h}\}]_{x=-n-\frac{1}{2}}^{x=n+\frac{1}{2}}$$

$$= \{\sigma^{2h+1}u_{n+\frac{1}{2}}-(n+\tfrac{1}{2},\, 1]\,\mu\sigma^{2h}u_{n+\frac{1}{2}}+[n+\tfrac{1}{2},\, 2)\,\sigma^{2h-1}u_{n+\frac{1}{2}}-\ldots$$
$$+[n+\tfrac{1}{2},\, 2h)\,\sigma u_{n+\frac{1}{2}}\}$$
$$+\{-\sigma^{2h+1}u_{-n-\frac{1}{2}}-(n+\tfrac{1}{2},\, 1]\,\mu\sigma^{2h}u_{-n-\frac{1}{2}}-[n+\tfrac{1}{2},\, 2)\,\sigma^{2h-1}u_{-n-\frac{1}{2}}-\ldots$$
$$-[n+\tfrac{1}{2},\, 2h)\,\sigma u_{-n-\frac{1}{2}}\}, \qquad (61)$$

$$\sum_{r=-n}^{r=n} [r,\, 2h)\, u_r$$

$$= [L\,\{\mu\sigma^{2h+1}u_0;\ \sigma^{2h+1}u_{x-h},\ \sigma^{2h+1}u_{x-h+1},\ \ldots,\ \sigma^{2h+1}u_{x+h}\}]_{x=-n-\frac{1}{2}}^{x=n+\frac{1}{2}}$$

$$= \{\sigma^{2h+1}u_{n+\frac{1}{2}}-[n+\tfrac{1}{2},\, 1)\,\sigma^{2h}u_{n+1}+[n+1,\, 2)\,\sigma^{2h-1}u_{n+\frac{1}{2}}-\ldots$$
$$+[n+1,\, 2h)\,\sigma u_{n+\frac{1}{2}}\}$$
$$+\{-\sigma^{2h+1}u_{-n-\frac{1}{2}}-[n+\tfrac{1}{2},\, 1)\,\sigma^{2h}u_{-n-1}-[n+1,\, 2)\,\sigma^{2h-1}u_{-n-\frac{1}{2}}-\ldots$$
$$-[n+1,\, 2h)\,\sigma u_{-n-\frac{1}{2}}\}, \qquad (62)$$

$$\sum_{r=-n+1}^{r=n} (r-\tfrac{1}{2}, 2h]\, u_r$$

$$= \left[L\left\{\sigma^{2h+1}u_{\frac{1}{2}};\ \sigma^{2h+1}u_{x-h},\ \sigma^{2h+1}u_{x-h+1},\ \ldots,\ \sigma^{2h+1}u_{x+h}\right\}\right]_{x=-n+\frac{1}{2}}^{x=n+\frac{1}{2}}$$

$$= \left\{\sigma^{2h+1}u_{n+\frac{1}{2}}-(n, 1]\,\mu\sigma^{2h}u_{n+\frac{1}{2}}+[n, 2)\,\sigma^{2h-1}u_{n+\frac{1}{2}}-\ldots \right.$$
$$\left. +[n, 2h)\,\sigma u_{n+\frac{1}{2}}\right\}$$

$$+\left\{-\sigma^{2h+1}u_{-n+\frac{1}{2}}-(n, 1]\,\mu\sigma^{2h}u_{-n+\frac{1}{2}}-[n, 2)\,\sigma^{2h-1}u_{-n+\frac{1}{2}}-\ldots \right.$$
$$\left. -[n, 2h)\,\sigma u_{-n+\frac{1}{2}}\right\}, \qquad (63)$$

$$\sum_{r=-n+1}^{r=n} [r-\tfrac{1}{2}, 2h)\, u_r$$

$$= \left[L\left\{\mu\sigma^{2h+1}u_{\frac{1}{2}};\ \sigma^{2h+1}u_{x-h},\ \sigma^{2h+1}u_{x-h+1},\ \ldots,\ \sigma^{2h+1}u_{x+h}\right\}\right]_{x=-n+\frac{1}{2}}^{x=n+\frac{1}{2}}$$

$$= \left\{\sigma^{2h+1}u_{n+\frac{1}{2}}-[n, 1)\sigma^{2h}u_{n+1}+[n+\tfrac{1}{2}, 2)\,\sigma^{2h-1}u_{n+\frac{1}{2}}-\ldots+[n+\tfrac{1}{2}, 2h)\sigma u_{n+\frac{1}{2}}\right\}$$

$$+\left\{-\sigma^{2h+1}u_{-n+\frac{1}{2}}-[n, 1)\,\sigma^{2h}u_{-n}-[n+\tfrac{1}{2}, 2)\,\sigma^{2h-1}u_{-n+\frac{1}{2}}-\ldots \right.$$
$$\left. -[n+\tfrac{1}{2}, 2h)\,\sigma u_{-n+\frac{1}{2}}\right\}. \qquad (64)$$

(ii) $g = 2h-1$.

$$\sum_{r=-n}^{r=n} (r, 2h-1]\, u_r$$

$$= \left[-L\left\{\sigma^{2h}u_0;\ \sigma^{2h}u_{x-h+\frac{1}{2}},\ \sigma^{2h}u_{x-h+\frac{3}{2}},\ \ldots,\ \sigma^{2h}u_{x+h-\frac{1}{2}}\right\}\right]_{x=-n-\frac{1}{2}}^{x=n+\frac{1}{2}}$$

$$= \left\{-\mu\sigma^{2h}u_{n+\frac{1}{2}}+[n+\tfrac{1}{2}, 1)\,\sigma^{2h-1}u_{n+\frac{1}{2}}-(n+\tfrac{1}{2}, 2]\,\mu\sigma^{2h-2}u_{n+\frac{1}{2}}+\ldots \right.$$
$$\left. +[n+\tfrac{1}{2}, 2h-1)\,\sigma u_{n+\frac{1}{2}}\right\}$$

$$+\left\{\mu\sigma^{2h}u_{-n-\frac{1}{2}}+[n+\tfrac{1}{2}, 1)\,\sigma^{2h-1}u_{-n-\frac{1}{2}}+(n+\tfrac{1}{2}, 2]\,\mu\sigma^{2h-2}u_{-n-\frac{1}{2}}+\ldots \right.$$
$$\left. +[n+\tfrac{1}{2}, 2h-1)\,\sigma u_{-n-\frac{1}{2}}\right\}, \qquad (65)$$

$$\sum_{r=-n}^{r=n} [r, 2h-1)\, u_r$$

$$= \left[-L\left\{\mu\sigma^{2h}u_0;\ \sigma^{2h}u_{x-h+\frac{1}{2}},\ \sigma^{2h}u_{x-h+\frac{3}{2}},\ \ldots,\ \sigma^{2h}u_{x+h-\frac{1}{2}}\right\}\right]_{x=-n-\frac{1}{2}}^{x=n+\frac{1}{2}}$$

$$= \left\{-\sigma^{2h}u_{n+1}+[n+1, 1)\,\sigma^{2h-1}u_{n+\frac{1}{2}}-[n+\tfrac{1}{2}, 2)\,\sigma^{2h-2}u_{n+1}+\ldots \right.$$
$$\left. +[n+1, 2h-1)\,\sigma u_{n+\frac{1}{2}}\right\}$$

$$+\left\{\sigma^{2h}u_{-n-1}+[n+1, 1)\,\sigma^{2h-1}u_{-n-\frac{1}{2}}+[n+\tfrac{1}{2}, 2)\,\sigma^{2h-2}u_{-n-1}+\ldots \right.$$
$$\left. +[n+1, 2h-1)\,\sigma u_{-n-\frac{1}{2}}\right\}, \qquad (66)$$

$$\sum_{r=-n+1}^{r=n} (r-\tfrac{1}{2},\ 2h-1]\, u_r$$

$$= \left[-L\left\{\sigma^{2h}u_{\frac{1}{2}};\ \sigma^{2h}u_{x-h+\frac{1}{2}},\ \sigma^{2h}u_{x-h+\frac{3}{2}},\ \ldots,\ \sigma^{2h}u_{x+h-\frac{1}{2}}\right\}\right]_{x=-n+\frac{1}{2}}^{x=n+\frac{1}{2}}$$

$$= \Big\{-\mu\sigma^{2h}u_{n+\frac{1}{2}}+[n,\ 1)\,\sigma^{2h-1}u_{n+\frac{1}{2}}-(n,\ 2]\,\mu\sigma^{2h-2}u_{n+\frac{1}{2}}+\ldots$$
$$+[n,\ 2h-1)\,\sigma u_{n+\frac{1}{2}}\Big\}$$
$$+\Big\{\mu\sigma^{2h}u_{-n+\frac{1}{2}}+[n,\ 1)\,\sigma^{2h-1}u_{-n+\frac{1}{2}}+(n,\ 2]\,\mu\sigma^{2h-2}u_{-n+\frac{1}{2}}+\ldots$$
$$+[n,\ 2h-1)\,\sigma u_{-n+\frac{1}{2}}\Big\}, \tag{67}$$

$$\sum_{r=-n+1}^{r=n} [r-\tfrac{1}{2},\ 2h-1)\, u_r$$

$$= \left[-L\left\{\mu\sigma^{2h}u_{\frac{1}{2}};\ \sigma^{2h}u_{x-h+\frac{1}{2}},\ \sigma^{2h}u_{x-h+\frac{3}{2}},\ \ldots,\ \sigma^{2h}u_{x+h-\frac{1}{2}}\right\}\right]_{x=-n+\frac{1}{2}}^{x=n+\frac{1}{2}}$$

$$= \Big\{-\sigma^{2h}u_{n+1}+[n+\tfrac{1}{2},\ 1)\,\sigma^{2h-1}u_{n+\frac{1}{2}}-[n,\ 2)\,\sigma^{2h-2}u_{n+1}+\ldots$$
$$+[n+\tfrac{1}{2},\ 2h-1)\,\sigma u_{n+\frac{1}{2}}\Big\}$$
$$+\Big\{\sigma^{2h}u_{-n}+[n+\tfrac{1}{2},\ 1)\,\sigma^{2h-1}u_{-n+\frac{1}{2}}+[n,\ 2)\,\sigma^{2h-2}u_{-n}+\ldots$$
$$+[n+\tfrac{1}{2},\ 2h-1)\,\sigma u_{-n+\frac{1}{2}}\Big\}. \tag{68}$$

By means of (61), (64), (66), and (67), we can obtain the expressions required in § 7 (v).

18. *Central differences of powers of* 0.—The expressions for the ordinary moments M_1, M_2, ... in terms of the ordinary factorial moments involve the "differences of powers of 0"; the formulæ are well known. There are similar formulæ for central moments in terms of central factorial moments. The fundamental formulæ are

$$x^{2h} = \sum_{t=0}^{t=h} (\delta^{2t}0^{2h})[x,\ 2t), \tag{69}$$

$$x^{2h-1} = \sum_{t=1}^{t=h} (\mu\delta^{2t-1}0^{2h-1})(x,\ 2t-1], \tag{70}$$

$$x^{2h} = \sum_{t=0}^{t=h} (\mu\delta^{2t}0^{2h})(x,\ 2t], \tag{71}$$

$$x^{2h+1} = \sum_{t=0}^{t=h} (\delta^{2t+1}0^{2h+1})[x,\ 2t+1). \tag{72}$$

The coefficients in these formulæ are connected by the relations

$$\delta^{2t}0^{2h} = 2t\,.\,\mu\delta^{2t-1}0^{2h-1}, \tag{73}$$

$$\delta^{2t+1}0^{2h+1} = (2t+1)\,\mu\delta^{2t}0^{2h}, \tag{74}$$

$$\frac{\delta^{2t}0^{2h}}{(2t)!} = t^2\frac{\delta^{2t}0^{2h-2}}{(2t)!} + \frac{\delta^{2t-2}0^{2h-2}}{(2t-2)!}, \tag{75}$$

$$\frac{\delta^{2t+1}0^{2h+1}}{(2t+1)!} = (t+\tfrac{1}{2})^2\frac{\delta^{2t+1}0^{2h-1}}{(2t+1)!} + \frac{\delta^{2t-1}0^{2h-1}}{(2t-1)!}. \tag{76}$$

Thus $\delta^{2t}0^{2h}/(2t)!$ and $2^{2t}\delta^{2t+1}0^{2h+1}/(2t+1)!$ are integers; and they are of course zero if $t > h$.

The following are some of the values of

$$P \equiv \delta^{2t}0^{2h}/(2t)! \quad \text{and} \quad Q \equiv \delta^{2t+1}0^{2h+1}/(2t+1)!.$$

$h = 0$. $t = 0$; $P = 1$; $Q = 1$.

$h = 1$. $t = 0, 1$; $P = 0, 1$; $Q = 1/4, 1$.

$h = 2$. $t = 0, 1, 2$; $P = 0, 1, 1$; $Q = 1/16, 5/2, 1$.

$h = 3$. $t = 0, 1, 2, 3$; $P = 0, 1, 5, 1$; $Q = 1/64, 91/16, 35/4, 1$.

$h = 4$. $t = 0, 1, 2, 3, 4$; $P = 0, 1, 21, 14, 1$; $Q = 1/256, 205/16, 483/8, 21, 1$.

$h = 5$. $t = 0, 1, 2, 3, 4, 5$; $P = 0, 1, 85, 147, 30, 1$; $Q = 1/1024, 7381/256, 12485/32, 2541/8, 165/4, 1$.

$h = 6$. $t = 0, 1, 2, 3, 4, 5, 6$; $P = 0, 1, 341, 1408, 627, 55, 1$; $Q = 1/4096, 33215/512, 631631/256, 68497/16, 18447/16, 143/2, 1$.

14. By means of (69)–(72) and (61)–(68) we can express the central moments in terms of the central sums.

(i) For $2n+1$ u's, from u_{-n} to u_n, the central moments up to the 5th are

$$M_0 = \sigma u_{n+\frac{1}{2}} - \sigma u_{-n-\frac{1}{2}},$$

$$M_1 = -(\sigma^2 u_{n+1} - \sigma^2 u_{-n-1}) + (n+1)(\sigma u_{n+\frac{1}{2}} + \sigma u_{-n-\frac{1}{2}}),$$

$$M_2 = 2(\sigma^3 u_{n+\frac{1}{2}} - \sigma^3 u_{-n-\frac{1}{2}}) - (2n+1)(\sigma^2 u_{n+1} + \sigma^2 u_{-n-1}) + (n+1)^2(\sigma u_{n+\frac{1}{2}} - \sigma u_{-n-\frac{1}{2}}),$$

$$M_3 = -6(\sigma^4 u_{n+1} - \sigma^4 u_{-n-1}) + 6(n+1)(\sigma^3 u_{n+\frac{1}{2}} + \sigma^3 u_{-n-\frac{1}{2}}) - (3n^2+3n+1)(\sigma^2 u_{n+1} - \sigma^2 u_{-n-1}) + (n+1)^3(\sigma u_{n+\frac{1}{2}} + \sigma u_{-n-\frac{1}{2}}),$$

$$M_4 = 24(\sigma^5 u_{n+\frac{1}{2}} - \sigma^5 u_{-n-\frac{1}{2}}) - 12(2n+1)(\sigma^4 u_{n+1} + \sigma^4 u_{-n-1})$$
$$+ (12n^2+24n+14)(\sigma^3 u_{n+\frac{1}{2}} - \sigma^3 u_{-n-\frac{1}{2}})$$
$$-(2n+1)(2n^2+2n+1)(\sigma^2 u_{n+1} + \sigma^2 u_{-n-1})$$
$$+(n+1)^4 (\sigma u_{n+\frac{1}{2}} - \sigma u_{-n-\frac{1}{2}}),$$

$$M_5 = -120(\sigma^6 u_{n+1} - \sigma^6 u_{-n-1}) + 120(n+1)(\sigma^5 u_{n+\frac{1}{2}} + \sigma^5 u_{-n-\frac{1}{2}})$$
$$-30(2n^2+2n+1)(\sigma^4 u_{n+1} - \sigma^4 u_{-n-1})$$
$$+10(n+1)(2n^2+4n+3)(\sigma^3 u_{n+\frac{1}{2}} + \sigma^3 u_{-n-\frac{1}{2}})$$
$$-\{5(n^2+n)^2+5(n^2+n)+1\}(\sigma^2 u_{n+1} - \sigma^2 u_{-n-1})$$
$$+(n+1)^5 (\sigma u_{n+\frac{1}{2}} + \sigma u_{-n-\frac{1}{2}}).$$

(ii) For $2n$ u's from u_{-n+1} to u_n the central moments are

$$M_0 = \sigma u_{n+\frac{1}{2}} - \sigma u_{-n+\frac{1}{2}},$$
$$M_1 = -(\sigma^2 u_{n+1} - \sigma^2 u_{-n}) + (n+\tfrac{1}{2})(\sigma u_{n+\frac{1}{2}} + \sigma u_{-n+\frac{1}{2}}),$$
$$M_2 = 2(\sigma^3 u_{n+\frac{1}{2}} - \sigma^3 u_{-n+\frac{1}{2}}) - 2n(\sigma^2 u_{n+1} + \sigma^2 u_{-n}) + (n+\tfrac{1}{2})^2 (\sigma u_{n+\frac{1}{2}} - \sigma u_{-n+\frac{1}{2}}),$$
$$M_3 = -6(\sigma^4 u_{n+1} - \sigma^4 u_{-n}) + 3(2n+1)(\sigma^3 u_{n+\frac{1}{2}} + \sigma^3 u_{-n+\frac{1}{2}})$$
$$-(3n^2+\tfrac{1}{4})(\sigma^2 u_{n+1} - \sigma^2 u_{-n}) + (n+\tfrac{1}{2})^3 (\sigma u_{n+\frac{1}{2}} + \sigma u_{-n+\frac{1}{2}}),$$

$$M_4 = 24(\sigma^5 u_{n+\frac{1}{2}} - \sigma^5 u_{-n+\frac{1}{2}}) - 24n(\sigma^4 u_{n+1} + \sigma^4 u_{-n})$$
$$+(12n^2+12n+5)(\sigma^3 u_{n+\frac{1}{2}} - \sigma^3 u_{-n+\frac{1}{2}})$$
$$-n(4n^2+1)(\sigma^2 u_{n+1} + \sigma^2 u_{-n})$$
$$+(n+\tfrac{1}{2})^4 (\sigma u_{n+\frac{1}{2}} - \sigma u_{-n+\frac{1}{2}}),$$

$$M_5 = -120(\sigma^6 u_{n+1} - \sigma^6 u_{-n}) + 60(2n+1)(\sigma^5 u_{n+\frac{1}{2}} + \sigma^5 u_{-n+\frac{1}{2}})$$
$$-15(4n^2+1)(\sigma^4 u_{n+1} - \sigma^4 u_{-n})$$
$$+5(n+\tfrac{1}{2})(4n^2+4n+3)(\sigma^3 u_{n+\frac{1}{2}} + \sigma^3 u_{-n+\frac{1}{2}})$$
$$-(5n^4+\tfrac{5}{2}n^2+\tfrac{1}{16})(\sigma^2 u_{n+1} - \sigma^2 u_{-n})$$
$$+(n+\tfrac{1}{2})^5 (\sigma u_{n+\frac{1}{2}} + \sigma u_{-n+\frac{1}{2}}).$$

FITTING OF POLYNOMIAL BY METHOD OF LEAST SQUARES (SOLUTION IN TERMS OF DIFFERENCES OR SUMS)

By W. F. SHEPPARD.

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1. The data are $m+1$ quantities $u_0, u_1, u_2, \ldots, u_m$; and the problem is to find v_q, a polynomial in q of degree j, such that

$$S \equiv \sum_{q=0}^{q=m} (v_q - u_q)^2 \tag{1}$$

may be a minimum.

2. The ordinary method is to take v_q to be of the form

$$v_q \equiv a_0 + a_1 q + a_2 q^2 + \ldots + a_j q^j. \tag{2}$$

We then get a set of equations

$$\left.\begin{array}{l} (m+1)\, a_0 + \Sigma q \,.\, a_1 \quad + \Sigma q^2 \,.\, a_2 \quad + \ldots + \Sigma q^j \,.\, a_j \quad = \Sigma u_q \\ \Sigma q \,.\, a_0 + \Sigma q^2 \,.\, a_1 \quad + \Sigma q^3 \,.\, a_2 \quad + \ldots + \Sigma q^{j+1} \,.\, a_j = \Sigma q\, u_q \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \Sigma q^j \,.\, a_0 + \Sigma q^{j+1} \,.\, a_1 + \Sigma q^{j+2} \,.\, a_2 + \ldots + \Sigma q^{2j} \,.\, a_j \quad = \Sigma q^j u_q \end{array}\right\}, \tag{3}$$

where Σ represents summation for $q = 0, 1, 2, \ldots, m$. These equations have to be solved separately for each separate value of j, and solutions for particular values will be found in text-books; but it does not seem possible to give the general solution, for this form of v_q, in a simple form.

3. I have recently* given a solution in terms of central differences, taking v_q of a suitable form; reference will be made to this later. A simpler solution is in terms of advancing differences. The principle of the solution is the same; the argument is as follows.

* *Fifth International Congress of Mathematicians*, Cambridge, 1912, ii, 368 *seqq.*

(i) We take*

$$v_q = A_0 + A_1(q,\ 1) + A_2(q,\ 2) + \dots + A_j(q,\ j). \tag{4}$$

It is then obvious that $$A_f = \Delta^f v_0. \tag{5}$$

(ii) Making

$$S \equiv \sum_{q=0}^{q=m} \{A_0 + A_1(q,\ 1) + A_2(q,\ 2) + \dots + A_j(q,\ j) - u_q\}^2 \tag{6}$$

a minimum, we obtain a set of equations ($g = 0,\ 1,\ 2,\ \dots,\ j$)

$$\sum_{f=0}^{f=j} \left\{ \sum_{q=0}^{q=m} (q,\ g)(q,\ f) \right\} A_f = \sum_{q=0}^{q=m} (q,\ g)\, u_q \equiv P_g\,; \tag{7}$$

and by solving these equations we obtain each A as a linear compound of $P_0,\ P_1,\ P_2,\ \dots,\ P_j$.

(iii) Hence each A is a linear compound of the moments $M_0,\ M_1,\ M_2,\ \dots,\ M_j$, where

$$M_f \equiv \sum_{q=0}^{q=m} q^f u_q\,;$$

and therefore (see § 6 of the preceding paper) it is of the form

$$\sum_{s=0}^{s=m} \frac{\phi_j(s)}{(s+1)(s+2)\dots(s+j+1)} (m,\ s)\, \Delta^s u_0, \tag{8}$$

where $\phi_j(s)$ is some polynomial in s of degree j.

(iv) We know that

$$u_q = u_0 + (q,\ 1)\,\Delta u_0 + (q,\ 2)\,\Delta^2 u_0 + \dots + (q,\ j)\,\Delta^j u_0 + R_j, \tag{9}$$

where $$R_j = (q,\ j+1)\,\Delta^{j+1} u_0 + (q,\ j+2)\,\Delta^{j+2} u_0 + \dots + (q,\ m)\,\Delta^m u_0. \tag{10}$$

Substituting in (1), and making

$$S \equiv \sum_{q=0}^{q=m} \{(A_0 - u_0) + (q,\ 1)(A_1 - \Delta u_0) + (q,\ 2)(A_2 - \Delta^2 u_0) + \dots + (q,\ j)(A_j - \Delta^j u_0) - R_j\}^2 \tag{11}$$

a minimum, we obtain a set of equations which give each of the quantities

$$A_0 - u_0,\ A_1 - \Delta u_0,\ A_2 - \Delta^2 u_0,\ \dots,\ A_j - \Delta^j u_0,$$

---

* The notation of the reduced factorials $(q,\ f)$, &c., is the same as in the paper preceding this.

as a linear compound of the R's. Hence $A_f \equiv \Delta^f v_0$ differs from $\Delta^f u_0$ by differences of u_0 of order $j+1$ and upwards.

(v) It follows from (iv) and (iii) that ($f = 0, 1, 2, \ldots, j$)

$$\Delta^f v_0 = \sum_{s=0}^{s=m} (-)^{j-f} \frac{s(s-1)\ldots(s-j)}{(s-f).f!(j-f)!} \frac{(f+1)(f+2)\ldots(f+j+1)}{(s+1)(s+2)\ldots(s+j+1)} \frac{(m,\, s)}{(m,\, f)} \Delta^s u_0. \tag{12}$$

Hence, for any given value of j, the value of $\Delta^f v_0$ is found from the formula for u_m in terms of $u_0, \Delta u_0, \ldots, \Delta^m u_0$ by dividing it by the coefficient of $\Delta^f u_0$ and then multiplying the terms by a series of coefficients which, for the particular values of j and f, is definite and does not involve m.

4. This formula for $\Delta^f v_0$ involves $m-j$ terms, besides $\Delta^f u_0$. When m is large, it is better to express $\Delta^f v_0$ in terms of the moments or the successive sums of the u's, since there are then only $j+1$ terms to be calculated. If we write

$$\Sigma'' u_r \equiv u_r + u_{r+1} + \ldots + u_m, \quad \Sigma''^{g+1} u_r \equiv \Sigma''^{g} u_r + \Sigma''^{g} u_{r+1} + \ldots + \Sigma''^{g} u_m, \tag{13}$$

so that $\Sigma''^{g+1} u_0 = \Sigma''^{g} u_0 + \Sigma''^{g} u_1 + \ldots + \Sigma''^{g} u_m$, and $\Sigma''^{g+1} u_r$ is obtained from $\Sigma''^{g+1} u_0$ by subtracting successively $\Sigma''^{g} u_0, \Sigma''^{g} u_1, \ldots, \Sigma''^{g} u_{r-1}$, then (see (11) and (16) of the preceding paper)

$$\Sigma''^{g+1} u_0 = \sum_{q=0}^{q=m} (q+g,\, g)\, u_q \tag{14}$$

$$= (m+g,\, g) \sum_{s=0}^{s=m} \frac{m+g+1}{s+g+1} (m,\, s)\, \Delta^s u_0. \tag{14a}$$

Hence, from (12), by splitting up the coefficient of $(m, s)\,\Delta^s u_0$ in $\Delta^f v_0$ into partial fractions, we have

$$\Delta^f v_0 = \sum_{g=0}^{g=j} (-)^{f+g} \frac{f+1}{f+g+1} \frac{(j,\, f)(j,\, g)(f+j+1,\, j)(g+j+1,\, j)}{(m+g+1,\, f+g+1)(f+g+1,\, f)} \Sigma''^{g+1} u_0 \tag{15}$$

$$= \sum_{g=0}^{g=j} (-)^{f+g} \frac{f+1}{f+g+1} \frac{(j,\, f)(j,\, g)(f+j+1,\, j)(g+j+1,\, j)}{(m+g+1,\, f+g+1)(f+g+1,\, f)} \times \sum_{q=0}^{q=m} (q+g,\, g)\, u_q. \tag{15a}$$

We can in (15a) express $\sum_{q=0}^{q=m} (q+g,\, g)\, u_q$ in terms of the moments

$M_0, M_1, \ldots, M_g$; but it is simpler to construct the successive sums $\Sigma'' u_0, \Sigma''^2 u_0, \ldots, \Sigma''^{j+1} u_0$.

5. The following are particular cases; $S_1, S_2, \ldots, S_{j+1}$ being written for $\Sigma'' u_0, \Sigma''^2 u_0, \ldots, \Sigma''^{j+1} u_0$.

(i) $j = 1$.
$$v_0 = \frac{4}{(m+1,\ 1)} S_1 - \frac{3}{(m+2,\ 2)} S_2,$$
$$\Delta v_0 = -\frac{3}{(m+1,\ 2)} S_1 + \frac{2}{(m+2,\ 3)} S_2.$$

(ii) $j = 2$.
$$v_0 = \frac{9}{(m+1,\ 1)} S_1 - \frac{18}{(m+2,\ 2)} S_2 + \frac{10}{(m+3,\ 3)} S_3.$$
$$\Delta v_0 = -\frac{18}{(m+1,\ 2)} S_1 + \frac{32}{(m+2,\ 3)} S_2 - \frac{15}{(m+3,\ 4)} S_3,$$
$$\Delta^2 v_0 = \frac{10}{(m+1,\ 3)} S_1 - \frac{15}{(m+2,\ 4)} S_2 + \frac{6}{(m+3,\ 5)} S_3.$$

(iii) $j = 3$.
$$v_0 = \frac{16}{(m+1,\ 1)} S_1 - \frac{60}{(m+2,\ 2)} S_2 + \frac{80}{(m+3,\ 3)} S_3 - \frac{35}{(m+4,\ 4)} S_4,$$
$$\Delta v_0 = -\frac{60}{(m+1,\ 2)} S_1 + \frac{200}{(m+2,\ 3)} S_2 - \frac{225}{(m+3,\ 4)} S_3 + \frac{84}{(m+4,\ 5)} S_4.$$
$$\Delta^2 v_0 = \frac{80}{(m+1,\ 3)} S_1 - \frac{225}{(m+2,\ 4)} S_2 + \frac{216}{(m+3,\ 5)} S_3 - \frac{70}{(m+4,\ 6)} S_4,$$
$$\Delta^3 v_0 = -\frac{35}{(m+1,\ 4)} S_1 + \frac{84}{(m+2,\ 5)} S_2 - \frac{70}{(m+3,\ 6)} S_3 + \frac{20}{(m+4,\ 7)} S_4.$$

(iv) $j = 4$. The coefficients of $S_1/(m+1,\ 1)$, $S_2/(m+2,\ 2)$, ... in v_0, of $S_1/(m+1,\ 2)$, $S_2/(m+2,\ 3)$, ... in Δv_0, and so on, are
$$v_0 = +\ 25 - \ 150 + \ 350 - \ 350 + 126,$$
$$\Delta v_0 = -150 + \ 800 - 1575 + 1344 - 420,$$
$$\Delta^2 v_0 = +350 - 1575 + 2646 - 1960 + 540,$$
$$\Delta^3 v_0 = -350 + 1344 - 1960 + 1280 - 315,$$
$$\Delta^4 v_0 = +126 - \ 420 + \ 540 - \ 315 + \ 70.$$

6. To illustrate the method, take the example given on p. 370 of the paper already mentioned. The u's and the successive sums are as below.

u	$\Sigma'' u$	$\Sigma''^{2} u$	$\Sigma''^{3} u$	$\Sigma''^{4} u$	$\Sigma''^{5} u$
31950	317616	1726573	6845458	22103236	61585427
32299	285666	1408957	5118885	15257778	39482191
32532	253367	1123291	3709928	10138893	24224413
32611	220835	869924	2586637	6428965	14085520
32516	188224	649089	1716713	3842328	7656555
32282	155708	460865	1067624	2125615	3814227
31807	123426	305157	606759	1057991	1688612
31266	91619	181731	301602	451232	630621
30594	60353	90112	119871	149630	179389
29759	29759	29759	29759	29759	29759

(i) For $j = 2$, the calculation is as follows.

	$+317616+1726573+6845458$			
$660 v_0$	$+$	$594-$	$216+$	30
$660 \Delta v_0$	$-$	$264+$	$128-$	20
$660 \Delta^2 v_0$	$+$	$55-$	$30+$	5

This gives $v_0 = 31951{\cdot}33, \qquad \Delta v_0 = +366{\cdot}00,$

$$\Delta^2 v_0 = -153{\cdot}06\,;$$

the resulting table (differing only in the second decimal figure) is given on p. 371 of the paper.

(ii) For $j = 4$.

	$+317616+1726573+6845458+22103236+61585427$				
$12012 v_0$	$+\ 30030-$	$32760+$	$19110-$	$5880+$	756
$12012 \Delta v_0$	$-\ 40040+$	$58240-$	$38220+$	$12544-$	1680
$12012 \Delta^2 v_0$	$+\ 35035-$	$57330+$	$40131-$	$13720+$	1890
$12012 \Delta^3 v_0$	$-\ 20020+$	$34944-$	$25480+$	$8960-$	1260
$12012 \Delta^4 v_0$	$+\ 6006-$	$10920+$	$8190-$	$2940+$	420

This gives $v_0 = 31945{\cdot}93, \quad \Delta v_0 = +361{\cdot}07, \quad \Delta^2 v_0 = -134{\cdot}82,$

$$\Delta^3 v_0 = -18{\cdot}21, \quad \Delta^4 v_0 = +7{\cdot}58;$$

these values differ only in the second decimal figure from those on p. 372 of the paper.

7. We might obtain similar formulæ from the solution, referred to in § 3 above, in terms of central differences. There are two cases, according as $m+1$ is odd or even.

(i) If $m+1 = 2n+1$, the fundamental identities are (see (37) and (38) of the preceding paper)

$$\sum_{r=-n}^{r=n} (r,\, 2h]\, u_r = (4h+2)(n+\tfrac{1}{2},\, 2h+1] \sum_{s=0}^{s=n} \frac{(n+\frac{1}{2},\, 2s]}{2s+2h+1} \delta^{2s} u_0, \tag{16}$$

$$\sum_{r=-n}^{r=n} [r,\, 2h-1)\, u_r = 2(n+\tfrac{1}{2},\, 2h-1] \sum_{s=1}^{s=n} \frac{2s\,(n+\frac{1}{2},\, 2s]}{2s+2h-1} \mu\delta^{2s-1} u_0; \tag{17}$$

and the resulting formulæ are

$(t = 0, 1, 2, \ldots, k)$

$$\delta^{2t} v_0 = (-)^{k-t} \frac{[t+\frac{1}{2},\, k+1]}{t!\,(k-t)!\,(n+\frac{1}{2},\, 2t]} \sum_{s=0}^{s=n} \frac{s(s-1)\ldots(s-k)}{(s-t)[s+\frac{1}{2},\, k+1]} \times (n+\tfrac{1}{2},\, 2s]\,\delta^{2s} u_0, \tag{18}$$

$(t = 1, 2, \ldots, k)$

$$\mu\delta^{2t-1} v_0 = (-)^{k-t} \frac{[t+\frac{1}{2},\, k]}{t!\,(k-t)!\,(n+\frac{1}{2},\, 2t]} \sum_{s=1}^{s=n} \frac{s(s-1)\ldots(s-k)}{(s-t)[s+\frac{1}{2},\, k]} \times (n+\tfrac{1}{2},\, 2s]\,\mu\delta^{2s-1} u_0. \tag{19}$$

Hence, if we write

$$C_{2h} \equiv \sum_{r=-n}^{r=n} (r,\, 2h]\, u_r, \qquad C_{2h-1} = \sum_{r=-n}^{r=n} [r,\, 2h-1)\, u_r, \tag{20}$$

we find that

$(t = 0, 1, 2, \ldots, k)$

$$\delta^{2t} v_0 = \sum_{h=0}^{h=k} (-)^{h+t} \frac{2t+1}{4h+4t+2} \frac{(k,\, t)(k,\, h)[h+\frac{3}{2},\, k][t+\frac{3}{2},\, k]}{(n+\frac{1}{2},\, 2t]\,(n+\frac{1}{2},\, 2h+1]} C_{2h}, \tag{21}$$

$(t = 1, 2, \ldots, k)$

$$\mu\delta^{2t-1} v_0 = \sum_{h=1}^{h=k} (-)^{h+t} \frac{2t+1}{4h+4t-2} \frac{(k,\, t)(k-1,\, h-1)[h+\frac{1}{2},\, k][t+\frac{3}{2},\, k-1]}{(n+\frac{1}{2},\, 2t]\,(n+\frac{1}{2},\, 2h-1]} \times C_{2h-1}. \tag{22}$$

These formulæ are for the case in which $j = 2k$. If $j = 2k+1$, we must replace k in (19) and (22) by $k+1$.

(ii) If $m+1 = 2n$, the fundamental identities are (see (39) and (40) of the preceding paper)

$$\sum_{r=-n+1}^{r=n} [r-\tfrac{1}{2}, 2h)\, u_r = 2(n, 2h] \sum_{s=0}^{s=n-1} \frac{(2s+1)(n, 2s+1]}{2s+2h+1} \mu\delta^{2s} u_{\frac{1}{2}}, \tag{23}$$

$$\sum_{r=-n+1}^{r=n} (r-\tfrac{1}{2}, 2h-1]\, u_r = 4h(n, 2h] \sum_{s=0}^{s=n-1} \frac{(n, 2s+1]}{2s+2h+1} \delta^{2s+1} u_{\frac{1}{2}}; \tag{24}$$

and the resulting formulæ are

$(t = 0, 1, 2, \ldots, k)$

$$\mu\delta^{2t} v_{\frac{1}{2}} = (-)^{k-t} \frac{[t+\frac{3}{2}, k]}{t!\,(k-t)!\,(n, 2t+1]} \sum_{s=0}^{s=n-1} \frac{s(s-1)\ldots(s-k)}{(s-t)[s+\frac{3}{2}, k]} (n, 2s+1]\, \mu\delta^{2s} u_{\frac{1}{2}}, \tag{25}$$

$(t = 1, 2, \ldots, k)$

$$\delta^{2t-1} v_{\frac{1}{2}} = (-)^{k-t} \frac{[t+\frac{1}{2}, k]}{(t-1)!\,(k-t)!\,(n, 2t-1]} \sum_{s=1}^{s=n} \frac{(s-1)(s-2)\ldots(s-k)}{(s-t)[s+\frac{1}{2}, k]} \times (n, 2s-1]\, \delta^{2s-1} u_{\frac{1}{2}}. \tag{26}$$

Hence, if we write

$$D_{2h} \equiv \sum_{r=-n+1}^{r=n} [r-\tfrac{1}{2}, 2h)\, u_r, \qquad D_{2h-1} \equiv \sum_{r=-n+1}^{r=n} (r-\tfrac{1}{2}, 2h-1]\, u_r, \tag{27}$$

we find that

$(t = 0, 1, 2, \ldots, k)$

$$\mu\delta^{2t} v_{\frac{1}{2}} = \sum_{h=0}^{h=k} (-)^{h+t} \frac{2h+1}{4h+4t+2} \frac{(k, t)(k, h)[t+\frac{3}{2}, k][h+\frac{3}{2}, k]}{(n, 2t+1](n, 2h]} D_{2h}, \tag{28}$$

$(t = 1, 2, \ldots, k)$

$$\delta^{2t-1} v_{\frac{1}{2}} = \sum_{h=1}^{h=k} (-)^{h+t} \frac{2h+1}{4h+4t-2} \frac{(k-1, t-1)(k, h)[t+\frac{1}{2}, k][h+\frac{3}{2}, k-1]}{(n, 2t-1](n, 2h]} D_{2h-1}. \tag{29}$$

This is, as in (i), for the case of $j = 2k$; if $j = 2k+1$, we must alter k in (26) and (29) to $k+1$.

8. These formulæ, however, require modification in order to fit them for practical use. The cases to which central-difference methods are applicable are those in which our data are a large number of terms, and our object is, for a succession of values of x, to determine v_x, $\mu\delta v_x$, ... from $u_{x-n}, u_{x-n+1}, \ldots, u_{x+n}$, or $\mu v_{x+\frac{1}{2}}$, $\delta v_{x+\frac{1}{2}}$, ... from $u_{x-n+1}, u_{x-n+2}, \ldots, u_{x+n}$. We therefore require to convert the formulæ already obtained, which would involve u_x, $\mu\delta u_x$, ..., $\delta^{2n}u_x$, or $\mu u_{x+\frac{1}{2}}$, $\delta u_{x+\frac{1}{2}}$, ..., $\delta^{2n-1}u_{x+\frac{1}{2}}$, into formulæ involving successive sums.

In the central-sum notation we have

$$\sigma u_{x+\frac{1}{2}} \equiv \ldots + u_{x-1} + u_x, \qquad \sigma^2 u_x \equiv \ldots + \sigma u_{x-\frac{3}{2}} + \sigma u_{x-\frac{1}{2}}, \ \ldots .$$

These sums involve arbitrary constants, and our formulæ will be such as to make the constants disappear. We have to consider separately the cases of $m+1 = 2n+1$ and $m+1 = 2n$, and we shall, as before, take u_0 to be the middle term, or u_0 and u_1 the two middle terms.

9. First take $m+1 = 2n+1$. Then (see (61) and (66) of the preceding paper)

$$\begin{aligned} C_{2h} = & \sum_{f=0}^{f=h} (n+1,\, 2f]\,(\sigma^{2h-2f+1}u_{n+\frac{1}{2}} - \sigma^{2h-2f+1}u_{-n-\frac{1}{2}}) \\ & - \sum_{f=1}^{f=h} (n+\tfrac{1}{2},\, 2f-1](\sigma^{2h-2f+2}u_{n+1} + \sigma^{2h-2f+2}u_{-n-1}), \end{aligned} \tag{30}$$

$$\begin{aligned} C_{2h-1} = & \sum_{f=1}^{f=h} [n+\tfrac{1}{2},\, 2f-2)(-\sigma^{2h-2f+2}u_{n+1} + \sigma^{2h-2f+2}u_{-n-1}) \\ & + \sum_{f=1}^{f=h} [n+1,\, 2f-1)(\sigma^{2h-2f+1}u_{n+\frac{1}{2}} + \sigma^{2h-2f+1}u_{-n-\frac{1}{2}}). \end{aligned} \tag{31}$$

Substituting in (21) and (22), and rearranging the terms,

$(t = 0, 1, 2, \ldots, k)$

$$\begin{aligned} \delta^{2t}v_0 = & \sum_{f=0}^{f=k} \left\{ \sum_{h=f}^{h=k} (-)^{h+t} \frac{2t+1}{4h+4t+2} \frac{(k,\, t)(k,\, h)[h+\frac{3}{2},\, k][t+\frac{3}{2},\, k]}{(n+\frac{1}{2},\, 2t](n+\frac{1}{2},\, 2h+1]} \right. \\ & \left. \times (n+1,\, 2h-2f] \right\} (\sigma^{2f+1}u_{n+\frac{1}{2}} - \sigma^{2f+1}u_{-n-\frac{1}{2}}) \\ & - \sum_{f=1}^{f=k} \left\{ \sum_{h=f}^{h=k} (-)^{h+t} \frac{2t+1}{4h+4t+2} \frac{(k,\, t)(k,\, h)[h+\frac{3}{2},\, k][t+\frac{3}{2},\, k]}{(n+\frac{1}{2},\, 2t](n+\frac{1}{2},\, 2h+1]} \right. \\ & \left. \times (n+\tfrac{1}{2},\, 2h-2f+1] \right\} (\sigma^{2f}u_{n+1} + \sigma^{2f}u_{-n-1}), \end{aligned} \tag{32}$$

$(t = 1, 2, \ldots, k)$

$$\mu\delta^{2t-1}v_0 = \sum_{f=1}^{f=k}\left\{\sum_{h=f}^{h=k}(-)^{h+t}\frac{2t+1}{4h+4t-2}\,\frac{(k,t)(k-1,h-1)[h+\tfrac{1}{2},k][t+\tfrac{3}{2},k-1]}{(n+\tfrac{1}{2},2t](n+\tfrac{1}{2},2h-1]}\right.$$

$$\left.\times[n+\tfrac{1}{2},2h-2f)\right\}(-\sigma^{2f}u_{n+1}+\sigma^{2f}u_{-n-1})$$

$$+\sum_{f=1}^{f=k}\left\{\sum_{h=f}^{h=k}(-)^{h+t}\frac{2t+1}{4h+4t-2}\,\frac{(k,t)(k-1,h-1)[h+\tfrac{1}{2},k][t+\tfrac{3}{2},k-1]}{(n+\tfrac{1}{2},2t](n+\tfrac{1}{2},2h-1]}\right.$$

$$\left.\times[n+1,2h-2f+1)\right\}(\sigma^{2f-1}u_{n+\frac{1}{2}}+\sigma^{2f-1}u_{-n-\frac{1}{2}}). \quad (33)$$

10. Next take $m+1 = 2n$. Then (see (64) and (67) of the preceding paper)

$$D_{2h} = \sum_{f=0}^{f=h}[n+\tfrac{1}{2},2f)(\sigma^{2h-2f+1}u_{n+\frac{1}{2}}-\sigma^{2h-2f+1}u_{-n+\frac{1}{2}})$$

$$-\sum_{f=1}^{f=h}[n,2f-1)(\sigma^{2h-2f+2}u_{n+1}+\sigma^{2h-2f+2}u_{-n}), \quad (34)$$

$$D_{2h-1} = \sum_{f=1}^{f=h}(n,2f-2](-\sigma^{2h-2f+2}u_{n+1}+\sigma^{2h-2f+2}u_{-n})$$

$$+\sum_{f=1}^{f=h}(n+\tfrac{1}{2},2f-1](\sigma^{2h-2f+1}u_{n+\frac{1}{2}}+\sigma^{2h-2f+1}u_{-n+\frac{1}{2}}). \quad (35)$$

Substituting in (28) and (29),

$(t = 0, 1, 2, \ldots, k)$

$$\mu\delta^{2t}v_{\frac{1}{2}} = \sum_{f=0}^{f=k}\left\{\sum_{h=f}^{h=k}(-)^{h+t}\frac{2h+1}{4h+4t+2}\,\frac{(k,t)(k,h)[t+\tfrac{3}{2},k][h+\tfrac{3}{2},k]}{(n,2t+1](n,2h]}\right.$$

$$\left.\times[n+\tfrac{1}{2},2h-2f)\right\}(\sigma^{2f+1}u_{n+\frac{1}{2}}-\sigma^{2f+1}u_{-n+\frac{1}{2}})$$

$$-\sum_{f=1}^{f=k}\left\{\sum_{h=f}^{h=k}(-)^{h+t}\frac{2h+1}{4h+4t+2}\,\frac{(k,t)(k,h)[t+\tfrac{3}{2},k][h+\tfrac{3}{2},k]}{(n,2t+1](n,2h]}\right.$$

$$\left.\times[n,2h-2f+1)\right\}(\sigma^{2f}u_{n+1}+\sigma^{2f}u_{-n}), \quad (36)$$

$(t = 1, 2, \ldots, k)$

$$\delta^{2t-1} v_{\frac{1}{2}} = \sum_{f=1}^{f=k} \left\{ \sum_{h=f}^{h=k} (-)^{h+t} \frac{2h+1}{4h+4t-2} \frac{(k-1, t-1)(k, h)[t+\frac{1}{2}, k][h+\frac{3}{2}, k-1]}{(n, 2t-1](n, 2h]} \times (n, 2h-2f] \right\} (-\sigma^{2f} u_{n+1} + \sigma^{2f} u_{-n})$$

$$+ \sum_{f=1}^{f=k} \left\{ \sum_{h=f}^{h=k} (-)^{h+t} \frac{2h+1}{4h+4t-2} \frac{(k-1, t-1)(k, h)[t+\frac{1}{2}, k][h+\frac{3}{2}, k-1]}{(n, 2t-1](n, 2h]} \times (n+\tfrac{1}{2}, 2h-2f+1] \right\} (\sigma^{2f-1} u_{n+\frac{1}{2}} + \sigma^{2f-1} u_{-n+\frac{1}{2}}). \quad (37)$$

11. The following are particular cases of the formulæ of § 9.

(i) $j = 1$ or 2.

$$n(n+1)(2n+1)\,\mu\delta v_0 = -3\,(\sigma^2 u_{n+1} - \sigma^2 u_{-n-1}) + 3\,(n+1)\,(\sigma u_{n+\frac{1}{2}} + \sigma u_{-n-\frac{1}{2}}).$$

(ii) $j = 2$ or 3.

$$\begin{aligned}(2n-1)(2n+1)(2n+3)\, v_0 = &-30\,(\sigma^3 u_{n+\frac{1}{2}} - \sigma^3 u_{-n-\frac{1}{2}}) \\ &+ 15(2n+1)(\sigma^2 u_{n+1} + \sigma^2 u_{-n-1}) \\ &- 3(n+2)(2n+3)(\sigma u_{n+\frac{1}{2}} - \sigma u_{-n-\frac{1}{2}}),\end{aligned}$$

$$\begin{aligned}n(n+1)(2n-1)(2n+1)(2n+3)\,\delta^2 v_0 \\ = &+180(\sigma^3 u_{n+\frac{1}{2}} - \sigma^3 u_{-n-\frac{1}{2}}) \\ &- 90\,(2n+1)(\sigma^2 u_{n+1} + \sigma^2 u_{-n-1}) \\ &+ 30\,(n+1)(2n+3)\,(\sigma u_{n+\frac{1}{2}} - \sigma u_{-n-\frac{1}{2}}).\end{aligned}$$

(iii) $j = 3$ or 4.

$$\begin{aligned}n(n+1)(2n-1)(2n+1)(2n+3)\,\mu\delta v_0 \\ = &+630\,(\sigma^4 u_{n+1} - \sigma^4 u_{-n-1}) \\ &- 630(n+1)(\sigma^3 u_{n+\frac{1}{2}} + \sigma^3 u_{-n-\frac{1}{2}}) \\ &+ 15(16n^2+16n+9)(\sigma^2 u_{n+1} - \sigma^2 u_{-n-1}) \\ &- 15(n+1)(n+3)(2n+3)(\sigma u_{n+\frac{1}{2}} + \sigma u_{-n-\frac{1}{2}}),\end{aligned}$$

$$(n-1)n(n+1)(n+2)(2n-1)(2n+1)(2n+3)\,\mu\delta^3 v_0$$
$$= -6300\,(\sigma^4 u_{n+1}-\sigma^4 u_{-n-1})$$
$$+6300\,(n+1)(\sigma^3 u_{n+\frac{1}{2}}+\sigma^3 u_{-n-\frac{1}{2}})$$
$$-1260(2n^2+2n+1)(\sigma^2 u_{n+1}-\sigma^2 u_{-n-1})$$
$$+210(n+1)(n+2)(2n+3)(\sigma u_{n+\frac{1}{2}}+\sigma u_{-n-\frac{1}{2}}).$$

(iv) $j=4$ or 5.

$$(2n-3)(2n-1)(2n+1)(2n+3)(2n+5)\,v_0$$
$$= +5670\,(\sigma^5 u_{n+\frac{1}{2}}-\sigma^5 u_{-n-\frac{1}{2}})$$
$$-2835\,(2n+1)(\sigma^4 u_{n+1}+\sigma^4 u_{-n-1})$$
$$+105\,(22n^2+49n+39)\,(\sigma^3 u_{n+\frac{1}{2}}-\sigma^3 u_{-n-\frac{1}{2}})$$
$$-210\,(2n+1)(n^2+n+3)(\sigma^2 u_{n+1}+\sigma^2 u_{-n-1})$$
$$+\tfrac{15}{2}(n+2)(n+3)(2n+3)(2n+5)(\sigma u_{n+\frac{1}{2}}-\sigma u_{-n-\frac{1}{2}}),$$

$$n(n+1)(2n-3)(2n-1)(2n+1)(2n+3)(2n+5)\,\delta^2 v_0$$
$$= -113400\,(\sigma^5 u_{n+\frac{1}{2}}-\sigma^5 u_{-n-\frac{1}{2}})$$
$$+56700\,(2n+1)(\sigma^4 u_{n+1}+\sigma^4 u_{-n-1})$$
$$-1260\,(38n^2+83n+60)\,(\sigma^3 u_{n+\frac{1}{2}}-\sigma^3 u_{-n-\frac{1}{2}})$$
$$+630\,(2n+1)(8n^2+8n+15)(\sigma^2 u_{n+1}+\sigma^2 u_{-n-1})$$
$$-210\,(n+1)(n+3)(2n+3)(2n+5)(\sigma u_{n+\frac{1}{2}}-\sigma u_{-n-\frac{1}{2}}),$$

$$(n-1)n(n+1)(n+2)(2n-3)(2n-1)(2n+1)(2n+3)(2n+5)\,\delta^4 v_0$$
$$= +1587600\,(\sigma^5 u_{n+\frac{1}{2}}-\sigma^5 u_{-n-\frac{1}{2}})$$
$$-793800\,(2n+1)(\sigma^4 u_{n+1}+\sigma^4 u_{-n-1})$$
$$+113400\,(6n^2+13n+9)(\sigma^3 u_{n+\frac{1}{2}}-\sigma^3 u_{-n-\frac{1}{2}})$$
$$-37800\,(2n+1)(2n^2+2n+3)(\sigma^2 u_{n+1}+\sigma^2 u_{-n-1})$$
$$+3780\,(n+1)(n+2)(2n+3)(2n+5)(\sigma u_{n+\frac{1}{2}}-\sigma u_{-n-\frac{1}{2}}).$$

12. The above formulæ are specially important for the purpose of reduction of errors of the terms of a sequence, where the errors are all independent and all have the same mean square. I have shown, in the

paper already referred to, that in such a case the best value obtainable for any one term u_0, by linear combination of the observed value with n values on each side of it, is—for a particular definition of "best"—the same as the value given by a polynomial fitted by the method of least squares; *i.e.*, is the value represented by v_0 above. The process of reduction of error therefore consists in constructing the successive sums σu, $\sigma^2 u$, $\sigma^3 u$, ..., and applying the appropriate formula of § 11.

For any specified value of n, the formula for v_0 may be altered, if necessary, so as to obtain an equivalent formula convenient for numerical calculation. For $n = 6$, for instance, if $j = 2$ or 3, we have (replacing u_0 and v_0 by u_x and v_x)

$$143v_x = -2(\sigma^3 u_{x+5\frac{1}{2}} - \sigma^3 u_{x-5\frac{1}{2}}) + 11(\sigma^2 u_{x+6} + \sigma^2 u_{x-6}) - 11(\sigma u_{x+6\frac{1}{2}} - \sigma u_{x-6\frac{1}{2}}),$$

by means of which the tabular calculation of $143v_x$, and thence of v_x, is very simple.

ON INTEGRATION WITH RESPECT TO A FUNCTION OF BOUNDED VARIATION

By Prof. W. H. YOUNG.

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1. Stieltjes* was the first to introduce into analysis the concept of the integral of a function with respect to another function, which is not even necessarily continuous. The important use which he made of this notion shows that it is one which arises in the natural course of events. In Stieltjes' treatment, however, the function to be integrated is continuous, while the function with respect to which integration is performed is a monotone increasing function. The integral of a function of the latter class with respect to a continuous function is then defined by means of a formula equivalent to that of integration by parts.

Lebesgue† has considered the question as to the possibility of extending the concept of Stieltjes in such a manner as to embrace the integration of all bounded summable functions with respect to a function of bounded variation, or, which comes to the same thing, with respect to a monotone increasing function. He gives, in fact, a definition of such integration, based on a change of the independent variable, involving considerations of a delicate nature, and he expresses the opinion that it would be difficult to construct a definition of a different kind.‡

In the present paper I propose to show that two of the methods I have already sketched for the theory of integration with respect to a continuous variable are applicable, almost as they stand, when this variable is replaced by a discontinuous function of bounded variation. It thus appears that *mutatis mutandis* all the theorems in the ordinary theory are true in

* "Recherches sur les fractions continues," 1894, *Annales de la Faculté de Science de Toulouse*, Ser. 1, Vol. 8, pp. 1–122.

† "Les intégrales de Stieltjes," 1910, *Comptes Rendus*, Vol. 150, pp. 86–88.

‡ He terminates his note with the words "On défine en somme l'intégrale de Stieltjes pour $f(x)$ sommable borné et $g(x)$ à variation bornée, ce qui parait difficile de faire sans changement de variable."

the more general one. Moreover, we are able to employ almost precisely the same reasoning. Indeed we may say that the only difference consists in the somewhat greater generality in the preliminary theorem, which forms the base of the definition of the integral of a semi-continuous function. I take, however, the opportunity to supply proofs of the subsequent theorems, where I have up to the present nowhere published them.

As to the importance of the results thus shown to be true, it will be sufficient perhaps to point out that the use of the new concept, in the most general form here given to it, appears to be indispensable in the theory of Fourier series and kindred subjects. The trigonometrical cosine series whose coefficients are all unity is the derived series of the Fourier series of a function of bounded variation, a function, moreover, which is not continuous. Such a series is the trivial example of a whole class of trigonometrical series, which, though they are not Fourier series, have some remarkable properties in common with Fourier series, and it is precisely by means of the process of term-by-term integration of Fourier series with respect to a function of bounded variation that these properties are obtained.*

The further question then presents itself as to the possibility of replacing the function of bounded variation by a more general function, and in particular by any continuous function. Except in the special case when the function to be integrated is a function of bounded variation difficulties of a kind analogous to those which arise in the theory of non-absolutely convergent integrals now present themselves. It is more convenient therefore to consider such possible extensions in connection with that subject.†

2. The methods I wish to explain are based on the theory of monotone sequences. I have given a sketch of the first of these methods as applied to the theory of Lebesgue integration in a paper in the *Proceedings of the Royal Society*, entitled "On the New Theory of Integration,"‡ while the

* W. H. Young, "On Fourier Series and Functions of Bounded Variation," 1913, *Roy. Soc. Proc.*, A, Vol. 89, pp. 150–157; "On the Usual Convergence of a Class of Trigonometrical Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 13, pp. 13–28; "On Trigonometrical Series whose Cesàro Partial Summations Oscillate Finitely," *Roy. Soc. Proc.*, A, Vol. 88, pp. 561–568.

† Various circumstances have prevented me from composing the present paper myself. The substance of it only was given to my wife, who has kindly put it into form. The careful elaboration of the argument is due to her.

‡ *Roy. Soc. Proc.*, A, Vol. 88, 1912, pp. 170–178.

second method in the same connexion is indicated in an earlier paper published in the *Proceedings* of this Society,* entitled "On a New Method in the Theory of Integration." Both these methods have in the present instance distinct advantages. In the earlier paper the definition of the integral of a continuous function is supposed already given, and it is then shown how that of another function may be based upon it. In the later paper an independent definition is constructed, continuous functions being regarded merely as a special case of other functions, which may be traced out by successive monotone sequences of functions, which are in the first instance simple functions, constant in a finite number of stretches.

In the present theory both these methods have their respective advantages, and it is not difficult, if desired, to construct a method which combines the peculiarities of the two. I shall refer to these methods by letter; that which corresponds to that in my later paper I shall denote by (A), that in the earlier paper by (B), and that corresponding to a combination of the two methods by (C). I shall consider these methods in alphabetical order.

3. Let $g(x)$ be a positive monotone increasing function throughout the open or closed interval (a, b) considered, with a finite upper bound M. Then at every point $g(x)$ has a definite limit on the right and on the left $g(x+0)$ and $g(x-0)$, where

$$g(x-0) \leqslant g(x) \leqslant g(x+0). \tag{1}$$

At a point of continuity of $g(x)$ these three quantities are equal. At a discontinuity their differences form the three discontinuities, namely the (total) discontinuity w_x which is the difference between the extremes in (1), and the right- and left-hand discontinuities,

$$w_{x+} = g(x+0)-g(x) \quad \text{and} \quad w_{x-} = g(x)-g(x-0).$$

The discontinuities of $g(x)$ are countable, and we shall suppose them arranged in the usual way in countable order, namely, so that every larger quantity comes before every smaller one, and of two equal quantities, that one comes first which corresponds to a point lying to the left of the point corresponding to the other. The sum of the discontinuities is convergent, since it cannot exceed M. Hence, given e, we can determine an integer k, the least integer such that the sum of all but the first k discontinuities of

* *Proc. London Math. Soc.*, Ser. 2, Vol. 9, 1910, pp. 15–50.

$g(x)$ is less than or equal to e. For brevity we shall speak of the first k discontinuities as "the k points."

The function $f(x)$ which we propose to integrate with respect to $g(x)$ is not necessarily either continuous or of bounded variation. Its discontinuities are accordingly not in general countable, and at each point of discontinuity we may have a plurality of limits on each side, which, however, are the same on the right and on the left, except at a countable set of points. Instead of the three quantities therefore which occur in the case of $g(x)$ we have to consider five quantities. In the present connexion, however, we do not need to consider the differences between all these quantities, but only the greatest of these differences, which will accordingly be called *the discontinuity at the point* x, and denoted by w_x; this is the upper double limit of $|f(x+h)-f(x+k)|$ as h and k approach zero, the value zero being included in the range of values of h and of k.

4. The integrands with which we start in the method (A) are functions which are constant in each of a finite number of completely open intervals into which the interval (a, b) of integration is divided by a finite number of points of division. At each point of division the value of the function is either always less than or equal to either of the values in the neighbourhood of the point considered, in which case the function is called a simple l-function, or the value of the function at a point of division is always greater than or equal to either of the values in the neighbourhood, in which case the function is called a simple u-function.

The graphical representation of these functions is a broken line, consisting of parallels to the axis of x, and dots at the points of division. The integral of such a function from a to b with respect to the independent variable x is the sum of the areas of the rectangles bounded by the parts of the broken line, the axis of x and the ordinates at the points of division, including a and b. Starting from such simple integrals the whole theory of integration follows by the Method of Monotone Sequences.

The principle used in forming the above integrals of the simple functions is that the integral over (a, b) is the sum of the integrals of the function taken over a finite number of completely open intervals which with their end-points fill up (a, b); the integral of a function over an open interval throughout which it is constant being the product of the length of the interval into the value of the function, that is the product of the value of the function into the increment of the independent variable.

The somewhat summary and arbitrary manner in which we have here

neglected the points of division is only justified by the fact that the increment of the independent variable at a single point is zero. The same is true if we integrate with respect to any continuous function. If, however, we integrate with respect to a monotone increasing function $g(x)$, and the end-points of the interval are points of discontinuity of $g(x)$, or this is the case at one of them, the increment of $g(x)$ at such an end-point is a positive quantity, and, when multiplied by the value of the function to be integrated at the point contributes a term which is as important as that contributed by the corresponding product in the case of the completely open interval.

Thus in the division of (a, b) into a finite number of parts, we are led to regard these parts as consisting of completely open intervals and of points which together fill up (a, b), and over each of which the function to be integrated is constant. In other words, if $f(x)$ is a simple l- or u-function, whose discontinuities are among the points of division

$$a = c_1,\ c_2,\ \ldots,\ c_{n-1},\ c_n = b,$$

$$\int_a^b f(x)\, dg(x) = \sum_{i=1}^{i=n-1} \{f(c_i)[g(c_i+0)-g(c_i)]+f(c_i+0)[g(c_{i+1}-0)-g(c_i+0)] \\ +f(c_{i+1})[g(c_{i+1})-g(c_{i+1}-0)]\}. \quad \text{(I)}$$

5. Starting with simple l- and u-functions, we now build up the theory by the Method of Monotone Sequences. The underlying principle is the following :—

(I) *A function is said to have an integral with respect to a monotone increasing function $g(x)$, if it can be expressed as the limit (finite or infinite with determinate sign) of a monotone succession of functions, belonging to a class of functions whose integrals have already been defined with respect to $g(x)$, provided only that the limit of the integrals of the functions of every such succession is the same, and this limit is then called the integral of the given function with respect to $g(x)$.*

It is convenient to add the gloss that the integral is not to be regarded as existing unless it is finite.

6. The functions which enter will therefore be included among those already discussed obtained from simple l- and u-functions by the Method of Monotone Sequences. It will be seen from the discussion given in the present paper (§ 17) that all such functions, provided they are bounded functions, have integrals with respect to any monotone increasing function $g(x)$.

The limit of a monotone ascending sequence of simple l-functions is a general l-function, which is the same as the lower semi-continuous function of Baire, and the limit of a descending sequence of simple u-functions is a general u-function, which is the upper semi-continuous function of Baire. The other kinds of monotone sequences of simple l- and u-functions do not have to be considered separately; they lead to the same limiting functions as similar sequences of general l- and u-functions.

The nomenclature which I have introduced now serves for a *memoria technica* of the facts underlying it. The letter l prefixed to the name of a function denotes the limit of a monotone ascending sequence of such functions, and the letter u prefixed denotes the limit of a monotone descending sequence. Two adjacent l's or u's are equivalent to one only. Thus we have for the limits of monotone sequences of l- and u-functions l's and u's together with two further types of functions, lu's and ul's. Monotone sequences of these lead to functions already considered, and two new types, lul's and ulu's, and so on.

The nomenclature fails, and the properties of the functions become more complicated, when we get to transfinite orders of repetitions of monotone sequences.

7. The theorem on which the above classification of functions is based is that* which states that the limit of a monotone ascending (descending) sequence of l-functions (u-functions) is a function of the same type. That on which the theory of the integration of these functions, whether with respect to the independent variable or with respect to a monotone increasing function, depends is closely connected with the above theorem, and states that if we have a sequence of the type specified, the upper and lower bounds of the constituent functions in any closed interval themselves describe sequences which are monotone in the same sense as the original sequence, and have for limit respectively the upper and lower bounds of the limiting function.† This latter theorem I call the Theorem of the Bounds.

8. In order to pass from the integrals of simple l- and u-functions, already defined, to those of general l- and u-functions, whether, as we shall take in the first instance (§§ 8–23) these be bounded, or unbounded, but finite at the discontinuities of $g(x)$, by the Method of Monotone

* W. H. Young, "On Functions Defined by Monotone Sequences and their Upper and Lower Bounds," 1908, *Messenger of Mathematics*, pp. 148–154.

† *Ibid.*, p. 149.

Sequences, in accordance with the principle laid down in § 5, we now require the following theorem :—

THEOREM.—*Given two bounded monotone ascending sequences of simple l-functions with the same limiting function $f(x)$, the integrals of the two sequences with respect to a given positive monotone increasing function $g(x)$ have the same limit.*

We need evidently only deal with the case when the functions of the sequence are all positive ($\geqslant 0$).

Let
$$b_1 \leqslant b_2 \leqslant \dots \to f,$$
and
$$f_1 \leqslant f_2 \leqslant \dots \to f,$$
be the two sequences, B and F the limits of the integrals of the constituent functions with respect to $g(x)$. B and F are of course finite, since the sequences are bounded.

Let B' denote any quantity less than B, and let us determine N so that
$$\int_a^b b_N(x)\, dg(x) \geqslant B',$$
and so that the values of b_N at the k points already considered (§ 3) differ from the corresponding values of $f(x)$ by less than e_1; this latter is possible since the values of $f(x)$ at the discontinuities of $g(x)$ are certainly finite, $f(x)$ being, like the sequences, bounded.

Let us divide the interval (a, b) up at the points of discontinuity of $b_N(x)$, r_N say in number. The integral considered will then be (§ 4) the sum of terms due to the completely open intervals, at most r_N+1 in number, into which (a, b) is divided, and terms due to the remaining points, viz., the r_N points themselves.

Let d be one of these completely open intervals, h the corresponding increment of $g(x)$, and L the lower bound of $f(x)$ in the open interval. Then, throughout the interval d, since $b_N(x)$ is constant and $\leqslant f(x)$,
$$b_N(x) \leqslant L,$$
so that
$$\int_d b_N(x)\, dg(x) \leqslant Lh.$$
Take a fixed closed interval D inside d, so that the increment of $g(x)$ over this closed interval differs from h by less than $e_1 h/L$; this is possible, since the increment over an open interval is defined to be the upper bound of the increment over closed intervals in the open interval. The lower bound of $f(x)$ in D is of course $\geqslant L$. By the Theorem of the Bounds

(§ 7) we can determine n_0, so that, for $n \geqslant n_0$, the lower bound of $f_n(x)$ in this closed interval differs from that of $f(x)$ by less than e_1, and is therefore $\geqslant L - e_1$. Thus

$$\int_d f_n(x)\,dg(x) \geqslant \int_D f_n(x)\,dg(x) \geqslant (L-e_1)(h - e_1 h/L) \geqslant Lh - 2e_1 h.$$

Hence $$\int_d f_n(x)\,dg(x) - \int_d b_N(x)\,dg(x) \geqslant -2e_1 h \;\ldots\; (n \geqslant n_0).$$

Corresponding to each of the intervals we get such an inequality. Let n_1 denote the greatest of the corresponding integers n_0. Then, summing over all the intervals, we get for the part of the difference between the integrals of f_n and b_N due to the intervals a quantity greater than

$$-2e_1 M, \tag{1}$$

provided $n \geqslant n_1$.

We have next to consider the contribution due to the points of division. Let x be one of these; the corresponding quota to the difference of the integrals is

$$w_x\,[f_n(x) - b_N(x)].*$$

Now we can find $n_2 \geqslant n_1$, so that, for $n \geqslant n_2$, the values of $f_n(x)$ at the k points differ from the corresponding values of $f(x)$ by less than e_1, and therefore differ from the corresponding values of $b_N(x)$ by less than $2e_1$. Thus, if our point of division x is one of the k points, the corresponding quota is greater than

$$-2e_1 w_x,$$

and the sum of all such terms is therefore greater than

$$-2e_1 M, \tag{2}$$

provided $n \geqslant n_2$.

If, however, our point of division x is one of the remaining discontinuities of $g(x)$, the corresponding quota is numerically less than $w_x Q$, where Q is a finite quantity depending on the two sequences, and the sum of all such is accordingly numerically less than

$$eQ. \tag{3}$$

If the point of division is a point at which $g(x)$ is continuous, it contributes

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* If a and b are among the points at which $b_N(x)$ is discontinuous, w_x is to be replaced by w_{x+} for a, or w_{x-} for b.

of course nothing. Thus, finally, adding our results (1), (2) and (3),

$$\int_a^b f_n(x)\,dg(x)-\int_a^b b_N(x)\,dg(x) \geqslant -4e_1 M - eQ \quad (n \geqslant n_2).$$

Here the right-hand side is numerically as small as we please, and the second integral on the left is $\geqslant B'$. Hence

$$\int_a^b f_n(x)\,dg(x)-B' \geqslant 0 \quad (n \geqslant n_2).$$

Proceeding to the limit, and remembering that B' is any quantity less than B,

$$F \geqslant B.$$

Similarly, reversing the rôles of the two sequences,

$$B \geqslant F;$$

and therefore

$$B = F,$$

which proves the theorem.

9. Similarly we can prove the following alternative theorem :—

THEOREM.—*Given any two bounded monotone descending sequences of simple u-functions with the same limiting function, the integrals of the sequences with respect to a positive monotone increasing function $g(x)$ have the same limit.*

Again subtracting corresponding members of the two sequences, and remembering that a succession of zeros constitutes a monotone sequence of simple l- or u-functions with zero as limit, we get as an immediate corollary the following :—

THEOREM.—*If a bounded monotone ascending sequence of simple l-functions and a bounded monotone descending sequence of simple u-functions have the same limiting function, which is then of course continuous, the limits of the integrals with respect to $g(x)$ of the two sequences are the same.*

10. The preceding theorems form the basis of the theory of the integration of semi-continuous functions with respect to positive monotone increasing functions, and hence with respect to functions of bounded variation.

DEFINITION.—*The integral of a bounded l-function (u-function) with respect to a positive monotone increasing function is defined to be the*

limit of the integrals of any monotone ascending (descending) sequence of simple l-functions (u-functions) having the given l-function (u-function) for limit.

With this definition the usual properties of integrals immediately follow, as long as the functions integrated are semi-continuous functions.

We shall not here verify the fact that all monotone sequences of simple l- and u-functions whose limits are l- and u-functions may be integrated term-by-term. This proposition is, however, included in that of § 12.

11. We shall require the following lemma in the proof of the theorem of the next article.

Lemma.—*If $f(x)$ is a bounded l-function, it is representable as the limit of an increasing sequence of simple l-functions, and as the limit of such a sequence of simple u-functions.*

It is evidently only necessary to discuss the case when $f(x) \geqslant 0$, since, if this is not the case, it becomes so by adding a constant.

Divide the fundamental segment (a, b) in successive stages into $r_1, r_2, \ldots$ parts, where $r_1, r_2, \ldots$ is a succession of continually increasing integers.

The points of division used at one stage remain points of division at all subsequent stages, and ultimately are to become dense everywhere.

Consider the k-th stage. In each of the (r_k+1) parts into which the segment (a, b) is divided, we put the auxiliary function $f_k(x)$ equal to the lower bound of $f(x)$ in that part, including its end-points. At the points of division we assign the lesser (greater) of the two values in the adjacent parts. Thus

$$f_k(x) \leqslant f(x). \tag{1}$$

These auxiliary functions evidently form a monotone increasing sequence,

$$f_1(x) \leqslant f_2(x) \leqslant \ldots,$$

each of them is constant in a finite number of stretches, and, at the points of division is lower (upper) semi-continuous; that is, each is a simple l-function (u-function).

Now in any closed segment there is a point at which the lower semi-continuous function $f(x)$ assumes its lower bound. Thus, if z be any point, there will, by the definition of the function $f_k(x)$, be a point x_k in the segment, or one of the two segments, containing the point z, at which $f(x)$ has the value $f_k(z)$; the point x_k may, or may not, coincide with the

point z, but, in any case, as $k \to \infty$, the points x_k have the point z as limit. Consequently, $f(x)$ being a lower semi-continuous function,

$$f(z) \leqslant \underset{k \to \infty}{\mathrm{Lt}} f(x_k) \leqslant \underset{k \to \infty}{\mathrm{Lt}} f_k(z) = \underset{k \to \infty}{\mathrm{Lt}} f_k(z),$$

using (1). This proves the Lemma.

COR.—*The monotone sequences so constructed may be integrated term-by-term.*

It is clear that we only have to consider the values of the corresponding simple l- and u-functions at the points of division, since in the completely open segments between these the functions at each stage are equal. A point of division which is not a point of discontinuity of $g(x)$ contributes zero to both integrals. Let us, as in § 3, divide up the points of discontinuities of $g(x)$ into the k-points and the remaining points of discontinuity. These latter contribute less than Be to each integral, where B is a finite constant. At each of the k points the two sequences have the same finite limit, therefore we can confine our attention to such functions of the successions as differ at each of the k points by less than e/k; the contribution of the k-points to the corresponding integrals will then be less than Me. Hence the integrals of corresponding functions of the two sequences differ by less than $(B+M)e$, which is as small as we please. Thus the limits of the integrals of the two sequences differ by as little as we please, and are therefore the same. By definition the limit in the case of the sequence of simple l-functions is the integral of $f(x)$; this therefore proves the corollary.

Similarly, or as a direct corollary from the above Lemma, we have the alternative Lemma and Corollary:—

LEMMA.—*If $f(x)$ is a bounded u-function, it is representable as the limit of a decreasing sequence of simple u-functions and as the limit of such a sequence of simple l-functions.*

COR.—*These sequences may be integrated term-by-term.*

12. THEOREM.—*Bounded monotone sequences of l-functions (u-functions), whose limits are l-functions or u-functions, may be integrated term-by-term with respect to a positive monotone increasing function $g(x)$.*

We shall as usual assume the functions to be all positive for the purposes of proof.

CASE 1.—Let $b_1 \leqslant b_2 \leqslant \dots$ be a monotone ascending sequence of l-

functions; the limiting function $b(x)$ is then an l-function. Each b_n is then the limit of a monotone ascending sequence of simple l-functions, and we may so choose these that they form a doubly monotone scheme of functions, as follows:—

$$\begin{array}{ccccccccc} b_{11} & \leqslant & b_{12} & \leqslant & b_{13} & \leqslant & \dots & \to & b_1, \\ \wedge\!\!\backslash & & \wedge\!\!\backslash & & \wedge\!\!\backslash & & & & \wedge\!\!\backslash \\ b_{21} & \leqslant & b_{22} & \leqslant & b_{23} & \leqslant & \dots & \to & b_2, \\ \dots & & \dots & & \dots & & & & \dots \end{array}$$

In fact, if b_{21} is not originally $\geqslant b_{11}$ everywhere, we need only replace b_{11} by the function which is at every point equal to the lesser of b_{11} and b_{21}. The new function will still be a simple l-function and will be $\leqslant b_{12}$ as well as $\leqslant b_{21}$. If we proceed in like manner to adjust successively the functions b_{1r} in the first row, no value having been introduced except values in the second row, the limiting function of the first row will be at every point b_1 or b_2, and must therefore be b_1, since we have nowhere raised the value of a function in the first row. Similarly we adjust each row in turn, till we have our scheme, as given above, in which each row and each column constitutes a monotone ascending sequence.

It now follows immediately that

$$b_{11} \leqslant b_{22} \leqslant \dots \leqslant b_{nn} \leqslant \dots \to b.$$

Hence by the definition of the integral of an l-function,

$$\int b(x)\,dg(x) = \mathop{\mathrm{Lt}}_{n\to\infty} \int b_{nn}(x)\,dg(x).$$

But since b_{nn} and b_n are both l-functions, and the former is not greater than the latter,

$$\int b_{nn}(x)\,dg(x) \leqslant \int b_n(x)\,dg(x).$$

Therefore
$$\mathop{\mathrm{Lt}}_{n\to\infty} \int b_n(x)\,dg(x) \geqslant \int b(x)\,dg(x).$$

But since b_n and b are both l-functions, and the former is not greater than the latter, the sign $>$ in the preceding inequality is inadmissible. Thus

$$\mathop{\mathrm{Lt}}_{n\to\infty} \int b_n(x)\,dg(x) = \int b(x)\,dg(x).$$

This proves the theorem in this case.

CASE 2.—Let the given sequence be a monotone descending sequence of u-functions.

This may be treated in the same way as Case 1, or it may be deduced

from Case 1, since by changing the signs of all the functions and adding a suitable constant, the same for all the functions, we make all the functions positive l-functions.

CASE 3.—Let $f_1 \leqslant f_2 \leqslant \dots$ be a bounded monotone ascending sequence of u-functions, whose limiting function $f(x)$ is also an u-function.

We may, by the preceding Lemma, regard each function f_n as the limit of a monotone descending sequence of simple l-functions, which may be integrated term-by-term (§ 11). From among these simple l-functions we choose one, say $b_n(x)$, whose integral with respect to $g(x)$ differs from that of $f_n(x)$ by less than $2^{-n-1}e$.* The succession of functions $b_1, b_2, \dots$ is not necessarily monotone increasing, but we can change it into such a sequence as follows. Wherever $b_1 > b_2$, we replace the value of b_2 by that of b_1, retaining the values of b_2 at the remaining points. Let the modified function so obtained be $c_2(x)$. Then c_2 will still be an l-function, and it will still be $\geqslant f_2$. We then modify b_3, and so on. We thus get a monotone ascending sequence of l-functions,

$$c_1 = b_1 \leqslant c_2 \leqslant c_3 \leqslant \dots,$$

where for all values of n,

$$c_n(x) \geqslant f_n(x); \tag{1}$$

and therefore, since both these functions are l-functions,

$$\int c_n(x)\, dg(x) \geqslant \int f_n(x)\, dg(x).$$

But the limiting function $c(x)$ of the c_n-sequence being an l-function, we can, by Case 1 of the present discussion, integrate term-by-term. Thus

$$\int c(x)\, dg(x) \geqslant \operatorname*{Lt}_{n\to\infty} \int f_n(x)\, dg(x). \tag{2}$$

Moreover, by (1), proceeding to the limit with n,

$$c(x) \geqslant f(x);$$

and therefore, since $c(x)$ is an l-function, *and $f(x)$ is a u-function*,

$$\int c(x)\, dg(x) \geqslant \int f(x)\, dg(x). \tag{3}$$

Now at every point either

$$c_2(x) - f_2(x) = b_2(x) - f_2(x), \quad \text{or} \quad = b_1(x) - f_2(x) \leqslant b_1(x) - f_1(x).$$

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* The choice of the functions $b_n(x)$ therefore depends on $g(x)$.

Hence at every point

$$c_2-f_2 \leqslant (b_2-f_2)+(b_1-f_1).$$

Both sides of this relation represent l-functions, therefore

$$\int (c_2-f_2)\, dg(x) \leqslant \int (b_2-f_2)\, dg(x)+\int (b_1-f_1)\, dg(x) \leqslant e(2^{-2}+2^{-3}) \leqslant \tfrac{1}{2}e.$$

A similar inequality holds for each value of n. Hence

$$\int c_n(x)\, dg(x) \leqslant \int f_n(x)\, dg(x)+\tfrac{1}{2}e.$$

Proceeding to the limit, and comparing with (3),

$$\int f(x)\, dg(x) \leqslant \int c(x)\, dg(x) \leqslant \underset{n\to\infty}{\mathrm{Lt}} \int f_n(x)\, dg(x)+\tfrac{1}{2}e. \tag{4}$$

Hence, remembering that, since $f_n \leqslant f$, and therefore, *both being u-functions,*

$$\int f_n(x)\, dg(x) \leqslant \int f(x)\, dg(x), \tag{5}$$

we get, proceeding to the limit, and using (4),

$$\underset{n\to\infty}{\mathrm{Lt}} \int f_n(x)\, dg(x) \leqslant \int f(x)\, dg(x) \leqslant \underset{n\to\infty}{\mathrm{Lt}} \int f_n(x)\, dg(x)+\tfrac{1}{2}e.$$

Since e is as small as we please, this proves the required result, viz.,

$$\int f(x)\, dg(x) = \underset{n\to\infty}{\mathrm{Lt}} \int f_n(x)\, dg(x).$$

CASE 4.—If we are given a monotone ascending sequence of u-functions whose limiting function is an l-function, the proof in Case 3 still applies, since the fact that $f(x)$ was an u-function only entered into the argument at the places italicized, in order to obtain equations (3) and (5), and the argument is equally true if in these places we make $f(x)$ an l-function.

CASES 5 AND 6.—If we are given a monotone descending sequence of l-functions whose limiting function is either an l-function or an u-function, the argument used in Cases 3 and 4 applies *mutatis mutandis.*

Thus in all cases the theorem is true.

13. The limiting function of a monotone ascending sequence of l-functions is always an l-function, and the limiting function of a descending sequence of u-functions is always a u-function. Thus there is no restriction in assuming in Cases 1 and 2 of the preceding proof that the limiting function is an l- or u-function. In Cases 3–6, however, there is a restriction. The nature of the limiting function, however, only enters in

two places into the argument, where it is italicized. If in the proof therefore we replace the symbol $\int f(x)\,dg(x)$ by the symbol B, used to denote the upper bound of the integrals of u-functions less than $f(x)$, the relations (3) and (5) will still hold. This is evident at once in the case of (5), since f_n is an u-function not greater than f, and appears after a moment's consideration in the case of (3), since $c(x)$ is greater than any u-function less than $f(x)$.

Thus it appears that the limit of the integrals of any bounded monotone ascending sequence of u-functions whose limiting function is $f(x)$ is the upper bound of the integrals of u-functions less than $f(x)$. Similarly we see that the limit is the lower bound of the integrals of l-functions greater than $f(x)$.

Hence we get the following theorem:—

THEOREM.—*The limit of the integrals with respect to a positive monotone increasing function $g(x)$ of all bounded monotone sequences of l- and u-functions having the same limiting function $f(x)$, whether $f(x)$ is or is not an l-function or an u-function, is the same.*

If the sequence is monotone ascending (descending) and contains an infinite number of l-functions (u-functions), the limiting function $f(x)$ is an l-function (u-function), so that the result follows at once from the preceding theorem. If the sequence contains only a finite number of l- or u-functions, these may be disregarded. Thus we are left with two kinds of sequences which have to be compared, namely monotone ascending sequences of u-functions and monotone descending sequences of l-functions, and we have to consider separately the cases when $f(x)$ is the limit of one only of these types of monotone sequences, or of both types.

When $f(x)$ is the limit of one only of these types, the required result has just been proved, and it has been pointed out that the limit in question is the upper bound of the integrals of u-functions less than $f(x)$, and the lower bound of the integrals of l-functions greater than $f(x)$, these two bounds being accordingly equal.

There remains therefore only to compare two sequences monotone in opposite senses.

Let $a_1 \leqslant a_2 \leqslant \ldots$ be a monotone ascending sequence of u-functions, and $b_1 \geqslant b_2 \geqslant \ldots$ a monotone descending sequence of l-functions with the same limiting function $f(x)$. Then, if we write

$$c_n = b_n - a_n,$$

the functions $c_n(x)$ form a monotone descending sequence of l-functions

with zero as limit, which may then be integrated term-by-term, by the case already disposed of. Thus

$$\underset{n\to\infty}{\mathrm{Lt}} \int c_n(x)\, dg(x) = 0.$$

Hence
$$\underset{n\to\infty}{\mathrm{Lt}} \int b_n(x)\, dg(x) = \underset{n\to\infty}{\mathrm{Lt}} \int a_n(x)\, dg(x).$$

This proves the theorem in this final case, and so completes the proof.

14. The functions which are the limits of monotone ascending sequences of u-functions are what I have called lu-functions, and those which are the limits of monotone descending sequences of l-functions are ul-functions. The functions which are both lu- and ul-functions are the functions of Baire's first class, that is the limits of continuous functions. The preceding theorem therefore gives us the definition of the integral with respect to a positive monotone increasing function $g(x)$ of functions of these two types, as the limit of the integrals of any monotone sequence of l- or u-functions having the required integrand as limit, and shows that this definition is consistent. Moreover, it also shows that the integral may also be defined in the following manner.

DEFINITION.—*The integral with respect to a positive monotone increasing function $g(x)$ of a bounded lu-function (ul-function) is the upper (lower) bound of the integrals with respect to $g(x)$ of u-functions (l-functions) which are less (greater) than the function.*

With this definition we may express the result of § 13 by saying that *any bounded monotone sequence of l- and u-functions may be integrated term-by-term with respect to any positive monotone increasing function.*

We also have the following theorem :—

THEOREM.—*Bounded monotone sequences of lu- and ul-functions whose limits are lu- or ul-functions may be integrated term-by-term.*

The proof of this theorem is exactly parallel to that of § 12, and is divided into similar cases. Wherever we had l-functions, we now have lu-functions, and where we had u-functions we have ul-functions. Wherever we had simple l- or u-functions we now have general l- and u-functions. In Case 3 we have no need for a lemma, such as that proved in § 11, since an ul-function is by definition the limit of a monotone descending sequence of u-functions.

Similarly, by the argument of § 13, we have the following theorem :—

THEOREM.—*The limit of the integrals with respect to a positive mono-*

tone increasing function $g(x)$ of all bounded monotone sequences of lu- and ul-functions having the same limiting function $f(x)$, whether $f(x)$ is or is not a lu- or an ul-function is the same.

Here again the unique limit which we obtain for the integrals of the sequence is the upper bound of the integrals of u-functions less than $f(x)$, and at the same time the lower bound of the integrals of l-functions greater than $f(x)$.

This theorem establishes the existence of the integral of $f(x)$ in the case when $f(x)$ is a *lul*- or an *ulu*-function, and gives us the same alternative form of the definition as that already enunciated in the case of *lu*- and *ul*-functions. Moreover with this definition bounded *monotone sequences of lu- and ul-functions may be integrated term-by-term.*

15. It is clear that we might continue in this way stage by stage. But it is preferable to take a new departure, and prove certain theorems of a different kind, by means of which we are able to deal at once with the most general type of bounded function obtained from simple l- and u-functions by the method of monotone sequences, functions which, for brevity, I sometimes refer to as *bounded functions of monotone type.*

THEOREM.—*Given a bounded lu-function, a bounded ul-function can be found, nowhere less than the lu-function, and having the same integral with respect to a given positive monotone increasing function $g(x)$.*

Indeed, if we examine the proof of Case 3 of the theorem of § 12, it will be found, as already remarked, that the fact that the limiting function $f(x)$ is there an u-function only enters in two places into the argument, where it is italicized. The assumption is indeed only used in order that the limiting function may be one whose integral is known to exist. If we omit the assumption, the limiting function is a *lu*-function, and, as such, has an integral with respect to $g(x)$, according to our definition. The argument shows then how to find an ascending sequence of l-functions

$$c_1 \leqslant c_2 \leqslant c_3 \leqslant \dots,$$

whose limiting function $c(x)$ is, of course, an l-function, such that

$$c(x) \geqslant f(x),$$

and
$$\operatorname*{Lt}_{n\to\infty} \int f_n(x)\,dg(x) \leqslant \int c(x)\,dg(x) \leqslant \operatorname*{Lt}_{n\to\infty} \int f_n(x)\,dg(x) + \tfrac{1}{2}e.$$

That is, by the definition of the integral of $f(x)$,

$$\int f(x)\,dg(x) \leqslant \int c(x)\,dg(x) \leqslant \int f(x)\,dg(x) + \tfrac{1}{2}e.$$

Let us denote this function $c(x)$ by $C_1(x)$, and let us similarly define $C_2(x)$, taking $\frac{1}{2}e$ instead of e, and so on. We thus get a succession of l-functions $\geqslant f(x)$, and such that, for each value of n,

$$\int f(x)\,dg(x) \leqslant \int C_n(x)\,dg(x) \leqslant \int f(x)\,dg(x) + 2^{-n}e.$$

If this C_n-succession is not monotone descending, we can, in the usual way, make it so. Wherever $C_3 > C_2$, we decrease its value to that of C_2, and similarly treat in turn the subsequent functions C_n. We may therefore assume that the C_n-succession is a monotone descending sequence of l-functions $\geqslant f(x)$, and satisfying the double inequality last written down. The limiting function $q(x)$* of this sequence will be an ul-function, by definition, and its integral, being the limit of $\int C_n(x)\,dg(x)$, will, by the last inequality, differ from $\int f(x)\,dg(x)$ by less than $2^{-n}e$ for any value of n, and must therefore be equal to $\int f(x)\,dg(x)$. This proves the theorem.

We also have the following theorem.

THEOREM.—*Given a bounded lul, we can always find an ul, nowhere less than the lul, and having the same integral with respect to a given positive monotone increasing function $g(x)$.*

The argument is precisely the same as in the proof of the preceding theorem, except that the functions $b_n(x)$ (§ 12) are now general l-functions instead of simple l-functions.

16. We can now prove the following general theorem :—

THEOREM.—*Given any bounded function, formed by any monotone process, such as those here described, we can find an ul-function not less than it, and a lu-function not greater than it,*† *each having the same integral with respect to a given positive monotone increasing function $g(x)$.*

Suppose, for definiteness, that the monotone succession defining the function $f(x)$ is an ascending one, say

$$f_1 \leqslant f_2 \leqslant \dots .$$

Then, without loss of generality, we may suppose that the theorem has been proved to be true for each function $f_n(x)$ of the sequence.

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* It will be noticed that the construction of the function $q(x)$ depends on the particular monotone function $g(x)$ with respect to which we integrate, since, as pointed out in § 12, this is the case with the auxiliary functions $b_n(x)$.

† These functions depending on $g(x)$, see footnotes to §§ 12 and 15.

Let us take a *lu*-function $h'_n(x) \leqslant f_n(x)$, and having the same integral with respect to $g(x)$. Doing this for each integer n, we get a new succession, which, if not already monotone increasing, we modify as follows:—At any point where $h'_1 > h'_2$, let us increase the value of the latter to that of the former. Denoting the modified function by h_2, it is still a *lu*-function; and, since

$$h'_2 \leqslant h_2 \leqslant f_2,$$

it has the same integral as before, equal to that of f_2. We then proceed to similarly modify h'_3, and so on. Thus we have a monotone ascending sequence of *lu*-functions h_n, whose integrals are equal to those of the functions f_n.

The limiting function $h(x)$ of this succession is an *llu*, that is a *lu*, and is, of course, like every h_n, not greater than $f(x)$. Also its integral may be obtained by term-by-term integration of the h_n-sequence, and is, accordingly, the limit of $\int f_n(x)\,dg(x)$. Thus we have found such a *lu*-function as was required.

Again, let us take an *ul*-function $q'_n(x) \geqslant f_n(x)$, and having the same integral. Doing this for each integer n we get a new succession, which, if not already monotone increasing, we proceed to modify as follows:— Let $q_1(x)$ be the function whose value at each point is the lower bound of all the functions $q'_n(x)$. This function is easily seen to be an *ul*-function, since it is the limit of a monotone descending sequence of *ul*-functions, got by taking the lower bound of a finite number of the functions $q'_n(x)$. Similarly, let q_2 be the function formed in like manner from all the functions $q'_n(x)$ except the first, and so on. We thus get a monotone ascending sequence of *ul*-functions $q_n(x)$, such that

$$f_n \leqslant q_n \leqslant q'_n,$$

so that the integrals of these three functions are the same. The limiting function of the q_n-sequence being a *lul*-function, which is nowhere less than $f(x)$, and whose integral with respect to $g(x)$ is, by definition, the limit of that of $q_n(x)$, that is, of $f_n(x)$, we can, by the preceding theorem, find an *ul*-function $q(x)$, nowhere less than $f(x)$, and having for integral with respect to $g(x)$ the limit of $\int f_n(x)\,dg(x)$, that is, $\int h(x)\,dg(x)$.

This proves the theorem.

17. From the preceding theorem it follows that all the bounded functions obtained by monotone processes from simple *l*- and *u*-functions possess integrals with respect to a given positive monotone increasing

function. Indeed since an u-function less than the auxiliary function $h(x)$ of the preceding proof is certainly less than $f(x)$, it follows from the equation

$$\underset{n\to\infty}{\mathrm{Lt}} \int f_n(x)\,dg(x) = \int h(x)\,dg(x),$$

by § 14, that the limit on the left of this equation $\leqslant A$, where A denotes the upper bound of the integrals of u-functions less than $f(x)$. Similarly by means of the auxiliary function $q(x)$, we see that that limit $\geqslant B$, where B is the lower bound of the integrals of l-functions greater than $f(x)$. Thus, denoting the limit by F,

$$B \leqslant F \leqslant A.$$

But since an u-function less than $f(x)$ is certainly less than an l-function greater than $f(x)$, the same is true of their integrals, whence, considering the upper and lower bounds of these integrals, we see that

$$A \leqslant B.$$

Hence $$B = F = A.$$

Since A and B are defined with respect to $f(x)$, and are independent of the particular sequence $f_1 \leqslant f_2 \leqslant \ldots \to f$, this proves that the integral is uniquely determined, so that the definition we have given of the integral is consistent. At the same time this gives us a new definition of the integral of any bounded function of monotone type, that is any bounded function defined by means of monotone sequences, starting with simple l- and u-functions :—

DEFINITION.—*Form the integrals with respect to $g(x)$ of all u-functions less than the given function, and take the upper bound of these integrals; form the integrals with respect to $g(x)$ of all l-functions greater than the given function, and take the lower bound of these integrals; then, if the upper bound of the former and the lower bound of the latter agree, the function is said to possess an integral with respect to $g(x)$, or to be summable with respect to $g(x)$, and the value of the integral is the common value of these two bounds.*

By the preceding theorem it then follows that *all bounded functions of monotone type have integrals in this sense with respect to $g(x)$*, and that such integrals are the same as those previously defined by means of monotone sequences.

If the upper and lower bounds referred to in the above definition do not agree, we may call the former the upper integral (generalized), and

the latter the lower integral of the function with respect to $g(x)$. This is a point of merely academic interest in so far as we are dealing with bounded functions, since all the bounded functions which actually occur belong to one of the classes which we have defined by means of monotone sequences.

18. The mode of proof adopted in § 16 gives us a simple proof of the following important theorem :—

THEOREM.—*Any bounded monotone sequence of functions, formed by any monotone process, such as those here described, may be integrated with respect to a given positive monotone increasing function $g(x)$ term-by-term.*

We may evidently suppose the functions of the sequence to be positive. Let then $f_1 \leqslant f_2 \leqslant, \ldots$ be a monotone ascending sequence of positive functions, and $f(x)$ the limiting function. By the theorem of § 16 we can find two auxiliary successions of functions, $h_1, h_2, \ldots$ consisting of *lu*-functions, each not greater than the corresponding function of the f_n-sequence, and $q_1, q_2, \ldots$ consisting of *ul*-functions, each not less than the corresponding function of the f_n-sequence, the three functions with the same index having the same integral with respect to $g(x)$. Moreover, as in the proof of § 16, we can ensure that the two auxiliary successions are themselves monotone ascending sequences.

Let $h(x)$ and $q(x)$ be the limiting functions of the auxiliary sequences; the former is then a *lu*-function, whose integral is the limit of

$$\int_a^b h_n(x)\, dg(x),$$

and the latter is a *lul*-function, whose integral is the limit of

$$\int_a^b q_n(x)\, dg(x).$$

Hence
$$\int_a^b h(x)\, dg(x) = \int_a^b q(x)\, dg(x) = \operatorname*{Lt}_{n\to\infty} \int_a^b f_n(x)\, dg(x).$$

But, since $f(x)$ is the limit of $f_n(x)$, and $f_n(x)$ lies between $h_n(x)$ and $q_n(x)$, it follows that $f(x)$ lies between $h(x)$ and $g(x)$, so that also the integral of $f(x)$ with respect to $g(x)$ lies between those of $h(x)$ and $q(x)$, and is accordingly equal to them. Thus

$$\int_a^b f(x)\, dg(x) = \operatorname*{Lt}_{n\to\infty} \int_a^b f_n(x)\, dg(x),$$

which proves the theorem in this case, and therefore, by the usual reasoning in the general case of a bounded sequence monotonely ascending or descending.

COR.—*Any bounded sequence of such functions may be integrated term-by-term with respect to* $g(x)$.

For, if $f_1, f_2, \ldots$ be the given sequence, and f the limiting function, we can form a monotone descending sequence of functions* of monotone type having f as limiting function, by taking $q_n(x)$ to denote the upper bound at the point x of the quantities $f_n(x), f_{n+1}(x), \ldots$. Then, by the theorem.

$$\int_a^b f(x)\,dg(x) = \operatorname*{Lt}_{n\to\infty} \int_a^b q_n(x)\,dg(x).$$

The functions $q_n(x)$ being by their definition not less than the corresponding functions $f_n(x)$, we get

$$\int_a^b f(x)\,dg(x) \geqslant \operatorname*{Lt}_{n\to\infty} \int_a^b f_n(x)\,dg(x).$$

Similarly, considering the lower, instead of the upper bounds, we get

$$\int_a^b f(x)\,dg(x) \leqslant \operatorname*{Lt}_{n\to\infty} \int_a^b f_n(x)\,dg(x).$$

Hence $$\int_a^b f(x)\,dg(x) = \operatorname*{Lt}_{n\to\infty} \int_a^b f_n(x)\,dg(x),$$

which proves the corollary.

19. Now let $f(x)$ be a continuous function. Then we know that $f(x)$ is the limit of a monotone ascending sequence of simple l-functions,

$$a_1(x) \leqslant a_2(x) \leqslant \ldots,$$

defined as follows:—$a_n(x)$ is the lower bound of $f(x)$ in each of the closed intervals got by dividing the whole interval of integration into $(n-1)$ equal parts at the points

$$a = c_1,\ c_2,\ \ldots,\ c_{n-1},\ c_n = b.$$

We then know, by the property of uniform continuity, that given e, we can find n_e so large that, whatever point x_r we choose in the closed

* "On the New Theory of Integration," p. 177.

interval (c_r, c_{r+1}),

$$|a_n(x)-f(x_r)| \leqslant e,$$

for all points x in that closed interval, provided only $n \geqslant n_e$.

Hence, referring to the expression given in § 4, for the integral of $a_n(x)$ with respect to $g(x)$, and denoting the upper bound of $g(x)$ by M, we have

$$\int_a^b a_n(x)\, dg(x) = \sum_{i=1}^{i=n-1} f(x_i)\left[g(c_{i+1})-g(c_i)\right]+3\theta eM,$$

where

$$-1 \leqslant \theta \leqslant 1.$$

Now we have seen that, as $n \to \infty$, the left-hand side of this equation has a unique and finite limit, namely $\int_a^b f(x)\, dg(x)$; therefore the summation on the right has, as $n \to \infty$, a limit or limits which differ from this finite quantity by less than $3e$. Since e is as small as we please, this proves that the expression

$$\sum_{i=1}^{n-1} f(x_i)\left[g(c_{i+1})-g(c_i)\right], \tag{II}$$

has as $n \to \infty$ a unique and finite limit, namely $\int_a^b f(x)\, dg(x)$; moreover this is true however the points x_i are chosen in the corresponding intervals (c_i, c_{i+1}).

This proves that the integrals defined in the present paper by the method of monotone sequences coincide in the case of continuous functions with the integrals of Stieltjes. Indeed the definition of the integral of a continuous function with respect to a monotone increasing function given by Stieltjes depends on the Lemma that the above summation (II) has such a unique limit as we have seen to exist, and this limit is defined to be the Stieltjes integral.

20. The summation (II) used by Stieltjes suggests itself of course by analogy with the ancient definition of integration. The extension of the realm of ordinary integration by Riemann so as to include, not only continuous functions, but also functions whose discontinuities, to use the modern form of expression, form a set of zero content, suggests that the summation (II) may be used for functions of a more extended nature than continuous functions.

When we replace the independent variable x by a monotone increasing function $g(x)$, the analogon of the content of a set of points is *the variation of $g(x)$ with respect to the set of points in question*, the variation here

being of course the positive variation. Similarly when we replace x by a function of bounded variation, the content is replaced by *the total variation of the function with respect to the set of points.*

The variation of a monotone increasing function over a set of points may, like the content, be defined in different ways. It has been defined as the lower bound of the variation of the function over a set of non-overlapping intervals containing the set of points;* this corresponds to the definition of the outer content.† It has again been defined‡ as the unique limit, when this exists, of the variation of the function over the sets of intervals of a sequence of such sets, each lying inside the preceding, and having the set for inner limiting set; this assumes that the set is an i-set, and corresponds to the definition of the content of an i-set; it remains here to discuss whether the limit in question is unique for all i-sets. In view of the classification of sets of points§ which I have recently given, we shall naturally discuss the variation of a monotone increasing function with respect to a set of points more completely, and answer all the questions which suggest themselves.

The variation of a monotone increasing function $g(x)$ with respect to a set of points S will then be defined as the integral with respect to $g(x)$ of the function which is unity at the points of the set and zero elsewhere, provided this integral exists; if it does not exist we shall have an upper and a lower variation, namely, the upper and the lower integral of the function with respect to the set of points (*vide supra*, § 17).

It then follows from § 17 that *a monotone increasing function $g(x)$ possesses a variation with respect to any set of points which can be obtained from a finite number of closed or open intervals by means of monotone processes.* For it has been shown in my discussion‖ of such sets that the function which is unity at the points of the set and zero elsewhere is a function which can be obtained from simple l- and u-functions by the method of monotone sequences. In particular this is the case if the set S is a closed set, a set of intervals, an i-set or an o-set. This answers the question indicated a few lines back.

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* W. H. Young, "On Functions of Bounded Variation," 1910, *Quarterly Journal*. Vol. 42, p. 71.

† W. H. Young, "Open Sets and the Theory of Content," 1904, *Proc. London Math. Soc.*, Ser. 2, Vol. 2, pp. 16–57.

‡ *Loc. cit.*, footnote *, *supra*.

§ "On Functions and their Associated Sets of Points," 1912, *Proc. London Math. Soc.*, Ser. 2, Vol. 12, pp. 260 *seq*.

‖ *Loc. cit.*, § 8.

If the variation of the monotone increasing function $g(x)$ over every set of points of content zero is zero $g(x)$ is, by a theorem of Lebesgue's,* a Lebesgue integral, and conversely, if $f(x)$ is a Lebesgue integral, this is the case. Even if the function $f(x)$, which we propose to integrate with respect to a general monotone increasing function $g(x)$, has only discontinuities at a set of zero content, it will not usually happen that the variation of $g(x)$ over the set of these discontinuities is zero. It may, however, happen that the variation of $g(x)$ over the set of these discontinuities is zero, without the set having zero content, and without $g(x)$ being specialised so as to be a Lebesgue integral; a simple example of this is when $g(x) = x$ at the points of a set G of positive content, and is constant in the black intervals of G, while $f(x)$ has all its discontinuities internal to the black intervals of G.

Corresponding to Riemann's extension of the notion of an integrable function, we now have a certain class of functions which may be said to possess a "Riemann" integral with respect to the monotone increasing function $g(x)$, that is to say a function such that the summation (II), used by Stieltjes, has a unique and finite limit, however the points x are chosen in their corresponding intervals, and however those intervals are constructed, provided only the length of the greatest of them approaches zero as $n \to \infty$. The necessary and sufficient condition that a function $f(x)$ should belong to this class of functions is precisely analogous to Riemann's condition of integrability, and the proof is closely similar to the proof of his result.

THEOREM.—*The necessary and sufficient condition that a function $f(x)$ should possess a "Riemann" integral with respect to the monotone increasing function $g(x)$ is that the variation of $g(x)$ over the set of points at which $f(x)$ is discontinuous should be zero.*

Let us divide the closed interval (a, b) in any way we please into a finite number of parts, consisting of points of division

$$c_1 = a,\ c_2,\ \ldots,\ c_{n-1},\ c_n = b,$$

and completely open intervals. Let U_i and L_i denote the upper and lower bounds of $f(x)$ in the completely open interval (c_i, c_{i+1}).

The lower summation of $f(x)$ with respect to $g(x)$ for this division is

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* H. Lebesgue, "Sur la Recherche des Fonctions Primitives par l'Intégration," 1907, *Rend. dei Lincei*, Vol. 16, p. 286.

then the following expression

$$u_k(x) = \sum_{i=1}^{n-1} L_i\,[g(c_{i+1}) - g(c_i)],$$

while the upper summation $v_k(x)$ only differs from this by U_i replacing L_i.

Let v_k and u_k denote the upper and lower summations with respect to any other division, and u_q and v_q the upper and lower summations with respect to the division whose points of division consist of all those belonging to the two former divisions. Then, evidently

$$u_k \leqslant u_q \leqslant v_q \leqslant v_K.$$

Hence, if v denote the lower bound of all the upper summations, and u the upper bound of all the lower summations

$$u \leqslant v\,;$$

and therefore

$$u_K \leqslant u \leqslant v \leqslant v_K. \qquad \text{(i)}$$

Now

$$v_K - u_K = \sum_{i=1}^{n-1} (U_i - L_i)\,[g(c_{i+1}) - g(c_i)].$$

But from the meaning of the term discontinuity, it follows that, corresponding to each point x, we can find a tile d_x with x as point of attachment,* such that, if y_1 and y_2 are any two points in the tile, the difference between $f(y_1)$ and $f(y_2)$ is less than $W_x + e$, where W_x is the discontinuity of $f(x)$ at the point of attachment x. By the Heine-Borel theorem we can find a finite number of these tiles, having the following properties:—

(1) The length of each tile is less than an assigned quantity H.

(2) Each point of (a, b) is covered by one or more of the tiles.

Let h be the length of the smallest of these chosen tiles. Then, if each division (c_i, c_{i+1}) is of length less than h, it will lie at most in two of the chosen tiles, and therefore, if y_1 and y_2 are any two points in the closed division (c_i, c_{i+1}),

$$|f(y_1) - f(y_2)| \leqslant W_{x_1} + W_{x_2} + 2e,$$

where W_{x_1} and W_{x_2} are the discontinuities at the points of attachment of the two tiles in question, so that the points x_1 and x_2 are internal to an interval of length $2H$ which contains (c_i, c_{i+1}).

Now let S_e denote the closed set of those discontinuities of $f(x)$ which

* That is, an interval d_x with x as an internal point of special character.

are $\geqslant e$. Then, since the variation of the monotone increasing function $g(x)$ over the whole set of discontinuities is zero, it is clearly zero over the closed set S_e. Also the set S_e, being closed, is the inner limiting set as $H \to 0$ of a set of intervals, each of length $2H$ with the points of S_e as centres.

These intervals tile over a finite number of non-overlapping intervals $Z_{e,H}$, and S_e is therefore also the inner limiting set of $Z_{e,H}$ as $H \to 0$, so that the variation of $g(x)$ over $Z_{e,H}$ has, as $H \to 0$, for unique limit the variation of $g(x)$ over S_e, that is zero. Thus we can choose H so small that the variation of $g(x)$ over the intervals $Z_{e,H}$ is less than e_1, where e_1 is as small as we please.

Now, as we have seen, each division (c_i, c_{i+1}) either is internal to an interval of length $2H$ containing a point of S_e, and is in that case internal to the intervals $Z_{e,H}$, or else it is such that, y_1 and y_2 being any two of its points

$$|f(y_1)-f(y_2)| \leqslant 4e.$$

Thus the summation representing $v_K - u_K$ may be divided into two parts, the first of which corresponds to divisions in which

$$U_i - L_i \leqslant 4e,$$

and the second corresponds to divisions which lie in the intervals $Z_{e,H}$, so that the variation of $g(x)$ over them, that is,

$$\Sigma\left[g(c_{i+1}) - g(c_i)\right]$$

is less than e_1. We have, therefore, denoting by M, the upper bound of $g(x)$, and by Q the difference between the upper and lower bounds of $f(x)$,

$$0 \leqslant v-u \leqslant v_K - u_K \leqslant 4eM + Qe_1.$$

Since e is as small as we please, and M and Q are fixed, this proves that

$$u = v.$$

Thus the expression $$\sum_{i=1}^{n-1} f(x_i)\left[g(c_{i+1}) - g(c_i)\right]$$

has, as $n \to \infty$, a unique and finite limit u, however the points x_i are chosen in the corresponding divisions (c_i, c_{i+1}), and however the divisions be made, provided only, as $n \to \infty$, the length of the greatest of them approaches zero as limit.

Now let $a(x)$ be the function which in each completely open division (c_i, c_{i+1}) is equal to the lower bound of $f(x)$ in the corresponding closed division, and at the points of division is equal to the greater of the two

values in the neighbourhood, so that $a(x)$ is a simple u-function $\leqslant f(x)$. We have then by the definition of the integral of a simple function

$$\int_a^b a(x)\,dx = \Sigma\left[a(c_i)\right]\left[g(c_i+0)-g(c_i)\right]+a(c_i+0)\left[g(c_{i+1}-0)-g(c_i+0)\right]$$
$$+a(c_{i+1})\left[g(c_{i+1})-g(c_{i+1}-0)\right].$$

Dividing the summation into two parts as before, we see that this lies between

$$\sum_{i=1}^{n-1} f(x_i)\left[g(c_{i+1})-g(c_i)\right] \pm (4eM+Qe_1),$$

and is therefore as near as we please to u, provided the divisions are sufficiently small. Hence the upper bound of the integrals of upper semi-continuous functions $\leqslant f(x)$ with respect to $g(x)$ is not less than u.

Similarly, taking instead of $a(x)$ the function $b(x)$ which in each of the completely open intervals (c_i, c_{i+1}) is equal to the upper bound of $f(x)$ in the corresponding closed interval, and at the points of division is equal to the lesser of the two values in the neighbourhood, we see that the lower bound of the integrals of lower semi-continuous functions $\geqslant f(x)$ with respect to $g(x)$ is not greater than u. Since this lower bound is clearly not less than the former upper bound, it follows that they are equal, and are both equal to u, and that therefore $f(x)$ has an integral with respect to $g(x)$, and this integral is equal to u. This proves the theorem.

COR.—*Any function of bounded variation has a "Riemann" integral with respect to any continuous positive monotone increasing function.*

In fact, if $f(x)$ is a function of bounded variation, its discontinuities form a countable set. Therefore the variation of $g(x)$ with respect to these discontinuities is the limit of the variation of $g(x)$ with respect to a finite number of points, which is zero, since the variation of $g(x)$ with respect to a single point is zero, and therefore the same is true of the sum of its variations with respect to a finite number of points, that is, its variation with respect to a finite number of points.

This argument shows that *any function whose discontinuities are countable has a "Riemann" integral with respect to a positive monotone increasing function.*

21. The preceding theorem allows us to prove the theorem of integration by parts as follows:—

THEOREM.—*If $u(x)$ and $v(x)$ are both positive monotone increasing*

functions, and each has a "Riemann" integral with respect to the other, then

$$\int u\,dv + \int v\,du = uv + \text{const.}$$

In fact, we may then write

$$\int u\,dv = \text{Lt}\left(\sum_{i=1}^{n-1} u(c_i)\left[v(c_{i+1}) - v(c_i)\right]\right),$$

$$\int v\,du = \text{Lt}\left(\sum_{i=1}^{n-1} v(c_{i+1})\left[u(c_{i+1}) - u(c_i)\right]\right),$$

whence

$$\int u\,dv + \int v\,du = \text{Lt} \sum_{i=1}^{n-1} \left[u(c_{i+1})\,v(c_{i+1}) - u(c_i)\,v(c_i)\right]$$

$$= \text{Lt}\left[u(c_n)\,v(c_n) - u(c_1)\,v(c_1)\right]$$

$$= uv + \text{const.}$$

COR.—*If u and v are both positive monotone increasing functions and one of them is continuous, the formula for integration by parts holds.*

This justifies Stieltjes' use of the formula for integration by parts to define the integral of a monotone function with respect to a continuous function, which, in his case, is extended so as not necessarily to be of bounded variation.

22. We have now completely discussed the method (A) of defining the integrals of functions with respect to a positive monotone increasing function. In the method (B) we start with continuous functions instead of simple l-functions and u-functions. The definition of the integral of a continuous function $f(x)$ with respect to a positive monotone increasing function $g(x)$ may be taken to be that of Stieltjes, which depends on a preliminary Lemma, a proof of which by the method (A) has already been given. The enunciation of the Lemma is as follows:—

LEMMA.—*If $f(x)$ is a continuous function and $g(x)$ a positive monotone increasing function in the closed interval (a, b), then the summation*

$$\sum_{i=1}^{n-1} f(x_i)\left[g(c_{i+1}) - g(c_i)\right], \tag{II}$$

where

$$c_1 = a,\ c_2,\ \ldots,\ c_{n-1},\ c_n = b,$$

and x_i is any point in the closed interval (c_i, c_{i+1}), differs from a determinate limiting value I by as little as we please, provided all the intervals (c_i, c_{i+1}) are conveniently small.

The limiting value I is then defined to be the integral of $f(x)$ with respect to $g(x)$.

The original proof of this Lemma does not differ in any essential particular from that of the corresponding Lemma when the monotone increasing function is the independent variable itself, and is identical in form with the proof of the same Lemma when the function $f(x)$ is not necessarily continuous, but is such that the variation of $g(x)$ over the set of points at which $f(x)$ is discontinuous is zero, a proof written out in § 20, as the first part of the proof of the theorem there given. The fact that the above Lemma may be given this more general form shows that we may, if we please, begin, instead of with continuous functions, with functions whose discontinuities form a set over which $g(x)$ has zero variation, using the limiting value of the summation (II) as the definition of the integral in this case. The integrals so defined, as well as the simpler case of them due to Stieltjes, have already been identified (§§ 9 and 20) with the integrals defined by the method (A).

We may, however, if we please, avoid the use of this Lemma by using instead Weierstrass's theorem that any continuous function is the uniform limit of a sequence of polynomials. This enables us to reduce the problem of the integration of continuous functions to that of the integration of a polynomial, and therefore to that of the integration of a power of the independent variable. We may then use the formula

$$\int_a^b x^n\,dg(x) = \Big[x^n g(x)\Big]_a^b - n\int_a^b x^{n-1} g(x)\,dx,$$

as giving us the definition of the symbol on the left of this equation in terms of an integral of the same form with x^{n-1} instead of x^n as integrand, and therefore, by repeated use of the formula, in terms of $\int_a^b g(x)\,dx$, which, since $g(x)$ is a monotone increasing function, may be regarded as a Riemann integral. The integral of a polynomial is of course defined to be the sum of the integrals of its separate terms. We then show without difficulty that any uniformly convergent series of polynomials when integrated leads to a convergent series, and we take the sum of this series to define the integral of the sum of the original series. We have thus defined anew the integral of any continuous function with respect to the monotone increasing function $g(x)$.

The method of monotone sequences can now be employed as in method (A) remembering that it has been proved* that any bounded *u*-

* W. H. Young. "On Monotone Sequences of Continuous Functions," 1908, *Proc. Camb. Phil. Soc.*, Vol. XIV, p. 523.

function or l-function is the limit of a monotone sequence of continuous functions. We thus arrive at the definition of the integrals of the same functions as by method (A) with respect to the monotone increasing function $g(x)$, and the integrals so defined are the same as those defined by the other method.

23. In the third method (C), which combines in a certain sense the advantages of the methods (A) and (B), we start with those simple l- and u-functions whose discontinuities are only at points of continuity of the monotone increasing function $g(x)$. If $f(x)$ is such a function we may define its integral by a simpler summation than the general formula (I) of § 4, namely,

$$\int_a^b f(x)\,dg(x) = \sum_{i=1}^{n-1} f(c_i+0)\left[g(c_{i+1})-g(c_i)\right], \tag{III}$$

which is, in the case when $f(x)$ is such a function as is here considered, formally identical with the summation (II) used by Stieltjes.

It is easily verified that this expression for the integral is the same as that given by the formula (I) under the circumstances specified.

We have now only to show that any simple l- or u-function is the limit of a monotone sequence of such special l- and u-functions.

Let $f(x)$ be a simple l-function having its discontinuities at the points $P_1, P_2, \ldots, P_r$, some or all of which may be discontinuities of $g(x)$.

Taking any monotone descending sequence of quantities with zero as limit

$$e_1 > e_2 > \ldots > e_n > \ldots \to 0,$$

we can then, within a distance e_k of the point P_s on either side find points $P'_{s,k}$ and $P''_{s,k}$ respectively, such that between them there is no discontinuity of $f(x)$ except P_s, and that the points $P'_{s,k}$ and $P''_{s,k}$ are not themselves among the countable set of points at which $g(x)$ is discontinuous.

Let us find a pair of such points for each value of s from 1 to r, k being fixed, and let us define a function $f_k(x)$ to be in the closed interval $(P'_{s,k}, P''_{s,k})$ equal to the lower bound of $f(x)$ in that interval, for each value of s, and in the remaining intervals to coincide with $f(x)$. Then $f_k(x)$ is evidently a simple l-function less than or equal to $f(x)$, and describes as $k \to \infty$ a monotone increasing sequence, having $f(x)$ as limit, except at the points $P_1, P_2, \ldots, P_r$, at which this is not evident. Now at P_s the limit of $f_k(x)$ as $k \to \infty$ is the limit when $k \to \infty$ of the lower bound of $f(x)$ in the closed interval $(P'_{s,k}, P''_{s,k})$, and is therefore equal to $f(P_k)$, since $f(x)$ is lower semi-continuous at P_k. Thus at the doubtful

points also $f(x)$ is the limit of the monotone ascending sequence of special simple l-functions constructed.

Similarly any simple u-function is the limit of a monotone descending sequence of special simple u-functions.

It follows therefore from the theory of these integrals as defined by the method (A) that, starting with these special simple l- and u-functions, and defining their integrals by the formula (III), we arrive by the method of monotone sequences at the same theory of integration with respect to the monotone increasing function $g(x)$ as before.

24. We now come to the integrals of unbounded functions. The method we have adopted enables us to treat these very simply. Writing

$$f(x) = |f(x)| - [|f(x)| - f(x)],$$

we have expressed $f(x)$ as the difference of two positive functions ($\geqslant 0$), of which the second is not greater than twice the first. It will at once appear from the definition to be given of the integral of a positive function with respect to $g(x)$, that a positive function is summable with respect to $g(x)$ if another function can be found which is not less than the first, and is known to be summable with respect to $g(x)$. Hence it follows that, if $|f(x)|$ has an integral with respect to $g(x)$, so has $[|f(x)| - f(x)]$, and therefore so has $f(x)$ itself, the latter integral being the difference of the two former integrals. In this case $f(x)$ is said to have *an absolutely convergent integral with respect to $g(x)$, or to be summable with respect to $g(x)$*. In the discussion of absolutely convergent integrals therefore it is only necessary to consider positive functions, and we shall in future restrict ourselves in this sense.

The positive functions to be considered may or may not assume the value $+\infty$ at certain points. It must, however, be clear that, in order to obtain a finite value for the integral with respect to $g(x)$, the function to be integrated must not be infinite throughout any interval, or at any point at which $g(x)$ is discontinuous. We shall therefore further restrict the class of functions to be considered, assuming that these inconveniences are excluded.

This being so, we next remark that a function $f(x)$ such as we are considering can always be considered as the limit of a monotone ascending sequence of bounded functions

$$f_1 \leqslant f_2 \leqslant \dots \rightarrow f,$$

where, for instance, as in de la Vallée Poussin's definition of the improper

integral, $f_n(x)$ is the function which is equal to $f(x)$ wherever this is less than n, and is equal to n elsewhere. Moreover, if $f(x)$ is a function of monotone type, $f_n(x)$ is a bounded function of the same type.

If $f(x)$ is defined as the limit of a monotone descending sequence, the constituent functions will be unbounded functions, and will therefore not at present have satisfactorily defined integrals. Thus in defining the integrals of unbounded positive functions in accordance with the principle of § 5, we shall only consider monotone ascending sequences of bounded functions, *the integral of $f(x)$ being defined as the limit of the integrals of any monotone ascending sequence of bounded functions having $f(x)$ as limit, provided this limit is finite*, it being shown as a preliminary that *this limit is independent of the particular such sequence chosen.* We shall in this way have a definition of the integrals of unbounded l's, u's, lu's, ul's, and so on, as well as of the more complicated functions of transfinite types. It will then be necessary, in accordance with the principle of § 5, to show that with our definitions monotone sequences, both ascending and descending, of positive functions, bounded and unbounded, may be integrated term-by-term, this being understood with the gloss that, if the limit of the integrals is infinite, the limiting function is non-summable with respect to $g(x)$, and *vice versa.*

In order to prove that the proposed definition of the integral is consistent, it will only be necessary to establish the definition in the case of an unbounded lu-function and lul-function, and to show that any monotone ascending sequence of unbounded lu-functions and ul-functions may be integrated term-by-term; we can then apply the reasoning of § 16 to show that whatever function of monotone type $f(x)$ may be, the value of its integral is independent of the particular sequence of bounded functions used to obtain it. This is done in §§ 25–28.

25. Now if we examine the proof of the theorem of § 8, we shall see that the hypothesis that the sequence is bounded only enters in order to establish the following points:—

(i) B and F are finite quantities.

(ii) $f(x)$ is finite at each of the points of discontinuity of $g(x)$.

In the case we are assuming (ii) is hypothecated to be true, and though B and F need not be finite, they are necessarily positive, the assumption of a quantity B' less than B therefore makes B' finite, which is all that is required in the argument. We have thus the following extension of

that theorem :—

THEOREM.—*Given two monotone ascending sequences of positive simple l-functions with the same limiting function, the integrals of the two sequences with respect to a given positive monotone increasing function* $g(x)$ *have the same limit.*

Hence, as in § 10, we give the following definition :—

DEFINITION.—*The integral of a positive l-function, which is not infinite throughout any interval, nor at any of the points of discontinuity of* $g(x)$, *is defined to be the limit of the integrals of any monotone ascending sequence of simple l-functions having the given l-function as limit, provided the limit of the integrals is finite.*

The Lemma of § 11 evidently holds when $f(x)$ is positive but unbounded, provided it is not infinite throughout any interval, since the lower bound of $f(x)$ in any interval being finite, the auxiliary simple l-function $f_k(x)$ is properly defined. Thus

LEMMA.—*Any positive l-function is representable as the limit of an increasing sequence of simple l-functions, and as the limit of such a sequence of simple u-functions.*

We now come to the theorem of § 12, Cases 1 and 4. The argument in Case 1 is valid as it stands, so that we may at once assert that monotone ascending sequences of positive l-functions, whose infinities are restricted in the manner already mentioned, may be integrated term-by-term. In Case 4 we must of course at this stage assume that the u-functions are bounded u-functions, since we have not yet defined the integrals of unbounded u-functions. The Lemma of § 11 may therefore be applied in the alternative form given at the end of that article, and the whole argument holds verbatim. The argument of § 13 holds as it stands. We have therefore the following theorem :—

The limit of the integrals with respect to $g(x)$ *of all monotone ascending sequences of positive bounded l-functions and u-functions having the same limiting function* $f(x)$, *whether* $f(x)$ *is bounded or unbounded, is the same.*

This establishes the proposed definition of the integral of an unbounded lu-function, giving as a special case that of an unbounded u-function and including that of an unbounded l-function. It also gives us the following equivalent definition corresponding to that given in § 14 :—

DEFINITION.—*The integral of a positive lu-function* $f(x)$ *with respect*

to $g(x)$ is the upper bound of the integrals with respect to $g(x)$ of all positive bounded u-functions less than $f(x)$, provided this upper bound is finite. It is also the lower bound of the integrals with respect to $g(x)$ of l-functions not less than $f(x)$.

Moreover the preceding theorem may now be stated in the following form :—

THEOREM.—*Monotone ascending sequences of bounded l- and u-functions may be integrated term-by-term.*

This statement has to be interpreted with the gloss already noticed (§ 24).

26. We also have the following theorem :—

THEOREM.—*Monotone ascending sequences of positive unbounded lu-functions may be integrated term-by-term.*

Here the same gloss has to be applied.

The proof is moulded closely on that of § 12. We express the given sequence by means of a doubly monotone scheme of functions, all but those in the last column being bounded u-functions, and those in the last column being unbounded lu-functions. We then get the limiting function $f(x)$ expressed as the limit of the monotone ascending sequence of bounded u-functions which form the diagonal,

$$f_{11} \leqslant f_{22} \leqslant \ldots \to f.$$

By a theorem already proved we may integrate here term-by-term, interpreting this expression with the proper gloss. Thus

$$\int f(x)\, dg(x) = \underset{n\to\infty}{\mathrm{Lt}} \int f_{nn}(x)\, dg(x).$$

But, since $f_{nn} \leqslant f_n$, by the definition of the integral,

$$\int f_{nn}(x)\, dg(x) \leqslant \int f_n(x)\, dg(x).$$

Hence
$$\int f(x)\, dg(x) \leqslant \underset{n\to\infty}{\mathrm{Lt}} \int f_n(x)\, dg(x).$$

But, since $f_n \leqslant f$, we have similarly from the definition of the integral,

$$\int f_n(x)\, dg(x) \leqslant \int f(x)\, dg(x).$$

Hence proceeding to the limit $n \to \infty$, we see that the sign $<$ is inad-

missible in the preceding inequality. That is,

$$\int f(x)\,dg(x) = \underset{n\to\infty}{\text{Lt}} \int f_n(x)\,dg(x),$$

understanding this equation with the proper gloss. This proves the theorem.

27. Similarly, by the argument of Case 3 of § 12, we have the following theorem :—

THEOREM.—*Monotone ascending sequences of positive bounded ul-functions having the same limiting function, have the same limit for their integrals, and this limit is at the same time the upper bound of the integrals of all bounded u-functions less than $f(x)$, and the lower bound of the integrals of all unbounded l-functions not less than $f(x)$.*

In fact, in the argument of Case 3 of § 12, we have only to change simple l-functions to general bounded l-functions. The functions b_n and c_n are then bounded l-functions, and $c(x)$ is an unbounded l-function, while $f(x)$ itself is an unbounded ul-function. As in § 13 we then interpret the symbol $\int f(x)\,dg(x)$ in either of the two senses suggested by the enunciation, the whole argument now holds, and the theorem is proved.

We thus establish the consistency of the definition of the integral of an unbounded ul-function, and obtain the equivalent form :—

DEFINITION.—*The integral of a positive ul-function $f(x)$ with respect to $g(x)$ is the upper bound of the integrals with respect to $g(x)$ of all positive bounded u-functions less than $f(x)$, provided this upper bound is finite. It is also the lower bound of the integrals of all l-functions not less than $f(x)$ with respect to $g(x)$.*

With this definition the same argument adapted from Case 3 of § 12 enables us at once to prove the theorem at the beginning of the present article without restraining the constituents of the sequence to be bounded functions. This establishes the definition of the integral of an unbounded lul-function and gives us the following theorem :—

THEOREM.—*Monotone ascending sequences of positive unbounded ul-functions may be integrated term-by-term with respect to $g(x)$.*

This has to be understood with the usual gloss.

28. We can now remove the restriction to bounded functions in the theorems of §§ 15 and 16. We have accordingly the following theorem :—

THEOREM.—*Given any positive function, formed by any monotone process from simple positive l- and u-functions, we can find an unbounded ul-function not less than it, and an unbounded lu-function not greater than it, each having the same integral with respect to a given positive monotone increasing function $g(x)$.*

This theorem establishes, as in § 17, the consistence of the definition of the integral of $f(x)$ with respect to $g(x)$, or the non-summability of $g(x)$ with respect to $g(x)$, according as these auxiliary functions are or are not summable, and at the same time verifies the fact that, *in the case of positive unbounded functions of monotone type, the integral exists in the sense given in the definition of* § 17, *provided the common value of the upper and lower generalised integrals is finite.*

We have thus completely established the theorem of the integration of positive functions with respect to $g(x)$. This gives us at the same time the definition of the absolutely convergent integral with respect to $g(x)$ of any function $f(x)$, whether positive or not, which is summable with respect to $g(x)$, that is to say which is such that $|f(x)|$ has an integral, finite of course in value, with respect to $g(x)$. We see also that *the term-by-term integration of monotone sequences is allowable for all summable functions with respect to $g(x)$*, this being interpreted with the proper gloss ; for, if $f_n(x)$ denote the typical constituent of the sequence, and we write

$$f_n(x) = u_n(x) - v_n(x),$$

where $u_n(x) = f_n(x)$, wherever $f_n(x) \geqslant 0$, and $u_n(x) = 0$ elsewhere, both $u_n(x)$ and $v_n(x)$ are positive functions, and, as n increases, they describe monotone sequences, which may accordingly be integrated term-by-term, and whose limiting functions $u(x)$ and $v(x)$ bear the same relation to $f(x)$ that $u_n(x)$ and $v_n(x)$ do to $f_n(x)$. Thus, if the monotone sequences described by $\int u_n(x)\,dg(x)$ and $\int v_n(x)\,dg(x)$ have finite limits, these limits are the integrals of $u(x)$ and $v(x)$ respectively. Hence $u(x)$ and $v(x)$, and therefore also their difference $f(x)$, are summable with respect to $g(x)$, and $\int f(x)\,dg(x)$, being the difference of $\int u(x)\,dg(x)$ and $\int v(x)\,dg(x)$, is the limit of $\int f_n(x)\,dg(x)$.

If, on the other hand, one of the auxiliary u and v sequences is such that its integrals have infinity for limit, we notice first of all that it is then impossible for both the sequences to have this property. For, if the

f-sequence is monotone ascending the v-sequence is monotone descending, and if the f-sequence is monotone descending, the u-sequence is monotone descending. Suppose then it is the u-sequence which is monotone ascending and has for its integrals the limit infinity; then the limiting function $u(x)$ is non-summable with respect to $g(x)$, and therefore the same is true of $f(x)$, which is not less than $u(x)$. Similarly when it is the v-sequence which has for its integrals the limit infinity, $f(x)$ is non-summable with respect to $g(x)$.

Thus we may say that the term-by-term integration with respect to $g(x)$ of monotone sequences of function which are summable with respect to $g(x)$ is always allowable; this being, as usual, interpreted with the gloss that if, and only if, the limit of the integrals of the sequence is infinite, the limiting function is non-summable with respect to $g(x)$.

29. Returning to the integral of a simple l- or u-function $f(x)$ given in § 4, we see that if the function $g(x)$ is continuous, the expression on the right reduces to

$$\sum_{i=1}^{n-1} f(c_i+0)\,[g(c_{i+1})-g(c_i)],$$

and if $g(x)$ is the integral of a function $\gamma(x)$, this may be written

$$\int_a^b f(x)\,dg(x) = \sum_{i=1}^{n-1} f(c_i+0)\left[\int_{c_i}^{c_{i+0}} \gamma(x)\,dx\right] = \int_a^b f(x)\,\gamma(x)\,dx.$$

Hence it follows that if $f(x)$ describes any monotone sequence, or any succession of monotone sequences, so that $f(x)\,\gamma(x)$ remains summable, not only during the sequences themselves but also in the limit, we shall continue to have

$$\int_a^b f(x)\,dg(x) = \int_a^b f(x)\,\gamma(x)\,dx,$$

a relation which accordingly holds for all functions of monotone type $f(x)$, when

$$g(x) = \int_a^b \gamma(x)\,dx.$$

30. It only remains to add a few words on the definition and properties of an integral with respect to a function of bounded variation. If $h(x)$ be a function of bounded variation it is expressible as the difference of two positive monotone increasing functions, say $g_1(x)-g_2(x)$. We may

therefore say that a function $f(x)$ is summable with respect to $h(x)$ if it is summable with respect both to $g_1(x)$ and to $g_2(x)$, and define the integral of $f(x)$ with respect to $h(x)$ as $\int f(x)\, dg_1(x) - \int f(x)\, dg_2(x)$. With this definition it is evident that when $f(x)$ is a simple l- or u-function the form of the integral given in § 4 is unaltered, we have indeed only to change g into h on each side to obtain the equation which now holds. Hence it follows that the integral of a simple l- or u-function $f(x)$ with respect to a function of bounded variation $h(x)$ is independent of the particular way in which we express $h(x)$ as the difference of two positive monotone increasing functions. Thus, as long as we confine ourselves to bounded functions $f(x)$, and bounded monotone sequences no difficulty will occur. When, however, we have to deal with unbounded monotone sequences of bounded functions, or with unbounded functions, the theory as here developed can only be applied in so far as the results when we divide up $h(x)$ into $g_1(x) - g_2(x)$ are not illusory, through the appearance of the indeterminate form $\infty - \infty$.

31. In conclusion one application of the present theory may be given.

Let $f_n(x)$ be a function which as n approaches $f(x)$ as limit, and let the integrals of f_n and f respectively with respect to the independent variable be denoted by $F_n(x)$ and $F(x)$. If then $F_n(x)$ converges boundedly to $F(x)$ we may integrate term-by-term with respect to a positive monotone increasing function $g(x)$, and write

$$\operatorname*{Lt}_{n\to\infty} \int F_n(x)\, dg(x) = \int F(x)\, dg(x).$$

Since $F_n(x)$ and $F(x)$ are continuous, we may integrate by parts, which gives (§ 21)

$$\operatorname*{Lt}_{n\to\infty} \left\{ \Big[g(x)\, F_n(x) \Big]_a^x - \int_a^x g(x)\, dF_n(x) \right\} = \left\{ \Big[F(x)\, g(x) \Big]_a^x - \int_a^x g(x)\, dF(x) \right\}.$$

Hence, since F_n and F are both integrals,

$$\operatorname*{Lt}_{n\to\infty} \int_a^x g(x)\, s_n(x)\, dx = \int_a^x g(x) f(x)\, dx.$$

We have in this simple manner by the use of integration with respect to a monotone increasing function reproved a theorem stated and proved in my paper on "The Application of Expansions to Definite Integrals,"

the direct proof of which required delicate and lengthy handling. This theorem is as follows :—

If $\int f_n(x)\,dx$ *converges boundedly to* $\int f(x)\,dx$, *and* $g(x)$ *is any function of bounded variation in the interval considered, then*

$$\underset{n\to\infty}{\mathrm{Lt}} \int s_n(x)\,g(x)\,dx = \int f(x)\,g(x)\,dx.$$

This is only one of a number of cases in which the present theory may be used with a gain of clearness and simplicity in the proof of results whose enunciations do not themselves involve the concept of integration with respect to a function of bounded variation.

32. In conclusion we may give the following theorem as an instance of theorems in two or more dimensions in which the present theory may be used :—

THEOREM.—*If* $f(x, y)$ *is a function of monotone type in the two independent variables* x *and* y *for* $(0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b)$, *then the repeated integrals of* $f(x, y)$ *with respect to a positive monotone function* $g(x)$ *and to* y *are equal, provided these integrals exist.*

It will be at once seen that the proof of this theorem by the method of monotone sequences is immediate; we have only to convince ourselves that the theorem is true when $f(x)$ is a simple l- or u-function of (x, y). Assuming in the first instance that this has been proved, we proceed by induction.

Let $f_n(x, y)$ be the constituent function of a monotone sequence with $f(x, y)$ as limit. We then have

$$\int_0^b dy \int_0^a f(x, y)\,dg(x) = \int_0^b dy \underset{n\to\infty}{\mathrm{Lt}} \int_0^a f_n(x, y)\,dg(x)$$

$$= \underset{n\to\infty}{\mathrm{Lt}} \int_0^b dy \int_0^a f_n(x, y)\,dg(x),$$

since the last inside integral describes a monotone sequence. Similarly,

$$\int_0^a dg(x) \int_0^b f(x, y)\,dy = \underset{n\to\infty}{\mathrm{Lt}} \int_0^a dg(x) \int_0^b f_n(x, y)\,dy.$$

Assuming therefore the theorem known to be true for all the functions

$f_n(x, y)$, this gives

$$\int_0^b dy \int_0^a f(x, y)\, dg(x) = \int_0^a dg(x) \int_0^b f(x, y)\, dy.$$

This proves the theorem by induction, supposing it known to be true when $f(x, y)$ is a simple l- or u-function of (x, y). It will evidently only be necessary to discuss the case when $f(x, y)$ is a simple l-function of (x, y).

The simple l-function $f(x, y)$ determines a set of completely open rectangles throughout which $f(x)$ is constant, and which, with their boundary points fill up the closed fundamental rectangle. Let us project the sides of these rectangles on to the axis of x; we thus have a finite number of points,

$$c_1 = 0,\ c_2,\ \ldots,\ c_{n-1},\ c_n = a.$$

Now whatever point x we take in a chosen completely open interval between two of these points of division the ordinate through the point x will meet the same rectangles; it is only when we pass from one of these completely open intervals to another that the rectangles met by the ordinate will vary. Consequently

$$F(x) = \int_0^b f(x, y)\, dy,$$

which is—since for fixed x, $f(x, y)$ is a simple l-function of y, constant inside the rectangles—a simple l-function of x, constant in each of the completely open intervals between the points of division $c_1, c_2, \ldots, c_n$; that it is an l-function is known* and may easily be proved.

Hence, by the definition of the integral of a simple l-function,

$$\begin{aligned}\int_0^a F(x)\, dg(x) = \sum_{i=1}^{i=n-1} \{F(c_i)[g(c_i+0)-g(c_i)] \\ +F(c_i+0)[g(c_{i+1}-0)-g(c_i+0)] \\ +F(c_{i+1})[g(c_{i+1})-g(c_{i+1}-0)]\}. \qquad (1)\end{aligned}$$

On the other hand, for y constant, $f(x, y)$ is a simple l-function of x, whose discontinuities are among the points $c_1, c_2, \ldots, c_n$. We shall therefore make no error if we write down the expression for the integral using all

* W. H. Young, "On Parametric Integration," 1910, *Monatshefte f. Math. u. Phys.*, Jan., pp. 1–24.

these points $c_1, c_2, \ldots, c_n$ as points of division. We then have

$$\int_0^a f(x, y)\, dg(x) = \sum_{i=1}^{i=n-1} \{f(c_i, y)[g(c_i+0) - g(c_i)] \\ + f(c_i+0, y)[g(c_{i+1}-0) - g(c_i+0)] \\ + f(c_{i+1}, y)[g(c_{i+1}) - g(c_{i+1}-0)]\}, \tag{2}$$

where the right-hand sides of (1) and (2) only differ in the substitution of f for F. It follows therefore, from the meaning of $F(x)$, that when we integrate the left-hand side of (2) with respect to y we get the right-hand side of (1). This proves the required result, viz.,

$$\int_0^b dy \int_0^a f(x, y)\, dg(x) = \int_0^a dg(x) \int_0^b f(x, y)\, dy.$$

THE SKEW ISOGRAM MECHANISM*

By G. T. BENNETT.

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1. A closed chain of n rigid bodies, each connected with its two neighbours by a hinge-line, has in general $n-6$ degrees of internal freedom; and so when $n = 4$ the chain is in general triply stiff. There are three cases in which, exceptionally, one degree of freedom exists: namely, (i) when the hinge-lines are parallel, (ii) when the hinge-lines are concurrent, (iii) when the hinge-lines, taken in order, have common normals which are consecutively intersecting and alternately equal in length. The first two cases are now long familiar. The last case seems to have been first published ten years ago, by the author (*Engineering*, December 4th, 1903, p. 777, " A New Mechanism "). It was found also independently by Borel (*Comptes Rendus*, December 19th, 1904; and *Mémoires*, Tome XXXIII, No. 1, p. 56). It has since been noticed by Bricard (*Nouvelles Annales*, 1906, p. 78). In this present paper some properties of the deformable skew isogram will be given, and some compound mechanisms will be exhibited in which the skew isogram occurs as a basis of their construction.

2. Let the consecutive hinge-lines be h, k, h', k'; let the common normals intersect on these at A, B, A', B'; let $AB = A'B' = a$ and $BA' = B'A = b$ (Fig. 1). The quadrilateral $ABA'B'$ is a skew isogram, and the hinge-lines at the corners are normal to the adjacent sides. The four rigid links lie each along one side of the isogram, with hinges normal to the link at its two ends. The figure is symmetric about a line z which perpendicularly bisects the diagonals AA', BB'. The alternate interior angles at A, B, A', B' are equal, say θ, ϕ, θ', ϕ'. The consecutive inclinations of the hinge-lines, to be called the twists of the links, are also alternately equal, say α for AB and $A'B'$, and β for BA' and $B'A$.

* The term " isogram " is used to describe any four-sided figure, whether plane, spherical or skew, which has its alternate sides equal in length.

The freedom of the mechanism appears as a consequence of the

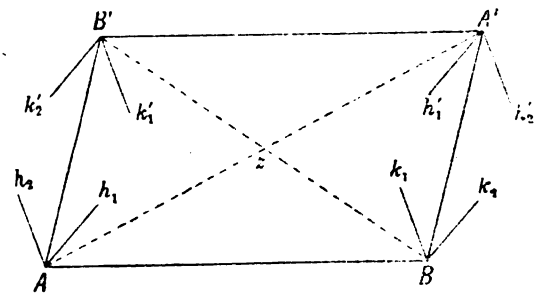

FIG. 1.—(*Vide* §§ 1–8). The skew isogram, seen from infinity on its axis of symmetry as a parallelogram; the angle-bisectors at its vertices appearing as two sets of four parallels.

symmetry of the tetrahedron $ABA'B'$ about the axis z. The volume of any tetrahedron may be written as $\frac{2}{3}\Delta\Delta' \sin a \div a$, where a is an edge, Δ, Δ' the triangles meeting in the edge, and a the inclination of their planes. Here the two triangles are of the same dimensions for each side of the isogram in turn, and hence $a/\sin a = b/\sin\beta = p$, say. If the isogram were merely articulated at the vertices, then a and β would be separately variable. The formula shows that both are constant if one is, and so proves the hinged mechanism deformable. The length p, taken as a scalar quantity, will be called the "index" of a link. The The links which compose the skew isogram consist of two congruent pairs all with the same index. It is to be observed that links of equal length and of the same index may have twists either equal or supplementary; so that they are either twins or images of each other.

The spherical indicatrix (Fig. 2) of the skew isogram is formed by

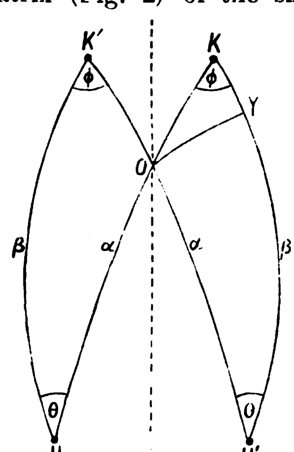

FIG. 2.—Spherical crossed isogram.

taking vertices H, K, H', K' representing the directions of h, k, h', k'. An arbitrary sense is given to h, and those of k, h', k' are derived by the successive twists, all measured in the same sense. The figure is a spherical crossed isogram with constant sides, and deforms simultaneously with the skew isogram.

Conversely, starting from any spherical quadrilateral, a skew quadrilateral of determinate form may be drawn with its sides in directions given by the vertices of the reciprocal spherical quadrangle; the sines of the solid angles of the four triangles giving the ratios of the lengths of the skew sides. For the case of the spherical isogram with constant sides α, β, α, β, these sines are equal to $\sin\theta\sin\phi\sin\alpha$ and $\sin\theta\sin\phi\sin\beta$, each occurring twice. The skew figure is an isogram with sides a, b, a, b, where $a/\sin\alpha = b/\sin\beta$; and so a and b may both be constant and the deformability is again proved.

3. Any one side of the quadrilateral $HKH'K'$ being omitted, Napier's formula for the remaining triangle gives

$$\tan\tfrac{1}{2}\theta\,\tan\tfrac{1}{2}\phi = \sin\tfrac{1}{2}(\alpha-\beta)/\sin\tfrac{1}{2}(\alpha+\beta). \qquad (1)$$

When $\theta = 0$, then $\phi = \pi$, and the links of the skew isogram are in one straight line $AB'BA'$, normal to all the hinge-lines; and when $\theta = \pi$, then $\phi = 0$, and the links lie in a straight line $B'AA'B$, normal to all the hinges. If θ increases continuously $\pi-\phi$ increases continuously, and if AB is fixed AB' and $B'A$ revolve continuously as cranks, with the skew connecting rod $B'A'$ transmitting the motion between the non-parallel and non-intersecting crank-axes h, k. The two rectilinear zero positions give neither "change-points" nor "dead-centres" as with parallel cranks. (For the mechanical aspects of the mechanism reference may be made to the article above quoted.)

Special cases of the mechanism occur:—

(i) $\alpha = \pi-\beta$, $a = b$. The isogram is equilateral. The four links consist of a twin pair, and another twin pair, images of the first. The diagonals AA', BB' are perpendicular and the figure is symmetrical about the plane through either diagonal and the axis z. Opposite hinges h and h' intersect at a point A_0 on z; and hinges k and k' intersect at a point B_0 on z. The formula (1) in this case gives $AA_0 . BB_0 = p^2\cos\alpha$, so that the product of the normals keeps constant during deformation.

(ii) $\alpha = \beta$, $a = b$. The four links are congruent. The motion

is discontinuous; $\theta = 0$ allows any value for ϕ, and $\phi = 0$ allows any value for θ.

(iii) $\alpha = 0$, $\beta = 0$. The plane crossed isogram.

(iv) $\alpha = 0$, $\beta = \pi$. The plane parallelogram.

It will be convenient later to use a notation which associates analogous mechanisms of the plane and spherical and skew varieties. Of those now being considered the plane isogram (whether a parallelogram or crossed) shall be denoted by (A1); the spherical isogram (whether crossed or not) shall be (A2); and the hinged skew isogram (A3). The mechanism (A2) occurs as the spherical indicatrix of (A3); and (A1) is a limiting case of either (A2) or (A3). (It is to be observed that a spherical crossed isogram becomes an uncrossed isogram if adjacent vertices are replaced by their antipodes.)

4. As a converse to the foregoing, if two rigid bodies revolve about fixed axes h and k, distant a apart and inclined at an angle α, and if the angles of rotation θ and ϕ are connected by any equation of the form

$$A \tan \tfrac{1}{2}\theta \tan \tfrac{1}{2}\phi + B \tan \tfrac{1}{2}\theta + C \tan \tfrac{1}{2}\phi + D = 0, \tag{2}$$

then a link of length a and twist α may connect them, consistently with their relative motion. For the equation (2) can be rewritten in a unique form

$$\tan \tfrac{1}{2}(\theta - \theta_0) \tan \tfrac{1}{2}(\phi - \phi_0) = \sin \tfrac{1}{2}(\alpha - \beta)/\sin \tfrac{1}{2}(\alpha + \beta), \tag{3}$$

differing from (1) only in the zero lines of the bodies from which the angles are to be measured, and thus determining the sides BA' and AB' of the isogram as lines of the bodies. The twist β satisfies the equation

$$\tan^2 \tfrac{1}{2}\beta \cot^2 \tfrac{1}{2}\alpha = [(A+D)^2 + (B-C)^2] / [(A-D)^2 + (B+C)^2], \tag{4}$$

and the only failing case occurs when the bodies rotate through equal angles; for then $B = C$, $A = -D$, $\beta = 0$, $b = 0$, or $B = -C$, $A = D$, $\beta = \pi$, $b = 0$, and the cranks are of zero length. If, however, the axes of rotation are parallel in this case, and rotation takes place in the same sense for the two bodies, then b is indeterminate and the isogram is a parallelogram.

5. Some useful formulæ connected with the geometry of rotating bodies may here be inserted. If two rigid bodies have a hinge line h, and if k and k' are any two lines, one of each body, then during the relative rotation about h the lines k and k' have a variable inclination ψ

and a variable distance δ, and these two variables are related. The relation is not simple unless, as will be supposed here, the common normals of h with k and k' meet on h. Projecting the closed pentagon $\delta kabk'$ on to δ, we then obtain a relation of the form

$$\delta \sin\psi = P - Q \cos\psi, \tag{5}$$

where P and Q are constants given by

$$P = a \cos\beta/\sin\alpha + b\cos\alpha/\sin\beta, \tag{6}$$

$$Q = a\cot\alpha + b\cot\beta. \tag{7}$$

The following special cases occur:—

$$\text{(i)}\quad a\cot\alpha = -b\cot\beta, \qquad Q = 0, \qquad \delta\sin\psi = P; \tag{8}$$

$$\text{(ii)}\quad a/\sin 2\alpha = -b/\sin 2\beta, \qquad P = 0, \qquad \delta\tan\psi = -Q; \tag{9}$$

$$\text{(iii)}\quad a/\sin\alpha = b/\sin\beta, \qquad P = Q, \qquad \delta\cot\tfrac{1}{2}\psi = P; \tag{10}$$

$$\text{(iv)}\quad a/\sin\alpha = -b/\sin\beta, \qquad P = -Q, \qquad \delta\tan\tfrac{1}{2}\psi = P. \tag{11}$$

In each case the condition for the constant value may be got by equating the values corresponding to the two zero positions, those for which a and b are in one line. In case (i) the constant product is the mutual moment of the lines. In case (ii) the constant is what may conveniently be called the relative pitch of the lines: it is the pitch of the screw having either line as axis and the other as a ray. Cases (iii) and (iv) are virtually the same, and they apply to the case of the hinged skew isogram. They show that the relative pitch of z and h, or z and h', is constant, and given by

$$(z, h) = \tfrac{1}{2}p(\cos\alpha + \cos\beta). \tag{12}$$

The relative pitches (z, k) and (z, k') have the same value.

The axis z has also a constant pitch relatively to any one of the sides of the isogram. The distance of z from a is half the common normal between the sides a, a; its length is therefore $\tfrac{1}{2}b\sin OY$ (Fig. 2). The inclination of z and a is $\tfrac{1}{2}KOH'$, and from the figure

$$\sin OY \tan\tfrac{1}{2}KOH' = (\cos\beta - \cos\alpha)/\sin\beta.$$

Hence $\quad (z, a) = \tfrac{1}{2}p(\cos\beta - \cos\alpha)$

and similarly $\quad (z, b) = \tfrac{1}{2}p(\cos\alpha - \cos\beta)$ $\qquad$ (13)

6. The figure of the skew isogram has certain quadric surfaces intimately associated with it. So much of their geometry as is connected

with the kinematics of the mechanism may be conveniently enunciated here. To the isogram are added first the internal angle bisectors at the vertices A, B, A', B', namely, h_1, k_1, h_1', k_1', and also the external angle bisectors $h_2 k_2 h_2' k_2'$. The following theorems then occur :—

(i) Two hyperboloids of revolution R_1, R_2 contain the sides of the isogram and touch each other at the vertices. The axes r_1, r_2 meet z normally at the respective centres C_1, C_2.

(ii) The lines $h_1 h_1' k_2 k_2'$ are meridian tangents of R_1 and meet r_1 : and $h_2 h_2' k_1 k_1'$ are meridian tangents of R_2 and meet r_2.

(iii) The lines $h_1 h_1' k_2 k_2'$ and z are generators of a rectangular hyperbolic paraboloid Π_1 and have a common normal r_2' parallel to r_2: and $h_2 h_2' k_1 k_1'$ and z are generators of a rectangular hyperbolic paraboloid Π_2 and have a common normal r_1' parallel to r_1.

(iv) The meridian plane $C_1 h_1$ of R_1 and the meridian plane $C_2 h_2$ of R_2 intersect perpendicularly in h, and pass respectively through r_1 and r_2. Similarly for h', k, k'. The orthogonal hyperboloid Ω generated by the line of intersection of perpendicular planes through r_1 and r_2 contains the four hinge lines h, h', k, k'. It has z as one axis, r_1, r_2 as generators at vertices C_1, C_2, and centre coincident with that of the sphere through the vertices A, B, A', B'. The tangents from C_1 or C_2 to the sphere have constant length (z, a). (13).

(v) Planes perpendicular to r_1 cut R_1 in latitude circles, and cut Ω in one set of circular sections. Planes perpendicular to r_2 cut R_2 in latitude circles, and cut Ω in the other set of circular sections.

7. The kinematics of the finite displacements may now be considered. First, with AB supposed fixed, consider the screw displacement which will take $A'B'$ to its zero position, given by $\theta = 0$, $\phi = \pi$. This may be got by compounding rotations (in succession) θ about h' and $\phi - \pi$ about k; as equivalent, half turns on h_1', $A'B$, $A'B$, k_2; as equivalent, therefore, half turns on h_1' and k_2 alone; and so [§ 6 (iii)] the screw has axis r_2', with shift and turn double of those which would take h_1' to k_2. Alternatively the same displacement of $A'B'$ is obtainable by taking rotations $\phi - \pi$ on k' and θ on h; as equivalent, half turns on k_2', AB', AB', h_1; as equivalent, therefore, half turns on k_2', h_1 alone; and so the screw has axis r_2' as before, and has the same shift and turn. The screw displacement of $A'B'$ to the other zero position is found, similarly, to have r_1' as axis.

Conversely, the screw displacement of $A'B'$ from a zero position to

any other may be considered. The geometry is simple, and may usefully be put in a form independent of all that precedes, as follows:—Take any rectangular hyperbolic paraboloid with ζ as one of the generators through the vertex, λ any other of the same system, and μ, μ', ν, ν' generators of the opposite system, normal to ζ and cutting λ in points A, A', B, B'. Let μ and μ' be equidistant from η, the second generator through the vertex, and let ν and ν' be also equidistant. So $AB = A'B' = a$, say; and $BA' = B'A = b$. Let the normals at A, B, A', B' be h, k, h', k'; and let their inclinations be $\alpha = \widehat{hk} = \widehat{h'k'}$ and $\beta = \widehat{kh'} = \widehat{k'h}$. Let μ and μ' cut λ at an angle $\frac{1}{2}\theta$, and let ν and ν' cut λ at an angle $\frac{1}{2}(\pi-\phi)$.

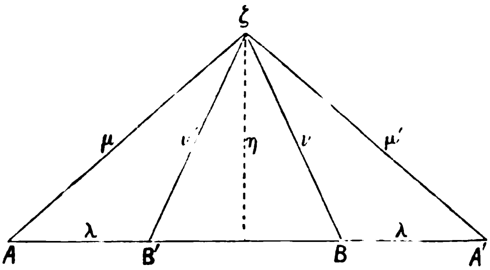

FIG. 3.—(*Vide* § 7.) Rectangular hyperbolic paraboloid seen from infinity on a principal generator.

(Fig. 3.) Regard then AB, BA', $A'B'$, $B'A$ as four links in one line, with the normals h, k, h', k' as hinges. The following results appear:—

(i) The four points A, B, A', B' of the generator λ have the same cross-ratio as the tangent planes at the points. Hence

$$a^2/b^2 = \sin^2\alpha/\sin^2\beta;$$

and the four links have all the same index.

(ii) Half-turns on μ' and ν are equivalent to half-turns on $\mu'\lambda\lambda\nu$, and so to rotations θ on h' and $\pi-\phi$ on k. Hence, if AB is fixed, and $A'B'$ moves, the link AB' may connect them without preventing these rotations. Similarly half-turns on ν' and μ are equivalent to half-turns on $\nu'\lambda\lambda\mu$, and so to rotations $\pi-\phi$ on k' and θ on h. The link $A'B$ allows these. Further, in virtue of the symmetry of $\mu\mu'\nu\nu'$, with ζ for normal, the two screw movements are the same.

(iii) Any rectangular hyperbolic paraboloid has a property expressed by the formula $\tan\chi \sin\psi = \cot\epsilon$; where, for any generator meeting λ, χ is their angle of intersection, ψ is the angle the normal

makes with the normal at $\lambda\eta$, and ϵ is the angle $\lambda\zeta$. For A', $\chi = \frac{1}{2}\theta$ and $\psi = \frac{1}{2}(\alpha+\beta)$; for B', $\chi = \frac{1}{2}(\pi-\phi)$ and $\psi = \frac{1}{2}(\alpha-\beta)$, and hence the formula (1).

(iv) A single infinity of rectangular hyperbolic paraboloids can be drawn with λ as generator and with given normals (h, k, h', k' and so all others) along λ. Each provides a principal generator ζ, a screw on which takes $A'B'$ from the zero position to one of the skew positions. One paraboloid, specially, has ζ coincident with λ; and the screw on λ leads to the other zero position.

8. The relative motion of a pair of opposite links may now be considered.

Let AB be supposed fixed, and let $A'B'$ have continuous motion. In any one of its positions a half-turn about z brings it to coincide with AB, end with end and hinge with hinge. Hence all positions of $A'B'$ are obtainable from the fixed piece AB by making half-turns about the different positions of the line z. The pitch of z relatively to h and k and AB are all constant (12)–(13); hence z generates a hyperboloid z. It has AB as one axis and the middle point of AB as centre. Any point P' carried by the moving link is derivable from the corresponding point P of the congruent fixed link by a half-turn on z. Hence the locus of P' is got by taking the doubled generator-pedal of the point P with regard to the hyperboloid Z. Specially, any point P of h gives a circular locus for P' on h', and any point P of k gives a circular locus for P' on k'. The geometry of any quadric Z, which associates itself with this property, may be briefly stated :—

(i) Taking one system of generators z, and one system of circular sections γ, planes through the points of a circle γ drawn perpendicular to the generators z all meet in a point P.

(ii) The locus of P for different circles γ is a line h normal to an axis a of z, and perpendicular to the planes of the other system of circles.

(iii) Taking either system of generators and either system of circles four such lines are obtained. They are generators of a confocal quadric, passing through its vertices on a.

(iv) The relation between the confocals is mutual, and the generators through the vertices of each are perpendicular to planes of circular sections of the other.

To find the axis of the small screw displacement for consecutive posi-

tions take adjacent generators z, z'. Half turns on these are equivalent to the screw displacement, and its axis is the common normal σ of z and z'. It touches the hyperboloid Z at a point of the line of striction and is perpendicular to z. It is also normal to the moving cranks b, b; for small rotations about h and k' give a screw motion with axis normal to AB', and small rotations about k and h' give a screw motion with axis normal to BA'. (The identity of these two screws demands that h, h', k, k' should be generators of a quadric. The quadric is Ω.) The line σ, normal to AB', is at distances b_1, b_2, say, from h and k', and makes angles β_1, β_2 with them; so that $b_1+b_2 = b$ and $\beta_1+\beta_2 = \beta$; and, from the pitch formulæ of a cylindroid $b_1 \cot \beta_1 = b_2 \cot \beta_2$. These are equally true for σ in relation to h and k (in place of k'), and suffice to define the locus of σ independently of the hyperboloid. The surface locus generated by σ is the space axode for the movement of the link $A'B'$. The body axode is similarly found, and is congruent with the first, the two being images in z. They have contact along σ and screw motion about it during the movement.

The familiar analogues in the case of the plane crossed isogram (A1) and the spherical isogram (A2) are these:—the instantaneous centre for the connecting link is the point of intersection of the revolving cranks; its locus is a conic (or sphero-conic) with the ends of the link for foci, either for the connecting link or the fixed; and the locus of any carried point is the doubled pedal of the corresponding fixed point with regard to the space-centrode conic.

9. Some compound mechanisms constructed by use of the skew isogram will now be described. To each one corresponds a spherical mechanism given by the directions of the hinge lines, and also a plane mechanism obtained as a degenerate case of the spherical mechanism. In cases where these are not already known they will be described in advance, and the skew mechanism afterwards.

A plane mechanism of twelve pieces (B1) composed of crossed isograms shall be described first.* Take (Fig. 4) concentric circles of arbitrary radius, centre O. Start from any point A on one circle and pass by any zigzag track $AB'CA'$, visiting the two circles alternately, from A to A'. Each step subtends an angle at O, and the resultant step subtends an angle equal to their sum. Retaining the angles but permuting their order six different tracks may be made from A to A', as in the figure. The four lengths BC', $B'C$, AD', $A'D$ are equal, and similarly for two other sets of four: $AB'DC'$ and $A'BD'C$ are a pair of crossed isograms, and similarly

* But see also Bricard, *Nouvelles Annales*, January 1913.

for two other such pairs. If these twelve lines are made links, and hinged at the eight points, the resulting mechanism (B1) has two degrees

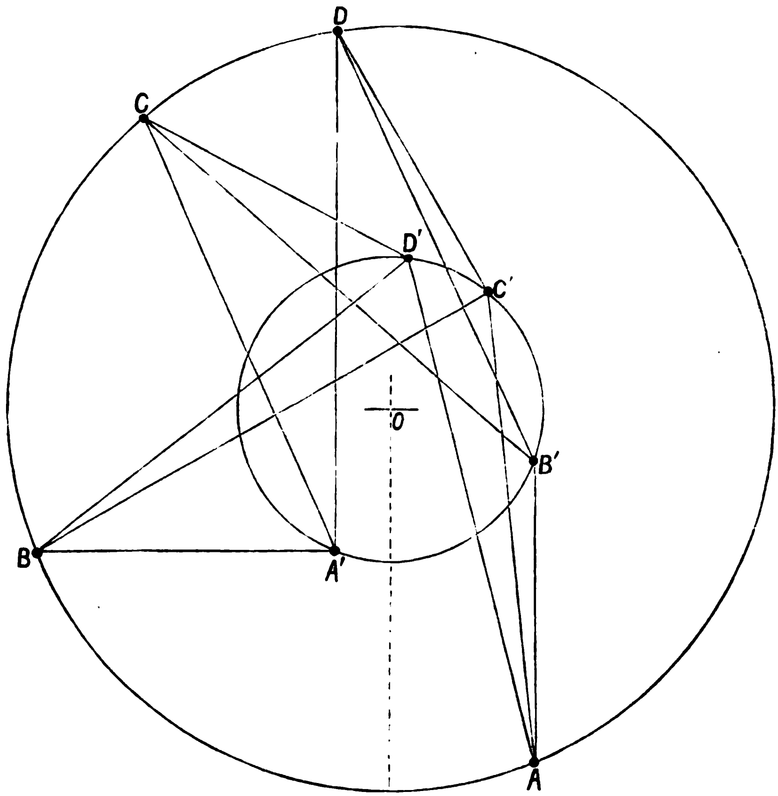

FIG. 4.—(*Vide* § 9.) Plane mechanism of twelve links hinged by threes at eight points, with two degrees of freedom.

of freedom; for the radii of the circles may vary independently. (The normal freedom for such a system as this would be unity.) The figure is the reciprocal, with respect to its centre, of the deformable double-four described in a paper on "Deformable Octahedra" [*Proc. London Math. Soc.*, 1911, Vol. 10, pp. 309–343, hereafter referred to as (D.O.)].

The figure depends for form and size on five parameters, and may be drawn, without any use of the circles, by starting with three concurrent links, inclined at arbitrary angles, as data. The rest of the figure is obtained by the drawing of crossed isograms. Changing the notation, let a base point 0 be taken, and three other points 1, 2, 3. A crossed isogram with consecutive vertices 1, 0, 2 gives for its fourth vertex a point 12; points 13 and 23 are obtained similarly. Next a crossed isogram with

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consecutive vertices 12, 1, 13 gives a fourth vertex 123; and three points obtained and named in this way are all identical. It may conveniently be said that the addition of three vecters by crossed isograms is commutative. Addition by parallelograms is of course intuitively commutative by virtue of the idea of direction.

Reverting to the mechanism and its notation, it is clear that the quadrangles $ABCD$ and $A'B'C'D'$ have corresponding-opposite sides parallel, such as BC and $A'D'$; and their product is constant and equal to $CA'^2 \sim BA'^2$, with the same value for the product $B'C'.AD$. Any four equal links, such as BC', $B'C$, AD', $A'D$ touch a circle with centre O; and there are three such circles.

For the kinematics, consider the instantaneous centres of relative motion of four equal links BC', $B'C$, AD', $A'D$. The relative centre for BC' and AD' is the intersection of $C'A$ and $D'B$, that of $B'C$ and $A'D$ is the intersection of CA' and DB', that of BC' and DA' is the intersection of $C'D$ and $A'B$, that of $B'C$ and $D'A$ is the intersection of CD' and AB'. By stiffening the isogram to which they belong any consecutive two of the cycle of four BC', AD', $B'C$, $A'D$ may be made to move as one piece, leaving one degree of freedom in the mechanism. Hence, as the instantaneous centres of three pieces taken in pairs are always collinear, it follows here that all four centres are collinear. They are the intersections of corresponding-opposite sides of the isograms $BA'CD'$ and $B'AC'D$. There are three such axes of perspective, and they may be shown to be concurrent. They correspond to the three centres of perspective found for the reciprocal figure (D.O., §§ 8, 9). Further, the relative centre for BC' and $B'C$ is somewhere on the first axis of perspective, its position depending upon the particular motion selected from those possible with the two degrees of freedom available; and similarly for AD' and $A'D$. The six relative centres of four pieces are in general the vertices of a quadrilateral; they are here collinear, and are hence three pairs of points in involution.

The mechanism with one degree of freedom got by stiffening one isogram, as a fixed base, leaves the companion isogram moving with its vertices connected to fixed points by four equal links. It occurs among a class discussed by A. C. Dixon. (*Messenger of Mathematics*, Vol. XXIX, p. 18.) Very specially the stiff isogram may be given either of its rectilinear forms. The doubly-free mechanism may come to have all its links in one straight line in four different ways.

10. The commutative addition of vectors by plane crossed isograms may be extended to any number.

Take, at starting, vectors joining a point 0 to points 1, 2, 3, 4. At the first stage of crossed-isogram addition there arise six points 12, 13, 14, 23, 24, 34; and at the second stage four points 123, 124, 134, 234. Then any pair of these lead to a point 1234 which is the same as for any other pair. For consider the three points 123, 124, 134; they belong to a three-vector figure, with 1 as a starting point, from which vectors are drawn to 12, 13, 14; at the next stage the points 123, 124, 134 are obtained; and the three-vector figure completes itself (§ 9) in three coincident points here termed 1234. The six possible points 1234 are therefore coincident in sets of three, and are therefore all one point.

The corresponding mechanism has thirty-two links, equal in sets of eight, hinged by sets of four (all unequal) at sixteen hinge points, and has three degrees of freedom. (It would normally be quadruply stiff.) It may, in four ways, be regarded as composed of two mechanisms of class (B1), made of the same material, with the hinge-points connected in pairs by an additional set of eight equal links. If one of the two (B1) mechanisms were stiffened there would remain one degree of freedom.

The proof of the commutative addition may be extended by induction to any number of vectors n. Combined $n-1$ at a time, by crossed addition, they lead (by hypothesis) to a unique point. For each vector entirely omitted there is one such point. Taking the three terminals arrived at by omission of any one of the first three vectors, there are then constructible three isograms which all give rise to the same final point: for they complete a three-vector figure whose own starting point is arrived at by the entire omission of all three vectors. The final points are therefore coincident by threes, and are therefore all one single point.

The corresponding mechanism (C1) has $2^{n-1}.n$ links, 2^{n-1} of each length; they are hinged at 2^n junctions, n at each junction. There are $n-1$ degrees of freedom. [The normal freedom would be $(4-n)\,2^{n-1}-3$.] In 2^{n-1} ways the links may come all in one straight line. From each of these rectilinear forms the mechanism may pass into the figure representing the ordinary commutative addition by parallelograms.

Two hinge points may be termed "opposite" when either is the final terminal corresponding to the other as starting point: they are the terminals of unclosed polygons with sides consisting of one link of each length. Two equal links are then "opposite" when their terminals are opposite. If $n = 4$, opposite links have complete (three degrees of) freedom relatively to each other. If either is fixed the other may be

brought to any position (within a limited range) by adjustment of the angles made by the fixed link with the three revolving links at either end.

11. The method of § 9 used in establishing the mechanism (B1) with two degrees of freedom is equally applicable to spherical geometry and produces a figure and mechanism (B2). It is composed of three sets of four equal arcs, hinged in sets of three at eight points. The eight points lie four on a small circle and four on another concentric small circle. There are three pairs of crossed isograms. The mechanism has two degrees of freedom. The six instantaneous centres of relative motion of any four equal links lie on a great circle, the axis of perspective of a pair of isograms. The figure is the spherical reciprocal of the spherical double-four. (D.O., § 26.)

In other terms, spherical addition of three vectors by crossed isograms is commutative. Further, the proof of § 10 holds good for the sphere; and hence addition by crossed isograms on the sphere is commutative for any number of vectors. A mechanism (C2) is given by such a figure, the spherical counterpart of (C1), with $n-1$ degrees of freedom.

It is noticeable that for crossed isograms addition is commutative for both the plane and the sphere: but for uncrossed isograms commutative addition is intuitive for the plane and is non-existent for the sphere.

12. The plane and spherical double-fours composed of crossed isograms, giving mechanisms with one degree of freedom (D.O., §§ 26, 27) will here be denoted (D1) and (D2). They are special cases of more general mechanisms (E1), (E2) which are reciprocal to (C1) and (C2), and may be derived from the figure of (C2) as follows.

In a spherical crossed isogram with angles θ, ϕ and sides α, β, the equation

$$\tan \tfrac{1}{2}\alpha / \tan \tfrac{1}{2}\beta = \cos \tfrac{1}{2}(\theta - \phi) / \cos \tfrac{1}{2}(\theta + \phi) \tag{14}$$

is equivalent to (1) of § 3. Hence, if the sides are altered in length, but the tangents of their halves preserve the same ratio, then the angles may remain unaltered. It follows that in the figure (C2), showing commutative spherical crossed-isogram addition of n vectors, the figure may preserve the angles at all the junctions unaltered while the tangents of the halves of all the vectors are altered in a fixed ratio. A new figure which is its spherical reciprocal will therefore have constant lengths and variable angles, and gives a mechanism (E2) with one degree of freedom. When the sphere becomes a plane the corresponding mechanism (E1) arises.

The figure (E1) may be constructed (cf. § 9) by starting with a line 0,

and lines 1, 2, 3, ..., n crossing it. To each line a sense is attributed. A crossed isogram with consecutive sides 1, 0, 2 is completed by a line 12; the line 12, with sense included, being obtained by reflecting 0 in the external angle bisector of 1 and 2. The same process continues throughout. The same method applies to (E2). The mechanism has 2^n links with n hinges on each, $2^{n-1}.n$ hinges in all, and has one degree of freedom. [The normal freedom would be $(3-n)\,2^n-3$.]

If the lines 1, 2, 3, ..., n are made tangents to a circle, then a special case of (E1) arises, in which half the system of lines touch this circle and the remaining half, including the line 0, touch a concentric circle. Similarly also for (E2).

13. A skew mechanism (B3) is now to be described. Take four points A, B, C, D with cylindrical coordinates (R, z, θ) consisting of a common radius R, and heights and azimuths given by the table

A	$z_1-z_2-z_3$	$\theta_1-\theta_2-\theta_3$
B	$-z_1+z_2-z_3$	$-\theta_1+\theta_2-\theta_3$
C	$-z_1-z_2+z_3$	$-\theta_1-\theta_2+\theta_3$
D	$z_1+z_2+z_3$	$\theta_1+\theta_2+\theta_3$

and four points A', B', C', D' with common radius R' and heights and azimuths got by reversing all the signs in the table. The points B', C', A, D have heights and azimuths exceeding those of C, B, D', A' respectively by $2z_1$ and $2\theta_1$. The four lines BC', CB', $D'A$, $A'D$ have all the same length a_1; and two other sets of four have lengths a_2 and a_3. The first four are rays of a screw with pitch ϖ and the axis z for axis, if

$$RR' \sin 2\theta_1 = 2z_1 \varpi.$$

So if $$z_1/\sin 2\theta_1 = z_2/\sin 2\theta_2 = z_3/\sin 2\theta_3 = RR'/2\varpi, \qquad (15)$$

then the twelve lines joining the vertices of the tetrahedron $ABCD$ to non-corresponding vertices of the tetrahedron $A'B'C'D'$ are rays of the same screw, and the two tetrahedra are mutually inscribed. Rays AB', AC', AD', concurrent at A, are coplanar; and so for all the eight points. The normals at A, B, C, D make the same angle ρ with z, given by the

pitch equation $R \cot \rho = \varpi$, and the normals at A', B', C', D' make an angle ρ' with z given by $R' \cot \rho' = \varpi$. The normals at B', C', A, D make equal angles α_1 with the normals at C, B, D', A' respectively; and similarly for angles α_2 and α_3. The spherical indicatrix given by the eight normals is the figure (B2).

Projecting BC' on to z gives

$$2z_1 = a_1 \sin \rho \sin \rho' \sin 2\theta_1 / \sin \alpha_1,$$

so
$$\varpi / \cos \rho \cos \rho' = a_1/\sin \alpha_1 = a_2/\sin \alpha_2 = a_3/\sin \alpha_3 = p, \tag{16}$$

and
$$R = p \sin \rho \cos \rho', \quad R' = p \cos \rho \sin \rho', \quad \varpi = p \cos \rho \cos \rho', \tag{17}$$

The figure (B2), with α_1, α_2, α_3 all constant, has two degrees of freedom. Hence, if p is constant the above equations keep a_1, a_2, a_3 constant in the skew figure. The line BC' may be made a link, with the normals at B and C' for hinges, and has constant length a_1 and constant twist α_1. Similarly for the rest. The mechanism (B3) so obtained has two degrees of freedom. (Its normal freedom would be -14.) It contains three pairs of skew isograms. Their axes are three pairs of lines normal to z, with coordinates (z_1, θ_1), $(-z_1, -\theta_1)$, ... The material of which the mechanism is composed consists of twelve links, with lengths and twists a_1 and α_1, a_2 and α_2, a_3 and α_3, four of each sort, and all with the same index.

The figure is such that (i) its orthogonal projection on any plane perpendicular to the screw axis is a figure of type (B1) with its centre on the axis, (ii) the twelve lines of the links are all rays of the screw. These two conditions alone suffice for the description and construction of the figure.

The figure is determined, in size and shape, by six parameters. Starting with three links AB', AC', AD' of equal index, with a common hinge at A, and arbitrary angles of inclination for the links, the figure may be completed by the successive construction of skew isograms, formed with links congruent with the three given links. The final point A' is reached by three different isograms and is the same for all three; and the same is true of the hinge-line associated with A'. It may be said, briefly but sufficiently intelligibly, that skew isogram addition is commutative for three skew vectors of arbitrary lengths and twists if their indices are equal.

The relative motions of four equal links may be treated as for (B1) in § 9. Taking in cyclic order the links BC', AD', CB', DA', any consecutive two are opposite sides of an isogram, and have a relative motion on a screw whose axis is the common normal of the other two sides: and as

any one of the four isograms may be stiffened (cf. § 9) the four screw axes have all a common normal. Hence common normals of corresponding-opposite sides of the isograms $AC'DB'$ and $A'CD'B$ have a common normal. It is (D.O., § 4) the axis of a polarizing screw which reciprocates these two tetrahedra. There are three such polarizing screws, each reciprocating a pair of isograms.

14. The last theorem may be extended to the case of n vectors and gives a mechanism (C3). The argument proceeds as for (C1) in § 10. The coincidences of vertices are accompanied by coincidence of the hinge lines associated with them, in virtue of the property of the figure (B3). The commutative addition for skew vectors of equal index holds good for n vectors. The mechanism (C3) has $n-1$ degrees of freedom, the number of links and hinges being the same as for (C1) and (C2). [The normal freedom would be of degree $(5-2n)2^n-6$.] In 2^{n-1} ways the links may all come into one straight line.

Pairs of hinges and pairs of links may be termed "opposite" as for (C1) in § 10. If $n = 7$, and one link is fixed, then, by suitable rotation of the six links hinged to it at either end, the link opposite to the fixed link may, within a limited range, be brought to any position in space.

15. A skew isogram double-four (D3) may now be constructed. Take points A, B, C, D with coordinates as in § 13, and draw lines 1, 2, 3, 4 perpendicular to R at a slope ρ to the plane $z = 0$: and draw similarly lines 1′, 2′, 3′, 4′ through A', B', C', D' perpendicular to R' at a slope ρ'. By suitably relating the coordinates these lines may be made to intersect so as to form a double-four. Let the lengths along 1, 2, 3, 4 from A, B, C, D up to the intersections with 1′, 2′, 3′, 4′ be given by the table

	1′	2′	3′	4′
1		$-a_3$	$-a_2$	a_1
2	$-a_3$		$-a_1$	a_2
3	$-a_2$	$-a_1$		a_3
4	a_1	a_2	a_3	

with a similar table in a_1', a_2', a_3' for lengths on 1′, 2′, 3′, 4′. For the

intersection of 2 and 3′,

$$a_1 \cos\rho + a_1' \cos\rho' = (R-R') \cot\theta_1, \tag{18}$$

$$-a_1 \cos\rho + a_1' \cos\rho' = (R+R') \tan\theta_1, \tag{19}$$

$$-a_1 \sin\rho + a_1' \sin\rho' = 2z_1, \tag{20}$$

and the same conditions hold for points 2′3, 14′, 1′4. With three such sets of conditions satisfied the lines form a double-four. The quadrilaterals 12′3′4 and 1′234′ are skew isograms, with the lines $z = z_1$, $\theta = \theta_1$ and $z = -z_1$, $\theta = -\theta_1$ as axes; and similarly for two other such pairs.

Let the normals at the intersections on 1 with 2′, 3′, 4′ make twist-angles $-\alpha_3$, $-\alpha_2$, α_1 with the normal to 1 and R at A; and similarly for the other normals, precisely as in the table for the lengths a_1, a_2, a_3. The normals give rise to a spherical double-four (D2); and if it deforms with α_1, α_2, α_3 all constant, then (D.O., § 26)

$$\cos\rho'/\cos\rho = \cos\alpha_1/\cos\alpha_1' = \cos\alpha_2/\cos\alpha_2' = \cos\alpha_3/\cos\alpha_3', \tag{21}$$

and $$\tan\theta_1 \tan\tfrac{1}{2}(\alpha_1+\alpha_1') = \sin\tfrac{1}{2}(\rho-\rho')/\cos\tfrac{1}{2}(\rho+\rho'), \tag{22}$$

$$\cot\theta_1 \tan\tfrac{1}{2}(\alpha_1-\alpha_1') = -\sin\tfrac{1}{2}(\rho+\rho')/\cos\tfrac{1}{2}(\rho-\rho'). \tag{23}$$

The skew figure becomes a mechanism if the equations can be satisfied by constant values for α_1, α_1', ...; and this may be secured by putting

$$R = \varpi \cos\rho \sin\rho', \qquad R' = \varpi \sin\rho \cos\rho', \tag{24}$$

$$a_1 = -\varpi \tan\alpha_1', \qquad a_1' = -\varpi \tan\alpha_1, \quad \text{etc.}, \tag{25}$$

where ϖ is constant.

The mechanism has one degree of freedom. (The normal freedom would be -18.) During the movement the points of contact of the links with the coaxial cylinders R, R' are permanent points of the links, and the normal to link and cylinder is also permanently the same line of the link. At two stages of the movement it happens that $\rho = \pm\rho' = \pi/2$, and the links come all in one line. (D.O., § 26.)

The present mechanism (D3) and the skew deformable octahedron (D.O., §§ 28–30) both have a spherical double-four (D2) as indicatrix. If this last is made to be the same in the two cases, then the octahedron and the skew double-four may deform simultaneously so that each tri-

angular face of the octahedron keeps perpendicular to a link of the double-four; and the hinge-edges of the triangular face keep parallel to the hinge-lines of the link. The octahedron flattens into one plane on each of the two occasions on which the double-four straightens into one line.

A special case of the mechanism occurs if $\rho = \rho'$. Then

$$R = R' = \varpi \sin \rho \cos \rho,$$

and $$a_1 = -a_1', \quad \alpha_1 = -\alpha_1' = \varpi \tan \alpha_1, \quad \text{etc.}$$

The links are congruent in pairs, and all touch the same circular cylinder.

Another case resembling the last in some respects is got by taking

$$\rho = \rho', \quad -R = R' = \varpi \cot \rho, \quad a_1 = -a_1', \quad \alpha_1 = \alpha_1' = \varpi \cot \alpha_1, \quad \text{etc.}$$

These satisfy the equations (18)–(23) and make z_1, z_2, z_3 all zero. The lines of the double-four are then generators of a hyperboloid of revolution, four of one system and four of the other. The twelve hinge-lines are normals to the surface. Each pair of links are images of each other. The axis of each of the six isograms is an equatorial diameter of the hyperboloid. (Cf. § 6, i.) When $\rho = 0$ or $\pi/2$, the links are in one straight line, lying along the equator of an infinite hyperboloid, or along the axis of one with zero equatorial radius.

16. The isogram $1'234'$ of the skew double-four (D3) has on each of its sides a point of intersection with a side of the companion isogram $12'3'4$. The product of the ratios in which the sides of the first are divided is equal to

$$(a_1^2 - a_2^2)(a_3^2 - a_1'^2) / (a_3^2 - a_1^2)(a_1'^2 - a_2'^2);$$

and this is equal to unity in virtue of (21), (25), § 15. Hence the four points $23'$, $2'3$, $14'$, $1'4$ are coplanar. There are three such sets of coplanar points.

For any double-four of rods which are merely articulated, and not hinged, it is known (Fontené, *Nouvelles Annales de Mathématiques*, 1904, p. 105) that two degrees of freedom occur if these three sets of four points, as above, are coplanar. The mechanism (D3), therefore, gives a figure with two degrees of freedom if the hinges are replaced by articulations.

A few comments may be added applicable alike to the above special double-four and to the general articulated double-four. The figure has twelve diagonals consisting of three sets of four; such as the four which join the vertices of $13'42'$ to the corresponding-opposite vertices of $1'34'2$. These four are in any case consecutively intersecting lines: but they are

also alternately intersecting in virtue of the points 31′, 3′1, 24′, 2′4 being coplanar and the points 12′, 1′2, 3′4, 34′ being coplanar. Hence these four diagonals are concurrent: and similarly for the other two sets of four. The double freedom of the figure becomes thus kinematically apparent; for if the first set of four diagonals (the squares of whose lengths are obviously linearly related) are kept constant, the rods 1, 2, 3, 4 make constant angles with 4′, 3′, 2′, 1′ respectively, and the whole system reduces virtually to four rigid plates 23′, 2′3, 14′, 1′4 hinged consecutively along the diagonals. The concurrency of these shows that there is one residual degree of freedom.

It is noteworthy that the eight lines of this double-four together with its twelve diagonals constitute a complete configuration, consisting of twenty lines meeting by sets of three in fifteen points. A simple metrical instance of such a configuration is afforded by taking eight lines to be mutual images in three orthogonal planes; they meet each plane in the corners of a rectangle; and the twelve sides of the rectangles and the eight lines themselves constitute the configuration of twenty.

17. If in the formulæ relating to the skew double-four (D3) of § 15, each set of three symbols are extended to a set of n symbols, then a more general mechanism (E3) is obtained. A link touches a circular cylinder of radius R at a point with coordinates

$$z_1+z_2+z_3+\ldots+z_n \quad \text{and} \quad \theta_1+\theta_2+\theta_3+\ldots+\theta_n;$$

it is furnished with hinges at points distant $a_1, a_2, a_3, \ldots, a_n$ from the contact point, and with twists (measured from the normal to the link which is also tangent to the cylinder at the point) $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$. For any other link of the same system any even number of symbols of each set, with the same suffixes, have their signs reversed. For the second system of links the cylinder radius is R': an odd number of symbols with the same suffixes have their signs changed, and the symbols for the lengths and twists are accented. The common slope of the first system is ρ and of the second is ρ'. Then any two links, one of each system, with only one sign different in their specifications, have the corresponding points and hinges united, in virtue of equations of the same form as (18)–(25) in § 15. The whole forms a mechanism with one degree of freedom. There are 2^{n-1} links of each system and each has n hinges.

It is to be observed that the spherical indicatrix is not the most general type of (E2), but is the case, noticed in § 12, in which its links all touch two concentric circles.

18. The case of the hyperboloid of revolution of § 15 may be extended in the same way, giving another (E3) mechanism. Generators of one system 2^{n-1} in number cross the equator in longitudes given by $\theta_1+\theta_2+\theta_3+\ldots+\theta_n$, together with all other values got by an even number of changes of sign; and 2^{n-1} of the opposite system cross in longitudes got by any odd number of changes of sign. Then any two generators with only one sign different are furnished with a hinge normal to both (and to the hyperboloid) at their intersection. The hinges fall in north and south latitudes corresponding to longitude differences $2\theta_1$, $2\theta_2$, $2\theta_3$, ..., $2\theta_n$. The mechanism has one degree of freedom. During the deformation $R \tan \rho$ is constant. (The length of the imaginary axis is constant. Any two such hyperboloids may have contact along a generator.)

It may be added that if the hinges are replaced by mere articulations there are then two degrees of freedom, R and ρ being independent. If any one other point of intersection is articulated the one freedom then left gives the well-known confocal deformation, for which $R \sec \rho$ is constant; and all the other intersections may be then articulated also. For both the hinged and the articulated hyperboloid the longitude differences vary, the ratios of $\tan \theta_1$, $\tan \theta_2$, ..., keeping constant. Two generators of the same system differ in longitude by the sum of an even number of the differences $\pm 2\theta_1$, $\pm 2\theta_2$, $\pm 2\theta_3$, ... If this sum passes through the value 0 (mod 2π) the two generators pass each other.

A simple case for a model of the articulated hyperboloid with two degrees of freedom may be suggested. It is made by arranging, in a plane, eight rods to cross eight, draught-board fashion; with articulations as shown in the figure (Fig. 5). For this model the central generators first pass one another, and next the outermost come into line.

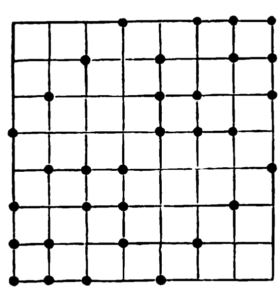

FIG. 5.—(*Vide* § 18.) Articulated rods giving hyperboloids of revolution of all sizes and shapes.

19. A very general type of skew isogram mechanism (F3) is derivable

from the converse method given in § 4. Suppose a fixed body A, and rotating bodies B_1, B_2, B_3, ..., B_n hinged to A and turning through angles whose halves have their tangents homographically related. Then B_1 and B_2 may be linked together by a piece C_{12}, and so for any two of the pieces B. Consider then C_{12} and C_{13}. The bodies B_1, A, B_2, C_{12} form a skew isogram mechanism, and so the rotation of C_{12} relatively to B_1 is the same as that of B_2 relatively to A: and similarly, for B_1, A, B_3, C_{13}, the rotation of C_{13} relatively to B_1 is the same as that of B_3 relatively to A. Hence the pieces C_{12} and C_{13} may be connected by a link D. All the pieces C_{12}, C_{13}, ..., C_{1n}, hinged to B_1, may be linked in pairs as were, originally, all the pieces B_1, B_2, ..., B_n hinged to A; and similarly for all pairs of C's hinged to B_2 or to B_3, ... The process may be continued. Plane and spherical mechanisms (F1) and (F2) may be similarly constructed. The process is subject to modification by the possibility of bodies which are to be connected having (i) equal rotations, or (ii) having no relative movement. An instance of the last has occurred in the case of the skew double-four (D3): to the piece $4'$ are hinged pieces 1, 2, 3; and these are connected in pairs by $1'$, $2'$, $3'$; then the three connectors of $1'$, $2'$, $3'$ in pairs are one and the same piece 4.

20. Finally, and briefly, some skew isogram mechanisms may be considered in which some or all of the sides of the isograms become infinitesimal in length.

A ladder-work of n isograms may be taken, for which the rungs P_0Q_0, P_1Q_1, P_2Q_2, ..., P_nQ_n are all congruent links of length a and twist α. The side-pieces forming polygons $P_0P_1P_2P_3 \dots P_n$ and $Q_0Q_1Q_2 \dots Q_n$ may, conveniently, also be made equal, with length b, twist β, and index $p = a/\sin\alpha = b/\sin\beta$. The system has n degrees of freedom, and one degree of freedom remains if either polygon is stiffened.

If now all the lengths b are supposed infinitesimal, the polygons become twisted curves, and the following results appear :—

(i) The curves have equal arcs and constant torsion $\tau = 1/p$; the binormals being the hinge-lines of the isograms.

(ii) The curves have their osculating planes cutting at a constant angle α in a chord PQ of constant length a, joining corresponding points and making equal angles with the curves.

(iii) The isogram formula (1), § 3, leads to the equations

$$\kappa_1 - \kappa_2 = 2\tau \cot \tfrac{1}{2}\alpha \sin \theta,$$

$$\kappa_1 + \kappa_2 = 2\,\frac{d\theta}{ds},$$

where κ_1, κ_2 are the curvatures.

If either curve is kept stiff the other assumes a series of forms all related in this same way to the fixed curve. If, specially, the fixed curve is made a straight line or a helix, the moving curve has a wave motion, and remains as a whole invariable.

21. If a set of m identical ladders each made of n skew isograms are united, side by side, to form a skew quadrilateral sheet of mn isograms, the mechanism has $m+n-1$ degrees of freedom. It is made stiff if two adjacent polygonal edges of the quadrilateral are made one rigid piece.

Suppose now that the isograms, kept of finite index p, are made equilateral and infinitesimal and nearly plane parallelograms. Then the sheet becomes a curved surface, with the following properties :—

(i) The normals of the surface are the hinge lines of the isograms.

(ii) The curves given by the sides of the isograms have their osculating planes tangent to the surface; they are the inflexion lines of the surface, and have constant torsion $\pm 1/p$.

(iii) The hinges of the isograms are normals to the surface, and they consecutively intersect (§ 3, i) along any line of diagonals of the isograms. The diagonal lines give the lines of curvature of the surface.

(iv) The product of the lengths of the intersecting hinges at pairs of opposite vertices of an isogram (§ 3, i) gives $\kappa_1 \kappa_2 = -1/p^2$ for the product of the principal curvatures.

(v) In virtue of (ii) or (iv) the surface is of constant negative curvature and remains so for all deformation of the mechanism.

(vi) The surface stiffens if two intersecting inflexion lines are given curves.

The mechanism thus verifies mechanically some of the known results of differential geometry. If the isograms are not equilateral, the diagonal lines are no longer lines of curvature; but the sides of the isograms still give the inflexion lines, and (i), (ii), (v), (vi) still hold good, and the deforming mechanism is capable of giving any form of pseudospherical surface.

Note added January 26th, 1914.—The commutative addition of § 14 and the differential methods of § 21 may be brought into simultaneous use. If the vectors commuted are $a_1 a_2 \dots a_m b_1 b_2 \dots b_n$, the restriction may be introduced that the order of the a's is fixed by the suffixes and also that of the b's. So much of the complete mechanism as is then retained may be described as consisting of a skew quadrilateral sheet S, composed of mn isograms, each isogram having its sides given by an a and a b. If an additional vector c is now introduced, with no restriction of order, the mechanism consists of two separate mechanisms S, S', composed of the same material, with the corresponding hinge-lines connected by $(m+1)(n+1)$ equal links c. If S is kept stiff, S' and the links c have one degree of freedom. If the a's and b's are made infinitesimal and c remains finite, S becomes a pseudospherical surface of arbitrary form; S' is of the same type and of the same curvature and has its asymptotic curves equal, arc for arc, to those of S, and with the same constant torsion. The links c give all the common tangents of S and S', each with tangent planes intersecting in c at the same angle γ, the twist of the links c. A single infinity of such surfaces S' are associated with S in virtue of the one degree of freedom; and if c and γ (subject to $c = p \sin \gamma$) vary also, there are a double infinity of surfaces S'. In the special case when $\gamma = \pi/2$ and $c = p$, the surfaces S and S' constitute the two sheets of the surface of centres of a Weingarten surface with its principal radii of curvature having a constant difference p. In this case, therefore, there occurs a mechanical presentation of the theorems of Ribaucour and Bianchi in regard to surfaces of constant negative curvature. If, very specially, the mechanism S takes a degenerate rectilinear form, then S' becomes a screw tractricoid or a tractricoid of revolution, with S for axis.

As a further generalization it may be observed that the introduction of additional skew vectors $c'c'' \dots$, unrestricted as to sequence, like c, gives a mechanism consisting of a number of sheets such as S, S', ..., all composed of the same material $a_1 a_2 \dots a_m b_1 b_2 \dots b_n$, united in commutative sequence by sets of equal links, each set being given by one of the vectors c, c', c'', ... The whole, compared with a complete mechanism given by the commutative addition of c, c', c'', ... alone, has a sheet in place of each junction point, and a set of equal links connecting two sheets in place of a single link connecting two junctions.*

* Mention may here be made of two recent papers dealing with the skew isogram mechanism. Bricard (*Comptes Rendus*, January 12th, 1914) has found independently the mechanism of § 13, and gives a concise and simple proof of its two degrees of freedom. Kœnigs (*Comptes Rendus*, November 24th, 1913) treats of movements in two parameters that are doubly resoluble, with the skew isogram as an instance.

TAUBERIAN THEOREMS CONCERNING POWER SERIES AND DIRICHLET'S SERIES WHOSE COEFFICIENTS ARE POSITIVE*

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1. The general nature of the theorems contained in this paper resembles that of the "Tauberian" theorems which we have proved in a series of recent papers.† They have, however, a character of their own, in that they are concerned primarily with series of positive terms.

Let
$$f(x) = \Sigma a_n x^n$$
be a power series convergent for $|x| < 1$. We shall consider only positive values of x less than 1.

Let
$$s_n = a_0 + a_1 + \ldots + a_n,$$
$$L(u) = (\log u)^{a_1} (\log \log u)^{a_2} \ldots,$$
where the a's are real. Then it is known that, *if*
$$s_n \sim A n^a L(n),$$
where $A \neq 0$, *as* $n \to \infty$, *the indices* $a, a_1, a_2, \ldots$ *being such that* $n^a L(n)$ *tends to a positive limit or to infinity, then*
$$f(x) \sim A \frac{\Gamma(a+1)}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right),$$
as $x \to 1$.‡

* A short abstract of some of the principal results of this paper was published under the title "Tauberian Theorems concerning Series of Positive Terms" in the *Messenger of Mathematics*, Vol. 42, pp. 191, 192.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 301–320; Vol. 9, pp. 434–448; Vol. 11, pp. 1–16 and pp. 411–478.

‡ The first of the a's which is not zero must be positive. If $a = a_1 = a_2 = \ldots = 0$, the theorem reduces to Abel's well known theorem. The special theorem in which $a_1 = a_2 = \ldots = 0$

The principal object of this paper is to prove that *the converse of the theorem is true when the coefficients a_n are positive.* We shall prove, for example, that *if* $a_n \geqslant 0$, *and*

$$f(x) = \Sigma a_n x^n \sim \frac{A}{(1-x)^a} \quad (A > 0,\ a > 0),$$

then
$$s_n \sim \frac{An^a}{\Gamma(1+a)}.$$

This result is a very curious one, largely because it lies much deeper and is much harder to prove than a first impression might tempt one to believe. Its appearance is that of a "special"* (or "o") Tauberian theorem. In reality, as will appear in the sequel, it is a theorem of the "general" (or "O") type, and left-handed"† in addition. It is, in fact, of the same order of difficulty as the theorem "*if* $a_n > -K/n$, *and* $f(x) \to A$, *then* Σa_n *converges to the sum* A."‡ The proof therefore naturally involves all the apparatus of repeated differentiation on which the proofs of such theorems ultimately depend.§

2. We begin by proving some subsidiary theorems which are interesting in themselves. It is hardly necessary to remark that all variables and functions considered in them are real. We suppose first that x is a variable which tends to infinity.

We shall begin by proving a theorem which is due to Landau, and on which nearly all our subsequent analysis depends. The theorem is of great interest in itself, inasmuch as its general character is that of an "O" Tauberian theorem, and it was the first theorem of this nature stated explicitly.

THEOREM 1.—*Suppose that* (i) $f(x)$ *is differentiable, and* (ii) $xf'(x)$

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was proved by Appell, *Comptes Rendus*, Vol. 87, p. 689. The substance of the general theorem is due to Lasker, *Phil. Trans. Roy. Soc.*, (A), Vol. 196, p. 444: Lasker does not actually state it, but it is a trivial deduction from the theorem which he proves. The theorem was first stated explicitly in the form we have adopted by Pringsheim, *Acta Mathematica*, Vol. 28, p. 29. Pringsheim, however, proves a more general theorem, inasmuch as he considers paths of approach of x to 1 other than the real axis.

* *Cf.* Hardy and Littlewood, *Proc. London Math. Soc.*, Ser. 2, Vol. 11, p. 413.

† *L.c.*, p. 416.

‡ We stated this theorem without proof at the end of our paper already quoted (*l.c.*, p. 478).

§ Littlewood, "The Converse of Abel's Theorem," *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 434 *et seq.*

steadily increases. Then

$$f(x) \sim x,$$

involves

$$f'(x) \sim 1.^*$$

Suppose that the theorem is untrue. Then it must be possible to find a number h different from 1, and a sequence (x_ν) of values of x tending to infinity, such that

$$f'(x_\nu) \to h,$$

as $\nu \to \infty$. Let us suppose, for example, that $h > 1$; and let δ be a positive number. Then, as $\nu \to \infty$,

$$\frac{f(x_\nu+\delta x_\nu)-f(x_\nu)}{\delta x_\nu} = \frac{1}{\delta x_\nu}\int_{x_\nu}^{x_\nu+\delta x_\nu} f'(x)\,dx \geqslant \frac{x_\nu f'(x_\nu)}{\delta x_\nu}\int_{x_\nu}^{x_\nu+\delta x_\nu}\frac{dx}{x}$$

$$\sim \frac{h}{\delta}\int_{x_\nu}^{x_\nu+\delta x_\nu}\frac{dx}{x} = \frac{h\log(1+\delta)}{\delta}.$$

But, since $f(x) \sim x$,

$$\frac{f(x_\nu+\delta x_\nu)-f(x_\nu)}{\delta x_\nu} \sim 1;$$

and these two results are contradictory if δ is sufficiently small. The hypothesis that $h < 1$ may be disposed of similarly.

3. THEOREM 2.—*Let $\phi(x)$ be a function which tends to infinity with x and has a continuous and positive derivative, and suppose that*

$$\text{(i)} \quad \frac{\phi(x)}{\phi'(x)} \sim x,$$

$$\text{(ii)} \quad xf'(x) \text{ steadily increases.}$$

Then

$$f(x) \sim \phi(x)$$

involves

$$f'(x) \sim \phi'(x).$$

This theorem follows at once from Theorem 1 by means of the substitution

$$x = \phi(y).$$

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* The converse inference may, of course, *always* be made. Theorem 1 was proved by Landau (*Rendiconti di Palermo*, Vol. 26, p. 218). We insert a proof for the sake of completeness.

By means of one or other of the substitutions

$$x = \frac{1}{y-c}, \quad x = \frac{1}{c-y}$$

we deduce

THEOREM 2a.—*Let $\phi(x)$ be a function of x which tends to infinity as x tends to c from above or from below; and suppose that*

(i) $\dfrac{\phi(x)}{\phi'(x)} \sim -(x-c)$,

(ii) $(x-c)f'(x)$ *steadily decreases or increases.**

Then $f(x) \sim \phi(x)$ *involves* $f'(x) \sim \phi'(x)$.

Suppose, in particular, that

$$\phi(x) = \frac{A}{(1-x)^a} L\left(\frac{1}{1-x}\right),$$

where $A > 0$, $a > 0$, and $x \to 1$ from below. Then $1-x \sim a\phi/\phi'$. Hence we have

THEOREM 3.—*If* $$f(x) \sim \frac{A}{(1-x)^a} L\left(\frac{1}{1-x}\right),$$

where $A > 0$, $a > 0$, *and* $(1-x)f'(x)$ *increases as* $x \to 1$, *then*

$$f'(x) \sim \frac{aA}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right).$$

4. From Theorem 3 we can deduce a preliminary theorem concerning power series which seems of considerable interest in itself.

THEOREM 4.—*If* $f(x) = \Sigma a_n x^n$ *is a power series with positive*† *coefficients, and*

$$f(x) \sim \frac{A}{(1-x)^a} L\left(\frac{1}{1-x}\right) \quad (A > 0,\ a > 0),$$

then $$f^{(r)}(x) \sim \frac{\Gamma(a+r)}{\Gamma(a)} \frac{A}{(1-x)^{a+r}} L\left(\frac{1}{1-x}\right).$$

* $(x-c)f'(x)$ must be an increasing or a decreasing function according as x tends to its limit from below or above.

† We use "positive" to include "zero."

We have

$$g(x) = \Sigma s_n x^n = \frac{f(x)}{1-x} \sim \frac{A}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right).$$

Also

$$(1-x)g'(x) = (1-x)\,\Sigma n s_n x^{n-1} = s_1 + (2s_2 - s_1)\,x + (3s_3 - 2s_2)\,x^2 + \dots$$

has all its coefficients positive, since s_n increases steadily with n. Hence $(1-x)\,g'(x)$ increases with x, and so, by Theorem 3,

$$g'(x) \sim \frac{(a+1)A}{(1-x)^{a+2}} L\left(\frac{1}{1-x}\right),$$

$$f'(x) = (1-x)\,g'(x) - g(x) \sim \frac{aA}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right).$$

A repetition of the argument leads to a complete proof of the theorem.

It is important to observe that this process of differentiation is *not* legitimate when $a = 0$. Suppose, *e.g.*, that

$$f(x) \sim \log\left(\frac{1}{1-x}\right).$$

We cannot infer that

$$f'(x) \sim \frac{1}{1-x};$$

all that the argument leads to is

$$f'(x) = o\left\{\frac{1}{1-x}\log\left(\frac{1}{1-x}\right)\right\}.$$

We can show, moreover, by actual examples, that the suggested inference would be invalid. Suppose, for example, that

$$f(x) = \Sigma x^{a^n} \quad (a \geqslant 2).$$

Then it is easy to see that $s_n \sim \log_a n$, and so

$$f(x) \sim \frac{1}{\log a}\log\left(\frac{1}{1-x}\right).$$

But it is not true that $(1-x)\,f'(x)$ leads to a limit as $x \to 1$. This is most easily proved by means of Theorem 8 below. Since

$$xf'(x) = \Sigma\, a^n x^{a^n}$$

is a series of positive terms, $f'(x) \sim A/(1-x)$ would involve

$$t_\nu = \sum_{a^n \leqslant \nu} a^n \sim A\nu;$$

and this is obviously untrue, since whenever ν passes through a value equal to a power of a, a new term is introduced into t_ν which is greater than the sum of all which precede.*

* See Hardy, *Quarterly Journal*, Vol. 38, pp. 279 *et seq.*, for analytical formulæ which show in an explicit manner the behaviour as $x \to 1$ of the series Σx^{a^n}, $\Sigma(-1)^n x^{a^n}$ and their derivatives.

In the sequel we shall use not Theorem 4 itself, but the theorem into which it is transformed by the substitutions

$$x = e^{-t}, \quad f(x) = F(t).$$

THEOREM 4a.—*If* $a_n \geqslant 0$, *and*

$$F(t) = \Sigma a_n e^{-nt} \sim A t^{-a} L\left(\frac{1}{t}\right),$$

as $t \to 0$, *then*

$$(-1)^{(r)} F^{(r)}(t) \sim \frac{\Gamma(a+r)}{\Gamma(a)} A t^{-a-r} L\left(\frac{1}{t}\right).$$

5. Theorem 4 is capable of various interesting generalisations.

THEOREM 5.—*The condition that* $a_n \geqslant 0$ *of Theorem* 4 *may be replaced by the more general condition that*

$$na_n = O_L\{n^a L(n)\},$$

i.e., that

$$na_n > -Kn^a L(n).$$

Let

$$g(x) = \Sigma b_n x^n = \Sigma \{a_n + Kn^{a-1} L(n)\} x^n.$$

Then $b_n > 0$, and

$$g(x) \sim \left\{A + \frac{K}{\Gamma(a)}\right\} \frac{1}{(1-x)^a} L\left(\frac{1}{1-x}\right).$$

Hence, by Theorem 4,

$$g'(x) \sim \left\{A + \frac{K}{\Gamma(a)}\right\} \frac{a}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right);$$

and so

$$f'(x) \sim \frac{Aa}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right).$$

THEOREM 6.—*The condition that* $a_n \geqslant 0$ *of Theorem* 4 *may also be replaced by* $s_n \geqslant 0$, *or by* $s_n^k \geqslant 0$, *where* s_n^k *is any one of Cesàro's means formed from the series* Σa_n; *or, more generally, by* $s_n = O_L\{n^a L(n)\}$ *or*

$$s_n^k = O_L\{n^{a+k} L(n)\}.$$

In the proof of Theorem 4, the condition $a_n \geqslant 0$ is used only to justify the differentiation of the asymptotic equality

$$g(x) = \Sigma s_n x^n \sim \frac{A}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right).$$

And the result of the theorem shows that $s_n \geqslant 0$ is a sufficient condition for this. Repeating this argument we see that $s_n^k \geqslant 0$ is a sufficient condition.

The more general results may be established in the same way, if we appeal at each stage to Theorem 5 instead of to Theorem 4.

6. Suppose that the conditions of Theorem 4 (or of one of its generalisations) are satisfied. Then

$$\Sigma na_n x^n = xf'(x) \sim \frac{Aa}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right).$$

Operating repeatedly in this manner, we see that

$$\Sigma n^r a_n x^n \sim \frac{\Gamma(a+r)}{\Gamma(a)} \frac{A}{(1-x)^{a+r}} L\left(\frac{1}{1-x}\right)$$

for all positive integral values of r. We shall now show that *this result holds for all values of r greater than $-\alpha$.*

It is plainly enough to prove this when $r = -\beta$, $0 < \beta < \alpha$. We write

$$x = e^{-t}, \quad f(x) = F(t),$$

so that
$$F(t) \sim \frac{A}{t^{\alpha}} L\left(\frac{1}{t}\right),$$

as $t \to 0$. Then

$$\Sigma n^{-\beta} a_n e^{-nt} = \frac{1}{\Gamma(\beta)} \int_0^{\infty} u^{\beta-1} \Sigma a_n e^{-n(t+u)} du = \frac{1}{\Gamma(\beta)} \int_0^{\infty} u^{\beta-1} F(t+u)\, du.$$

Also
$$\int_0^{\infty} u^{\beta-1} F(t+u)\, du \sim A \int_0^{\infty} \frac{u^{\beta-1}}{(t+u)^{\alpha}} L\left(\frac{1}{t+u}\right) du$$

$$\sim \frac{A\Gamma(\beta)\Gamma(\alpha-\beta)}{\Gamma(\alpha)} t^{\beta-\alpha} L\left(\frac{1}{t}\right).*$$

The result is thus established for $-\alpha < r < 0$; the general result then follows by using the special result in which r is a positive integer. We have thus proved

THEOREM 7.—*If $f(x) = \Sigma a_n x^n$ is a power series with positive coefficients (or subject to the more general conditions of Theorems 5 or 6), and*

$$f(x) \sim \frac{A}{(1-x)^{\alpha}} L\left(\frac{1}{1-x}\right) \quad (A > 0,\ \alpha > 0),$$

then
$$f_r(x) = \Sigma n^r a_n x^n \sim \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \frac{A}{(1-x)^{\alpha+r}} L\left(\frac{1}{1-x}\right),$$

for any value of r, integral or not, greater than $-\alpha$.

7. We pass now to the proof of the principal theorem of the paper.

THEOREM 8.—*If $f(x) = \Sigma a_n x^n$ is a power series with positive coefficients, and*

$$f(x) \sim \frac{A}{(1-x)^{\alpha}} L\left(\frac{1}{1-x}\right),$$

where $A > 0$ and the indices $\alpha, \alpha_1, \alpha_2, \ldots$ are such that $n^{\alpha} L(n)$ tends to a positive limit or to infinity as $n \to \infty$, then

$$s_n \sim \frac{A}{\Gamma(\alpha+1)} n^{\alpha} L(n).$$

We suppose $A = 1$, and write $x = e^{-t}$, so that $t \to 0$, and

(1) $$f(x) = F(t) = \Sigma a_n e^{-nt} \sim t^{-\alpha} L(1/t).$$

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* These transformations, of course, merely express the general lines of a straightforward proof, the details of which will easily be supplied by anyone accustomed to work of this character.

In the first place, we have

$$s_n \leqslant e \sum_0^n a_\nu e^{-\nu n} \leqslant eF(1/n),$$

and so, from (1),

(2) $$s_n = O\{n^a L(n)\}.$$

Next, we have

(3) $$\Sigma s_n e^{-nt} \sim t^{-a-1} L(1/t).$$

Differentiating this relation r times, as we may do in virtue of Theorem 4, since $a+1>0$, we obtain

(4) $$\Sigma s_n n^r e^{-nt} \sim \frac{\Gamma(a+r+1)}{\Gamma(a+1)} t^{-a-r-1} L\left(\frac{1}{t}\right).$$

We shall now prove that, *if any positive* ϵ *is given, we can choose, first* r *and* ζ, *and then*

$$t_0 = t_0(\epsilon, r, \zeta),*$$

in such a way that

(5) $$\sum_{(1+\zeta)(a+r)/t}^{\infty} s_n n^r e^{-nt} < \epsilon \frac{\Gamma(a+r+1)}{\Gamma(a+1)} t^{-a-r-1} L\left(\frac{1}{t}\right),$$

(6) $$\sum_0^{(1-\zeta)(a+r)/t} s_n n^r e^{-nt} < \epsilon \frac{\Gamma(a+r+1)}{\Gamma(a+1)} t^{-a-r-1} L\left(\frac{1}{t}\right),$$

for $0 < t \leqslant t_0$.†

We shall suppose that r and ζ are functions of one another such that $\zeta^2 r \rightarrow \infty$ and $\zeta^3 r \rightarrow 0$ as $r \rightarrow \infty$ and $\zeta \rightarrow 0$. We may suppose, for example, that $\zeta^5 r^2 = 1$. The condition that $\zeta^3 r \rightarrow 0$ will not be used until § 9.

8. It follows from (2) that the left-hand side of (5) is of the form

$$O \sum_{(1+\zeta)(a+r)/t}^{\infty} n^{a+r} L(n) e^{-nt}.‡$$

The maximum of the function under the sign of summation, considered as

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* r will be large, ζ small, and t_0 small; and r and ζ will be connected in a way which will be defined precisely in a moment.

† The limits of summation are not in general integral. The extreme terms, considered as functions of t, are not of higher order than $t^{-a-r} L(1/t)$, and the argument is in no way affected by including or excluding an additional term or two.

‡ Here and in the sequel the constant implied by the O is independent of r, ζ, and t.

a function of n, occurs for a value of n given by an equation

$$a+r+\epsilon_n = nt,$$

where ϵ_n is a function of n only which tends to zero as $n \to \infty$. Hence the function decreases steadily throughout the limits of the summation, and

$$\sum_{(1+\zeta)(a+r)t}^{\infty} n^{a+r} L(n)\, e^{-nt} < \nu^{a+r} L(\nu)\, e^{-\nu t} + \int_{(1+\zeta)(a+r)t}^{\infty} u^{a+r} L(u)\, e^{-ut}\, du,$$

where $n = \nu$ corresponds to the first term of the sum. The isolated term may be neglected.*

The integral we write in the form

$$\int_{(1+\zeta)(a+r)/t}^{\infty} u^{a+r} L(u)\, e^{-ut/(1+\zeta)}\, e^{-\zeta ut/(1+\zeta)}\, du.$$

The maximum of the function $u^{a+r} L(u) e^{-ut/(1+\zeta)}$ is given by an equation of the form

$$(1+\zeta)(a+r+\epsilon_u) = ut.$$

Writing $(1+\zeta)(a+r+\epsilon_u)/t$ for u in the first three factors of the subject of integration, and observing that the functions

$$\left(1+\frac{\epsilon_u}{a+r}\right)^{a+r}, \qquad L\left\{\frac{(1+\zeta)(a+r+\epsilon_u)}{t}\right\}$$

are, when r is large enough and t small enough, certainly less than constant multiples of 1 and $L(1/t)$ respectively, we see that our integral is of the form

$$O\left[\left\{\frac{(1+\zeta)(a+r)}{t}\right\}^{a+r} L\left(\frac{1}{t}\right) e^{-a-r} \int_{(1+\zeta)(a+r)t}^{\infty} e^{-\zeta ut(1+\zeta)}\, du\right]$$

$$= O\left[\frac{1+\zeta}{\zeta}(a+r)^{a+r} e^{-(a+r)\{1+\zeta-\log(1+\zeta)\}} t^{-a-r-1} L\left(\frac{1}{t}\right)\right]$$

$$= O\left\{\frac{1+\zeta}{\zeta}(a+r)^{a+r} e^{-a-r} t^{-a-r-1} L\left(\frac{1}{t}\right)\right\},$$

since $\zeta - \log(1+\zeta) > 0$ when $\zeta > 0$. Our conclusion now follows from

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* See the last footnote.

the facts that

$$\Gamma(a+r+1) \sim (a+r)^{a+r+\frac{1}{2}} e^{-a-r} \sqrt{(2\pi)},$$

as $r \to \infty$, and that $\zeta^2 r \to \infty$.

The inequality (6) may be established in the same way. We write

$$e^{-nt} = e^{-nt(1-\zeta)} e^{\zeta nt(1-\zeta)},$$

and have finally to observe that $\zeta + \log(1-\zeta) < 0$. Otherwise the argument is practically the same.

From (4), (5), and (6) it follows that *when ϵ is given, we can choose r, ζ, and $t_0(\epsilon, r, \zeta)$ in such a way that*

$$(7) \quad (1-\epsilon)\frac{\Gamma(a+r+1)}{\Gamma(a+1)} t^{-a-r-1} L\left(\frac{1}{t}\right) < \sum_{(1-\zeta)(a+r)t}^{(1+\zeta)(a+r)t} s_n n^r e^{-nt} < (1+\epsilon)\frac{\Gamma(a+r+1)}{\Gamma(a+1)} t^{-a-r-1} L\left(\frac{1}{t}\right),$$

for $0 < t \leqslant t_0$.* Hence, as s_n is an increasing function of n, we obtain

$$(8) \qquad s_{(1-\zeta)(a+r)/t} \sum_{(1-\zeta)(a+r)/t}^{(1+\zeta)(a+r)/t} n^r e^{-nt} < (1+\epsilon)\frac{\Gamma(a+r+1)}{\Gamma(a+1)} t^{-a-r-1} L\left(\frac{1}{t}\right),$$

$$(9) \qquad s_{(1+\zeta)(a+r)/t} \sum_{(1-\zeta)(a+r)/t}^{(1+\zeta)(a+r)/t} n^r e^{-nt} > (1-\epsilon)\frac{\Gamma(a+r+1)}{\Gamma(a+1)} t^{-a-r-1} L\left(\frac{1}{t}\right).\dagger$$

9. Now write
$$n = \frac{a+r}{t} + \lambda,$$

so that $|\lambda| < \zeta(a+r)/t$. The next step in the proof consists in showing that we may replace $n^r e^{-nt}$, in the inequalities (8) and (9), by

$$\left(\frac{a+r}{t}\right)^r \exp\left\{-a-r-\frac{r\lambda^2 t^2}{2(a+r)^2}\right\}.$$

We have here to make use of the second relation between r and ζ, namely that $\zeta^3 r$ is small. We have

$$n^r e^{-nt} = \left(\frac{a+r}{t}\right)^r \exp\left\{-a-r-\lambda t + r\log\left(1+\frac{\lambda t}{a+r}\right)\right\}.$$

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* We do not imply that r, ζ, t_0 have the *same* values as before.

† We may interpret s_x, when x is not integral, as meaning $s_{[x]}$. But the truth of the inequalities would not be affected by the inclusion or exclusion of an additional term or two.

Also, as $|\lambda| < \zeta(a+r)/t$, we have

$$r\log\left(1+\frac{\lambda t}{a+r}\right)-\lambda t = -\frac{r\lambda^2t^2}{2(a+r)^2}+\frac{r\lambda^3t^3}{3(a+r)^3}-\ldots-\frac{a\lambda t}{a+r}$$

$$= -\frac{r\lambda^2t^2}{2(a+r)^2}+O(\zeta^3 r)+O(\zeta);$$

and the factor
$$e^{O(\zeta^3 r)+O(\zeta)}$$

tends to 1 as r tends to infinity and ζ to zero. If now we make this substitution in (8) and (9), and also substitute for $\Gamma(a+r+1)$ its asymptotic equivalent given by Stirling's theorem, we arrive at the following conclusion. *Given any positive ϵ, it is possible to choose first r and ζ, and then $t_0 = t_0(\epsilon, r, \zeta)$, in such a way that*

$$(10)\qquad s_{(1-\zeta)(a+r)/t}\sum e^{-\frac{1}{2}r\lambda^2t^2/(a+r)^2} < \frac{(1+\epsilon)\sqrt{(2\pi)}}{\Gamma(a+1)}(a+r)^{a+\frac{1}{2}}\,t^{-a-1}L\left(\frac{1}{t}\right),$$

$$(11)\qquad s_{(1+\zeta)(a+r)/t}\sum e^{-\frac{1}{2}r\lambda^2t^2/(a+r)^2} > \frac{(1-\epsilon)\sqrt{(2\pi)}}{\Gamma(a+1)}(a+r)^{a+\frac{1}{2}}\,t^{-a-1}L\left(\frac{1}{t}\right),$$

for $0 < t \leqslant t_0$, the values of λ included in the sums being those which differ from $(a+r)/t$ by an integer and are less in absolute value than $\zeta(a+r)/t$.

10. In the inequalities (10) and (11) we may suppose that λ ranges from $-\infty$ to $+\infty$. For, when this is so,

$$(12)\qquad \sum e^{-\frac{1}{2}r\lambda^2t^2/(a+r)^2} \sim \frac{a+r}{t}\sqrt{\left(\frac{2\pi}{r}\right)},$$

as $t\to 0$. On the other hand,

$$(13)\qquad \sum_{\lambda>\zeta(a+r)/t} e^{-\frac{1}{2}r\lambda^2t^2/(a+r)^2} = O(e^{-\frac{1}{2}\zeta^2 r})+\int_{\zeta(a+r)/t}^{\infty} e^{-\frac{1}{2}r\lambda^2t^2/(a+r)^2}\,d\lambda.$$

The integral is

$$\frac{\zeta(a+r)}{t}\int_1^{\infty} e^{-\frac{1}{2}\zeta^2\mu^2 r}\,d\mu = O\left\{\frac{a+r}{t\sqrt{r}}e^{-\frac{1}{2}\zeta^2 r}\right\}.$$

As $\zeta^2 r$ is large, the sum (13) is small compared with the sum (12).

A similar argument may, of course, be applied to the terms for which λ is large and negative. We may therefore suppose that λ ranges from $-\infty$ to $+\infty$ in (10) and (11).

We now use the asymptotic relation (12) to transform these inequalities. Observing that $r \sim a+r$ as $r \to \infty$, and that, when r is fixed,

$$L\left(\frac{1}{t}\right) \sim L\left(\frac{a+r}{t}\right)$$

as $t \to 0$, we see that *given* ϵ *it is possible to choose* r, ζ, *and* $t_0(\epsilon, r, \zeta)$, *so that*

$$s_{(1-\zeta)(a+r)/t} < \frac{1+\epsilon}{\Gamma(a+1)}\left(\frac{a+r}{t}\right)^a L\left(\frac{a+r}{t}\right), \tag{13}$$

$$s_{(1+\zeta)(a+r)/t} > \frac{1-\epsilon}{\Gamma(a+1)}\left(\frac{a+r}{t}\right)^a L\left(\frac{a+r}{t}\right), \tag{14}$$

for $0 < t \leqslant t_0$. Taking $n = (1-\zeta)(a+r)/t$ and $n = (1+\zeta)(a+r)/t$ in turn, and remembering that ζ is small, we see that *when* ϵ *is given it is possible to choose* n_0 *so that*

$$(1-\epsilon)\, n^a L(n) < \Gamma(1+a)\, s_n < (1+\epsilon)\, n^a L(n),$$

for $n \geqslant n_0$. Thus Theorem 8 is proved.

11. Theorem 8 has, as we remarked, in § 1, the appearance of a "special" (or "o") Tauberian theorem.* But we can at once deduce from it a theorem of an obviously "general" (or "O") character.

THEOREM 9.—*If we suppose, in Theorem* 8, *that* $a > 0$, *then the condition that* $a_n \geqslant 0$ *may be replaced by the condition that*

$$na_n = O_L\{n^a L(n)\};$$

i.e., that
$$na_n > -Kn^a L(n).$$

For let
$$g(x) = \Sigma\{a_n + Kn^{a-1}L(n)\}\, x^n = \Sigma b_n x^n.\dagger$$

Then $b_n > 0$, and

$$g(x) \sim \frac{A+K\Gamma(a)}{(1-x)^a} L\left(\frac{1}{1-x}\right).$$

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* When all the a's are zero, the theorem really is an "o" theorem. This may account for its having this appearance in general.

† We may suppress enough terms at the beginning to ensure that $L(n)$ is defined for all values of n in question.

Hence, by Theorem 8,

$$\sum^{n} \{a_\nu + K\nu^{a-1}L(\nu)\} \sim \frac{A+K\Gamma(a)}{\Gamma(a+1)} n^a L(n),$$

and so
$$s_n \sim \frac{A}{\Gamma(a+1)} n^a L(n).$$

It is also easy to see that, in all the theorems which we have been discussing, the function $L(u)$, instead of having the special form

$$(\log u)^{a_1} (\log\log u)^{a_2} \dots,$$

may be any logarithmico-exponential function,* such that

$$u^{-\delta} \prec L(u) \prec u^{\delta}.$$

Hence we deduce

THEOREM 10.—*If* $a_n \geqslant 0$, *and*

$$f(x) \sim \mathfrak{L}\left(\frac{1}{1-x}\right),$$

where $\mathfrak{L}(u)$ *is any logarithmico-exponential function such that*

$$1 \prec \mathfrak{L}(u) \prec u^{\Delta},$$

so that $\mathfrak{L}(u) = u^a L(u)$, *where* $a \geqslant 0$ *and* $u^{-\delta} \prec L(u) \prec u^{\delta}$, *then*

$$s_n \sim \frac{1}{\Gamma(a+1)} \mathfrak{L}(n).$$

12. Theorem 9 may be proved by another method which possesses considerable interest. It is less direct than that which we have followed, and involves an appeal to a theorem of which we have not published any proof. But it exhibits the relations between Theorem 9 and some of our former theorems in a very interesting light.

We suppose, for simplicity, that

$$a_1 = a_2 = \dots = 0.$$

It is easy to see that it is enough to prove that *if*

$$f(x) = o(1-x)^{-a},$$

* For an explanation of the terminology and notation of the next few lines see Hardy, "Orders of infinity," *Camb. Math. Tracts*, No. 12.

and $$a_n = O_L(n^{\alpha-1}),$$

then $$s_n = o(n^{\alpha}).$$

We write $$g(x) = \Sigma\,(a_n + Kn^{\alpha-1})x^n = \Sigma\, b_n x^n.$$

Then $b_n > 0$ and $g(x) = O(1-x)^{-\alpha}$; and it follows, as at the beginning of § 7, that

$$\sum_0^n b_\nu = O(n^{\alpha}),$$

and so that $s_n = O(n^{\alpha})$. But, from the equations

$$s_n = O(n^{\alpha}), \qquad \Sigma s_n x^n = o(1-x)^{-\alpha-1},$$

it follows, by Theorem 26 of our last paper,* that

$$\sigma_n = s_0 + s_1 + \ldots + s_n = o(n^{\alpha+1}).$$

Now we know that if $\phi(x)$ is a function of x which has a continuous second derivative $\phi''(x)$ for all sufficiently large values of x, and

$$\phi(x) = o(x^{\alpha+1}), \qquad \phi''(x) = O(x^{\alpha-1}),$$

then $$\phi'(x) = o(x^{\alpha}).\dagger$$

It is possible to generalise this result by writing

$$\phi''(x) = O_L(x^{\alpha-1})$$

for $$\phi''(x) = O(x^{\alpha+1}).$$

And this generalised result possesses an analogue for series, namely : *if*

$$s_0 + s_1 + \ldots + s_n = o(n^{\alpha+1}), \qquad a_n = O_L(n^{\alpha-1}),$$

then $$s_n = o(n^{\alpha}).$$

From this it is clear that we can deduce the result required. It must not be supposed, however, that in this proof we really dispense with the process of r-fold differentiation. This is involved in our former Theorem 26, by an appeal to which we covered the most difficult transition of our proof.

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* *L.c.*, p. 443.

† This follows from Theorem 1 of our last paper if $\alpha \geqslant 1$, and from Theorem 5 in any case.

13. The analogue of Theorem 9, in the case in which

$$a = a_1 = a_2 = \ldots = 0,$$

is as follows.

THEOREM 11.—*If $f(x) \to A$ as $x \to 1$, and $a_n > -K/n$, then Σa_n converges to the sum A.*

This theorem is true, and constitutes a very interesting extension of Littlewood's generalisation of Tauber's theorem. But a special proof is required.*

We have, as $x \to 1$, $$f(x) = A + o(1),$$

and $$f''(x) = \Sigma n(n-1) a_n x^{n-2} > -K \Sigma (n-1) x^{n-2} = O_L (1-x)^{-2},$$

From this it may be deduced that $$f' = o(1-x)^{-1}.$$

We write y for $1-x$, so that $y \to 0$, by positive values. The theorem we wish to prove is that *the equations* $$F(y) = A + o(1), \quad F''(y) = O_L (1/y)^2$$

imply $$F'(y) = o(1/y).$$

Suppose first that, if possible $$F'(y_s) > H/y_s \quad (H > 0),$$

for an infinity of values y_s of y whose limit is zero. We have also $F''(y) > -K/y^2$, and so, if $y > y_s$,

$$F'(y) = F'(y_s) + \int_{y_s}^{y} F''(u)\,du > \frac{H}{y_s} - \frac{K(y-y_s)}{y_s^2}.$$

It is clearly possible to choose a positive number δ, so that

$$F'(y) > \tfrac{1}{2}H/y_s,$$

for $$y_s \leqslant y \leqslant \eta_s = (1+\delta) y_s.$$

And then $$F(\eta_s) - F(y_s) = \int_{y_s}^{\eta_s} F'(y)\,dy > \tfrac{1}{2}H \frac{\eta_s - y_s}{y_s} = \tfrac{1}{2}\delta H,$$

which contradicts $$F(y) = A + o(1).$$

Similarly we can show that it is impossible that

$$F'(y_s) < -H/y_s \quad (H > 0).$$

In this case we start from the fact that, if $0 < y < y_s$,

$$F'(y) = F'(y_s) - \int_{y}^{y_s} F''(u)\,du < -\frac{H}{y_s} + \frac{K(y_s - y)}{y^2},$$

and argue in the same way. Hence $F'(y) = o(1/y)$.

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* If we write $$g(x) = \Sigma \left(a_n + \frac{K}{n}\right) x^n = \Sigma b_n x^n,$$

$g(x)$ is of higher order than $f(x)$. Hence we cannot prove that $s_n = O(1)$ in the way in which we proved $s_n = O(n^a)$ in § 12.

Hence $$\sum n a_n x^n = o\left(\frac{1}{1-x}\right),$$

and $$n a_n = O_L(1).$$

Hence, by Theorem 9, $$a_1 + 2a_2 + \dots + n a_n = o(n);$$

and the convergence of $\sum a_n$ now follows from Pringsheim's generalisation of Tauber's theorem.*

14. In Theorem 9 we supposed that $\alpha > 0$. An argument similar to that of the last section enables us to remove this restriction.

THEOREM 12.—*The result of Theorem* 9 *holds even when* $\alpha = 0$.†

We have $$f(x) \sim L\left(\frac{1}{1-x}\right),$$

and $$f''(x) > -K \sum (n-1) L(n) x^{n-2} = O_L\left\{\frac{1}{(1-x)^2} L\left(\frac{1}{1-x}\right)\right\}.$$

From this we deduce‡ $$f'(x) = o\left\{\frac{1}{1-x} L\left(\frac{1}{1-x}\right)\right\}.$$

Hence $$\sum n a_n x^n = o\left\{\frac{1}{1-x} L\left(\frac{1}{1-x}\right)\right\};$$

and therefore, by Theorem 9, $a_1 + 2a_2 + \dots + n a_n = o\{nL(n)\}$.

That $s_n \sim L(n)$ now follows from Theorem 45 of our last paper.§

15. Before leaving power series and passing on to Dirichlet's series we may add one further remark. The theorems which we have proved are all of what we have called an "Abel-Tauber" type; in all of them we start from (i) a hypothesis as to the behaviour of $f(x) = \sum a_n x^n$ as $x \to 1$, (ii) an inequality satisfied by a_n, and deduce information as to the behaviour of s_n. There are, of course, corresponding theorems of a "Cesàro-Tauber" type, in which the hypothesis (i) is replaced by a hypothesis as to the behaviour of one of Cesàro's means formed from $\sum a_n$. These theorems are naturally easier to prove. We may content ourselves with enunciating the simplest analogue of Theorem 9, viz.,

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* See Bromwich, *Infinite Series*, p. 251, Ex. 28.

† This theorem contains Theorem 11 as a particular case.

‡ The proof is similar to that of § 13.

§ The argument by which Theorem 9 itself was proved would only lead to the result with an unnecessarily severe restriction on a_n, viz., that

$$a_n = O_L\{\psi(n)\},$$

where $$\int^n \psi(u)\, du \sim L(n).$$

Thus, if $L(u) = \log u$, this argument would require $a_n = O_L(1/n)$, whereas the real condition is $a_n = O_L(\log n/n)$. The reason why a more elaborate argument is needed when $\alpha = 0$ than when $\alpha > 0$ lies in the fact that

$$\int^n u^\alpha L(u) \frac{du}{u}$$

is of order $n^\alpha L(n)$ when $\alpha > 0$, but of higher order when $\alpha = 0$.

THEOREM 13.—*If* $(s_0+s_1+\ldots+s_n)/(n+1) \sim An^a \quad (A>0,\ a>0)$,

and $na_n = O_L(n^a)$,

then $s_n \sim An^a$.

It was substantially this theorem which was assumed at the end of § 12. The reader will have no difficulty in framing further theorems of this type and of a more general character.

16. We conclude by a brief statement of the analogues of the most important of the preceding theorems for ordinary Dirichlet's series. There are corresponding theorems, for Dirichlet's series of the general type $\Sigma a_n e^{-\lambda_n s}$, which it is our intention to publish elsewhere, and we shall therefore not enter into the details of the proofs.

THEOREM 14.—*If* $f(s) = \Sigma a_n n^{-s}$ *is an ordinary Dirichlet's series with positive coefficients, convergent for* $s > 1$, *and*

$$f(s) \sim \frac{A}{(s-1)^a} L\left(\frac{1}{s-1}\right) \quad (a>0),$$

as $s \to 1$, *then*

$$(-1)^r f^{(r)}(s) = \Sigma a_n (\log n)^r n^{-s} \sim \frac{\Gamma(a+r)}{\Gamma(a)} \frac{A}{(s-1)^{a+r}} L\left(\frac{1}{s-1}\right).$$

The argument by which we prove this theorem is substantially the same as that which we used in proving Theorem 4. We have only to observe that, if $g(s) = f(s)/(s-1)$, then

$$(s-1)\,g'(s) = f'(s) - \frac{f(s)}{s-1} = -\Sigma \log n\, a_n n^{-s} - \frac{1}{s-1}\Sigma a_n n^{-s}$$

steadily decreases as $s \to 0$.

Theorem 14, though interesting in itself, does not give us precisely what is required for the proof of the analogue of Theorem 8. This is contained in

THEOREM 15.—*A result similar to that of Theorem* 14 *holds for series of the form*

$$\Sigma a_n \left\{\frac{1}{n^s} - \frac{1}{(n+1)^s}\right\} \quad (a_n \geqslant 0).$$

The proof of this theorem is very much the same as that of Theorem 14.

17. From Theorem 15 we can deduce the analogue of Theorem 8, viz.,

THEOREM 16.—*If* $f(s) = \Sigma a_n n^{-s}$ *is an ordinary Dirichlet's series with*

positive coefficients, convergent for $s > 1$, *and*

$$f(s) \sim \frac{A}{(s-1)^a} L\left(\frac{1}{s-1}\right),$$

the indices $a, a_1, \ldots$ *being such that* $(\log n)^a L(\log n)$ *tends to a positive limit or to infinity as* $n \to \infty$, *then*

$$s_n = \frac{a_1}{1} + \frac{a_2}{2} + \ldots + \frac{a_n}{n} \sim \frac{A}{\Gamma(a+1)} (\log n)^a L(\log n).$$

We have first

$$s_n \leqslant e \sum_1^n \frac{a_\nu}{\nu} e^{-s \log \nu/\log n} < ef\left(\frac{1}{\log n}\right),$$

and so

$$s_n = O\{(\log n)^a L(\log n)\}.$$

Next

$$f(s) = \Sigma s_n \left\{\frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}}\right\} \sim \frac{A}{(s-1)^a} L\left(\frac{1}{s-1}\right),$$

a relation which takes the place of (3) of § 7. We differentiate r times, as we are entitled to do in virtue of Theorem 15; and the rest of the argument follows the lines of the proof of Theorem 8, no new difficulty of principle occurring.

17. We do not propose to write this argument out at length, nor to discuss in detail the analogues of Theorems 9 *et seq.* It may, however, be observed that the left-handed condition which now occurs instead of

$$na_n = O_L\{n^a L(n)\}$$

is

$$\log n \, a_n = O_L\{(\log n)^a L(\log n)\}.$$

Further, the analogue of Theorem 11 deserves a separate statement. It is

THEOREM 17.—*If* $f(s) = \Sigma a_n n^{-s} \to A$ *as* $s \to 1$, *and* $a_n > -K/\log n$, *then* $\Sigma(a_n/n)$ *converges to the sum* A.

This is the "left-handed" form of Littlewood's* generalisation of Landau's† analogue of Tauber's theorem for ordinary Dirichlet's series.

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* Littlewood, *l.c.*, p. 438. As we are supposing s to tend to unity instead of to zero, our condition is $a_n > -K/\log n$ instead of $a_n > -K/n \log n$.

† Landau, *Monatshefte für Math.*, Vol. 18, p. 8.

NOTE ON LAMBERT'S SERIES

By G. H. HARDY.

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1. In a very interesting memoir "Über Lambertsche Reihen," published recently in *Crelle's Journal*,* Herr K. Knopp proves the following theorem :—

Suppose that the series

$$\Sigma \frac{a_{k\nu+l}}{k\nu+l} \quad (l = 0, 1, 2, \ldots, k-1),$$

are convergent, so that the series

$$\sum_1^\infty a_n \frac{x^n}{1-x^n}$$

is certainly absolutely convergent for $r = |x| < 1$. *Let* $f(x)$ *denote the sum of the latter series for* $r < 1$, *and let*

$$x = rx_0 = re^{2\kappa\pi i/k},$$

where κ *is prime to* k. *Then*

$$\lim_{r\to 1} (1-r) f(x) = \sum_1^\infty \frac{a_{k\nu}}{k\nu}.$$

In its simplest form, when $k = 1$, this theorem is the analogue for Lambert's series, that is to say, series of the form

$$\Sigma a_n \frac{x^n}{1-x^n},$$

of Abel's theorem on the continuity of power series. I gave a proof of

* *Journal für Math.*, Vol. 142, p. 283. A less general theorem of the same character was proved by Franel, *Math. Annalen*, Vol. 52, pp. 543 *et seq.*

this special form of the theorem in a paper published some years ago in these *Proceedings*;* and in a later paper† I stated without proof a generalised form of the result, in which the hypothesis of convergence is replaced by that of summability by some one of Cesàro's means. It is naturally suggested that Knopp's theorem should be capable of a similar generalisation, in which the hypothesis is that of the summability of the various series

$$\Sigma \frac{a_{k\nu+l}}{k\nu+l}.$$

That the theorem thus suggested is, in fact, true will appear in the sequel. It is not, however, precisely that which I propose to prove. A little reflection, in fact, shows that Knopp's hypothesis may be replaced by another which is more natural and also slightly more general. His theorem consists in reality of two parts. Let us write

$$f(x) = \sum_{1}^{\infty} a_{k\nu} \frac{x^{k\nu}}{1-x^{k\nu}} + \sum_{l=1}^{k-1} \sum_{0}^{\infty} a_{k\nu+l} \frac{x^{k\nu+l}}{1-x^{k\nu+l}} = f_0(x) + \sum_{l=1}^{k-1} f_l(x),$$

the first series containing those terms which become infinite as $x \to x_0$. It is clear that the limit assigned by the theorem arises solely from $f_0(x)$: the theorem is, in fact, equivalent to two theorems, expressed respectively by the equations

$$\text{(1)} \qquad \lim (1-r) f_0(x) = \Sigma \frac{a_{k\nu}}{k\nu},$$

$$\text{(2)} \qquad \lim (1-r) f_l(x) = 0.$$

Conditions for the truth of (1) are naturally expressed in terms of the series on the right-hand side; but there is nothing in (2) to suggest the introduction of the series

$$\Sigma \frac{a_{k\nu+l}}{k\nu+l}.\ddagger$$

I propose therefore to modify Knopp's condition. The condition

* *Proc. London Math. Soc.*, Ser. 2, Vol. 4, p. 253.

† *Math. Annalen*, Vol. 64, p. 91.

‡ The denominator $k\nu$ arises from the fact that

$$\lim \frac{1-r}{1-x^{k\nu}} = \lim \frac{1-r}{1-r^{k\nu}} = \frac{1}{k\nu}.$$

But

$$\lim \frac{1-r}{1-x^{k\nu+l}} = 0.$$

which I shall suppose satisfied is that

$$\sum_{\nu=0}^{m} a_{k\nu+l} = o(m).$$

This is more appropriate and also more general:* we shall find, moreover, that the adoption of the more general hypothesis leads to a simplification of the proof.

This modified form of Knopp's theorem is a particular case of the theorem which follows, and the proof of which is the object of this note.

2. THEOREM 1.—*Let* $a_{k\nu+l} = b_{\nu,l}$, *and suppose that an integer p exists, such that* (i) *the series*

$$\sum \frac{b_{\nu,0}}{k\nu} = \sum \frac{a_{k\nu}}{k\nu}$$

is summable (C, p), (ii) *the p-th Cesàro sum* $B^p_{\nu,l}$, *formed from the series* $\sum b_{\nu,l}$, *satisfies the relation*

$$B^p_{\nu,l} = o(\nu^{p+1}).$$

Then

$$\lim_{r\to 1}(1-r)f(x) = \sum \frac{a_{k\nu}}{k\nu}.\dagger$$

3. This theorem, like Knopp's theorem, is really equivalent to two. We first consider $f_0(x)$, and write $y = x^k$ (so that y is real), and $c_\nu = b_{\nu,0}/\nu$. Then it is clear that what we have to prove is

THEOREM 2.—*If* $\sum c_\nu$ *is summable* (C, p), *then*

$$\lim_{y\to 1} \sum c_\nu \frac{\nu y^\nu(1-y)}{1-y^\nu} = \sum c_\nu.$$

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* If s_n is the sum of the first n terms of a series $\sum u_n$ the convergence of $\sum (u_n/n)$ involves $s_n = o(n)$, whereas the converse is not true.

† That the series which represents $f(x)$ still converges absolutely for $r < 1$ is trivial. That the hypothesis (ii) is more general than the hypothesis that $\sum \frac{a_{k\nu+l}}{k\nu+l}$ is summable (C, p), follows from Theorem 14 of Mr. Littlewood's and my paper "Contributions to the Arithmetic Theory of Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 11, p. 435.

The *least* values of p for which (i) or (ii) is satisfied may differ according to the value of l. If one of them is satisfied for any special p, it is satisfied for any greater p; there is therefore no objection to supposing all the p's the same.

This is the theorem which I stated in my paper in the *Math. Annalen* already referred to. It may be proved as follows.

Write $$y = e^{-u}, \quad \phi_\nu(u) = \frac{\nu e^{-\nu u}(1-e^{-u})}{1-e^{-\nu u}}.$$

The result will follow from Theorem 3 of the paper quoted, if we can show that

(1) the differences $$\Delta^\lambda \phi_\nu(u) \quad (0 \leqslant \lambda \leqslant p+1)$$

can be divided into q_λ groups of successive terms of the same sign, where q_λ is a number which may depend upon u, but remains less than a constant as $u \to 0$;

(2) the absolute value of

$$|\nu^\lambda \Delta^\lambda \phi_\nu(u)| \quad (0 \leqslant \lambda \leqslant p)$$

is less than a constant for all values of ν and u in question.

Let $$\Phi(\xi) = \frac{\xi e^{-\xi u}}{1-e^{-\xi u}} = \frac{\xi}{e^{\xi u}-1} = \xi \Psi(\xi).$$

Then it is easily verified that

$$\Psi^{(\lambda)}(\xi) = (-u)^\lambda \left\{ \frac{A_{\lambda,1}}{e^{\xi u}-1} + \frac{A_{\lambda,2}}{(e^{\xi u}-1)^2} + \ldots + \frac{A_{\lambda,\lambda+1}}{(e^{\xi u}-1)^{\lambda+1}} \right\},$$

$$\Phi^{(\lambda)}(\xi) = (-u)^{\lambda-1} \left\{ \frac{\lambda A_{\lambda-1,1} - A_{\lambda,1}\xi u}{e^{\xi u}-1} + \frac{\lambda A_{\lambda-1,2} - A_{\lambda,2}\xi u}{(e^{\xi u}-1)^2} \right.$$
$$\left. + \frac{\lambda A_{\lambda-1,\lambda} - A_{\lambda,\lambda}\xi u}{(e^{\xi u}-1)^\lambda} - \frac{A_{\lambda,\lambda+1}\xi u}{(e^{\xi u}-1)^{\lambda+1}} \right\},$$

where the A's are positive constants and $A_{\lambda,1} = 1$, $A_{\lambda,\lambda+1} = \lambda!$ for all values of λ. If $\Phi^{(\lambda)}(\xi) = 0$, we obtain

$$(\lambda - \xi u)\, e^{\lambda \xi u} + \ldots + \ldots = 0,$$

an equation in ξu whose remaining terms contain powers of $e^{\xi u}$ lower than the λ-th. It is plain that the number s of positive roots of this equation depends only on λ. Let us denote these roots by $\eta_1, \eta_2, \ldots, \eta_s$. Then the roots of $\Phi^{(\lambda)}(\xi) = 0$ are

$$\xi = \eta_1/u, \quad \eta_2/u, \quad \ldots, \quad \eta_s/u.$$

Now $$\phi_\nu(u) = (1-e^{-u})\,\Phi(\nu),$$

$$\Delta^\lambda \phi_\nu(u) = (1-e^{-u})\Delta^\lambda \Phi(\nu) = (-1)^\lambda (1-e^{-u}) \Phi^{(\lambda)}(\xi),$$

where $\nu \leqslant \xi \leqslant \nu+\lambda$. Choose u small enough to ensure that no two of η_1/u, η_2/u, ..., η_s/u differ by less than λ, so that the interval $(\nu, \nu+\lambda)$ can include at most one root of $\Phi^{(\lambda)}(\xi) = 0$. As ν increases from 1 onwards, $\Delta^\lambda \phi_\nu(u)$ remains of fixed sign until $\nu+\lambda > \xi_1/u$. After this alternations may occur, but they must cease as soon as $\nu > \xi_1/u$. The total number of possible changes of sign associated with the root ξ_1/u is at most $\lambda+1$. Proceeding in this way we see that $\Delta^\lambda \phi_\nu(u)$ cannot change sign more than $(\lambda+1)s$ times in all. This proves the proposition (1).

In order to prove (2) we must show that

$$|(1-e^{-u})\nu^\lambda \Delta^\lambda \Phi(\nu)|$$

is less than a constant; and this will be so if the same assertion is true of

$$|u\xi^\lambda \Phi^{(\lambda)}(\xi)|.$$

Referring back to the explicit formula for $\Phi^{(\lambda)}(\xi)$, we see that what we have to prove is that the functions

$$\frac{(\xi u)^\lambda}{(e^{\xi u}-1)^\mu} \; (\mu \leqslant \lambda), \qquad \frac{(\xi u)^{\lambda+1}}{(e^{\xi u}-1)^\mu} \; (\mu \leqslant \lambda+1)$$

are less than constants, and this is obvious. Thus (2) is true, and the proof of Theorem 2 is accordingly completed.

4. In order to complete the proof of Theorem 1, we have to show that

$$f_l(x) = \Sigma b_{\nu, l} \frac{x^{k\nu+l}}{1-x^{k\nu+l}} = o\left(\frac{1}{1-r}\right).$$

The series may be written in the form

$$\Sigma b_{\nu, l} \frac{ar^{k\nu+l}}{1-ar^{k\nu+l}},$$

where $a = e^{2l\kappa\pi i/k}$. Hence our theorem will follow as a corollary of

THEOREM 3.—*If*

$$g(r) = \Sigma c_\nu \frac{r^{k\nu+l}}{1-ar^{k\nu+l}},$$

where a is any number other than a positive number not less than 1, *and the p-th Cesàro sum C_ν^p formed from Σc_ν satisfies*

$$C_\nu^p = o(\nu^{p+1}),$$

then

$$g(r) = o\left(\frac{1}{1-r}\right),$$

as $r \to 1$.

The proof of this is simple. Let $\rho = r^k$ and $\beta = ar^l$. Then

$$\Delta \frac{r^{k\nu}}{1-ar^{k\nu+l}} = \Delta \frac{\rho^\nu}{1-\beta\rho^\nu} = \frac{(1-\rho)\,\rho^\nu}{(1-\beta\rho^\nu)(1-\beta\rho^{\nu+1})},$$

$$\Delta^2 \frac{r^{k\nu}}{1-ar^{k\nu+l}} = \frac{(1-\rho)^2\,\rho^\nu(1+\beta\rho^{\nu+1})}{(1-\beta\rho^\nu)(1-\beta\rho^{\nu+1})(1-\beta\rho^{\nu+2})},$$

and generally

$$\Delta^{p+1} \frac{r^{k\nu}}{1-ar^{k\nu+l}} = \frac{(1-\rho)^{p+1}\rho^\nu\chi_{p+1}}{(1-\beta\rho^\nu)(1-\beta\rho^{\nu+1})\dots(1-\beta\rho^{\nu+p+1})},$$

where χ_{p+1} is a polynomial in ρ and β whose coefficients depend only on p. As β satisfies the same condition as a, the factors in the denominator are all greater in absolute value than a constant. Hence

$$\Delta^{p+1} \frac{r^{k\nu}}{1-ar^{k\nu+l}} = (1-\rho)^{p+1}\rho^\nu O(1);$$

and so $\Sigma c_\nu \dfrac{r^{k\nu}}{1-ar^{k\nu+l}} = \Sigma C_\nu \Delta \dfrac{r^{k\nu}}{1-ar^{k\nu+l}} = \dots = \Sigma C_\nu^p \Delta^{p+1} \dfrac{r^{k\nu}}{1-ar^{k\nu+l}}$

$$= O(1-r)^{p+1}\,\Sigma o(\nu^{p+1})\,r^{k\nu} = o\left(\frac{1}{1-r}\right).$$

5. It is easy to verify that all our conditions are satisfied if

$$a_n = (-1)^n n^s,$$

where s is any number real or complex, and k is odd. Hence, when x approaches the point $e^{2\kappa\pi i/k}$ along a radius vector,

$$f(x) = \Sigma \frac{(-1)^n n^s x^n}{1-x^n} \sim \frac{k^{s-1}}{1-r}\,\Sigma(-1)^{k\nu}\nu^{s-1}.$$

If k is even, the series on the right is no longer summable; and $f(x)$ is, in fact, of higher order than $1/(1-r)$. Suppose, *e.g.*, that s is positive, and $k=2$, $\kappa=1$, so that $x \to -1$. Then

$$f(x) = \Sigma \frac{n^s y^n}{1-y^n} \sim \frac{\Gamma(s+1)\,\zeta(s+1)}{(1-y)^{s+1}},$$

as

$$y = -x \to 1.*$$

6. It is natural to suppose that Theorems 1–3 retain their validity

* See Knopp, *Dissertation*, Berlin, 1907, p. 34.

when $x \to x_0$ along any "Stolz-path," *i.e.*, any curve which has a continuous tangent and does not touch the unit circle. Knopp* has extended his theorem to this case, but only under a narrower hypothesis, viz., that the series $\Sigma \left| \frac{a_n}{n} \right|$ is convergent. It is quite easy to see that Theorem 3 is still true under the more general hypothesis; but to make the corresponding extension of Theorem 2 (and so of Theorem 1) appears to be a less simple matter. The proof would presumably be based upon a theorem of Dr. Bromwich† which includes as a special case the theorem of mine used in § 3.

* *L.c.*, § 1, p. 300.
† *Math. Annalen*, Vol. 65, p. 359.

CLOSED LINKAGES AND PORISTIC POLYGONS

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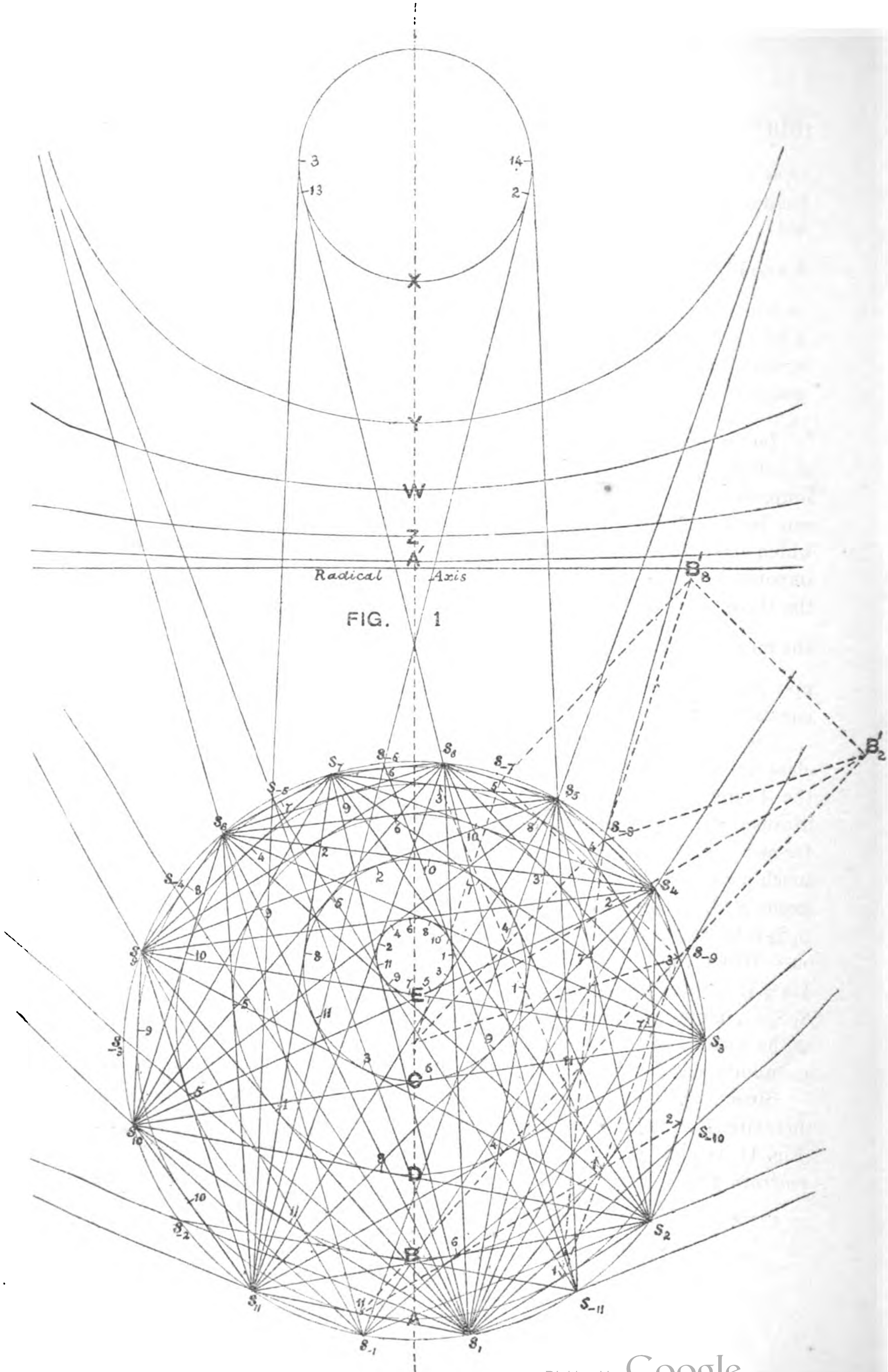
IN the *Proc. London Math. Soc.*, Ser. 2, Vol. 11, pp. 29–39, there is an article on "Closed Linkages" in which the author has pointed out the simplest way in which the connection between them and Poristic Polygons can be established. The present article deals with some consequences which arise from this connection. They are interesting on account of the importance of the ϕ angles, presently to be introduced, with reference to the theory of the transformation of elliptic functions. These ϕ angles are the amplitudes of the elliptic arguments $u+\dfrac{2rK}{n}$ or $u+\dfrac{4rK}{n}$ as the case may be, the argument ϑ of the article quoted being related to u by the equation $u = K-\vartheta-\frac{1}{2}\mu$.

Fig. 1 gives a diagram of a poristic polygon obtained from a third class linkage $O_1A_4B_2'O'$ calculated to close in 11 cells. The tangents to the A circle through the points A_1, A_2, ... intersect upon another circle because A_4B_2' is equal to O_1A_4.* S_1S_2 ... is the poristic polygon so formed. It has been shown by Prof. A. C. Dixon† that S_1S_3 and S_2S_4 touch a coaxial circle at the points where A_1A_3 cuts them, or in other words S_1S_3 touches this circle where $A_{11}A_2$ and A_1A_3 meet. Similarly S_1S_4 touches another coaxial circle where $A_{11}A_3$ and A_1A_4 meet, and so on. When the polygon has n sides (n being odd) there are therefore $\frac{1}{2}(n-1)$ interior circles A_1A_2 ..., B_1B_2 ..., C_1C_2 ..., &c., all coaxial with S_1S_2 ..., circumscribing a similar number of polygons; and the tangents at the vertices of these polygons, taken in the order in which the points are numbered on the diagram, intersect on the same circle S_1S_2 ... .

Since A_3B_2' and A_5B_8' (see Fig. 2) are both equal and parallel to O_1A_4, therefore A_3A_5 is parallel to $B_2'B_8'$ and consequently the polygon $B_2B_8B_3$ ... (Fig. 1) is similar to the $B_2'B_8'$ polygon. This latter may be said to *generate* the A polygon by means of the linkage $O_1AB'O'$, and in the

* *Proc. London Math. Soc.*, Ser. 2, Vol. 11, p. 38.

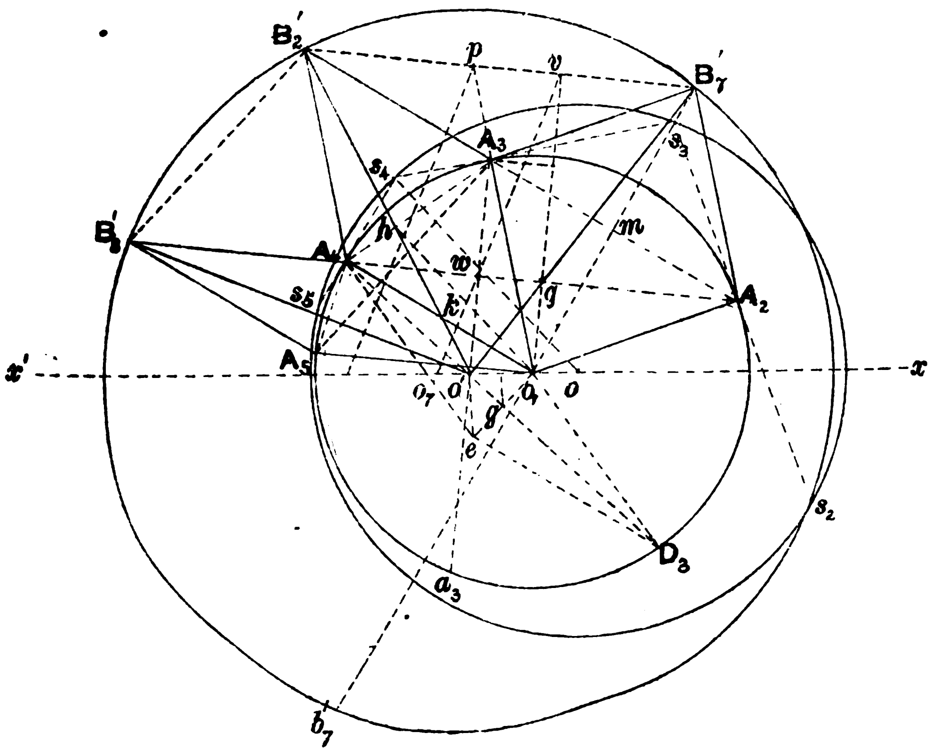
† *Quarterly Journal of Mathematics*, Vol. XLIV, No. 4.



X
Y
W
Z
A
Radical Axis
B_8
B_2'
E
C
D
B
A
S_1
S_2
S_3
S_4
S_5
S_6
S_7
S_8
S_9
S_{10}
S_{11}
S_{-1}
S_{-2}
S_{-3}
S_{-4}
S_{-5}
S_{-6}
S_{-7}
S_{-8}
S_{-9}
S_{-10}
S_{-11}

FIG. 1

FIG. 2



present article the notion of generation is restricted to the case when the generated polygon has tangents intersecting on a circle.

Suppose the radial angles of the points $A_1, A_2, \ldots$ from the centre O_1 are denoted by $a_1, a_2, \ldots$, and those of the points $B_1, B_2, \ldots$ from the centre O_2 by $\beta_1, \beta_2, \ldots$, while those of the points $S_1, S_2, \ldots$ are represented by $2\phi_1, 2\phi_2, \ldots$, then

$$a_1 = \phi_1+\phi_2, \quad a_2 = \phi_2+\phi_3, \quad \&c.; \qquad \beta_1 = \phi_1+\phi_3, \quad \beta_2 = \phi_3+\phi_5, \quad \&c.;$$

and therefore $O'B_7'$ is perpendicular to $S_2 S_4$ (Fig. 2). Hence we see that the A polygon whose angles are $\phi_1+\phi_2$, $\phi_2+\phi_3$, ... is generated by a polygon B' which is similar to the B polygon and whose radial angles are $\phi_1+\phi_3$, $\phi_2+\phi_4$. Since the tangents at the vertices of the B polygon also intersect on a circle, it too is capable of generation by a linkage in the same way. In fact, if we complete the rhombus $B_1 O_2 B_2$ (O_2 being the centre of the B circle) in the point U, U will describe a circle; for this is the converse of the proposition already proved,* viz., that B_7' describing a

* *Proc. London Math. Soc., loc. cit.*

circle and $O_1A_3B_7'A_2$ being a rhombus, the tangents at A_1, A_2, ... intersect on a circle. In the same way that $B_1'B_7'$ is parallel to A_1A_3 so is U_1U_7 parallel to B_1B_3. $B_1C_1C_7B_3$ are collinear like $A_1B_1B_7A_3$ are. Hence the radial angle which U_1 makes at the centre of its circle will be the same as C_1 makes at O_3 the centre of *its* circle, namely $\phi_1+\phi_5$; and the polygon which generates the $B_1B_2\ldots$ polygon will be similar to the C_1, C_7, ... polygon whose angles are $\phi_1+\phi_5$, $\phi_3+\phi_7$, ... In like manner the C_1C_2 polygon is generated by one similar to D_1D_7 whose angles are $\phi_1+\phi_9$, $\phi_5+\phi_{13}$, ..., *i.e.*, $\phi_1+\phi_9$, $\phi_5+\phi_2$, ..., and the $D_1D_2\ldots$ polygon by one similar to $E_1E_7\ldots$ whose angles are $\phi_1+\phi_{17}$, $\phi_9+\phi_{25}$, ..., *i.e.*, $\phi_1+\phi_6$, $\phi_9+\phi_3$, ...

Now the $E_1E_2\ldots$ polygon whose angles are $\phi_1+\phi_6$, $\phi_6+\phi_{11}$, ... is generated by a polygon whose angles are $\phi_1+\phi_{11}$, $\phi_6+\phi_{16}$, ..., *i.e.*, $\phi_1+\phi_{11}$, $\phi_6+\phi_5$, ... These are clearly those of the A polygon taken in the order 11, 5, 10, 4, ...

Fig. 3 gives these linkages combined in one diagram, and to recapitulate we have

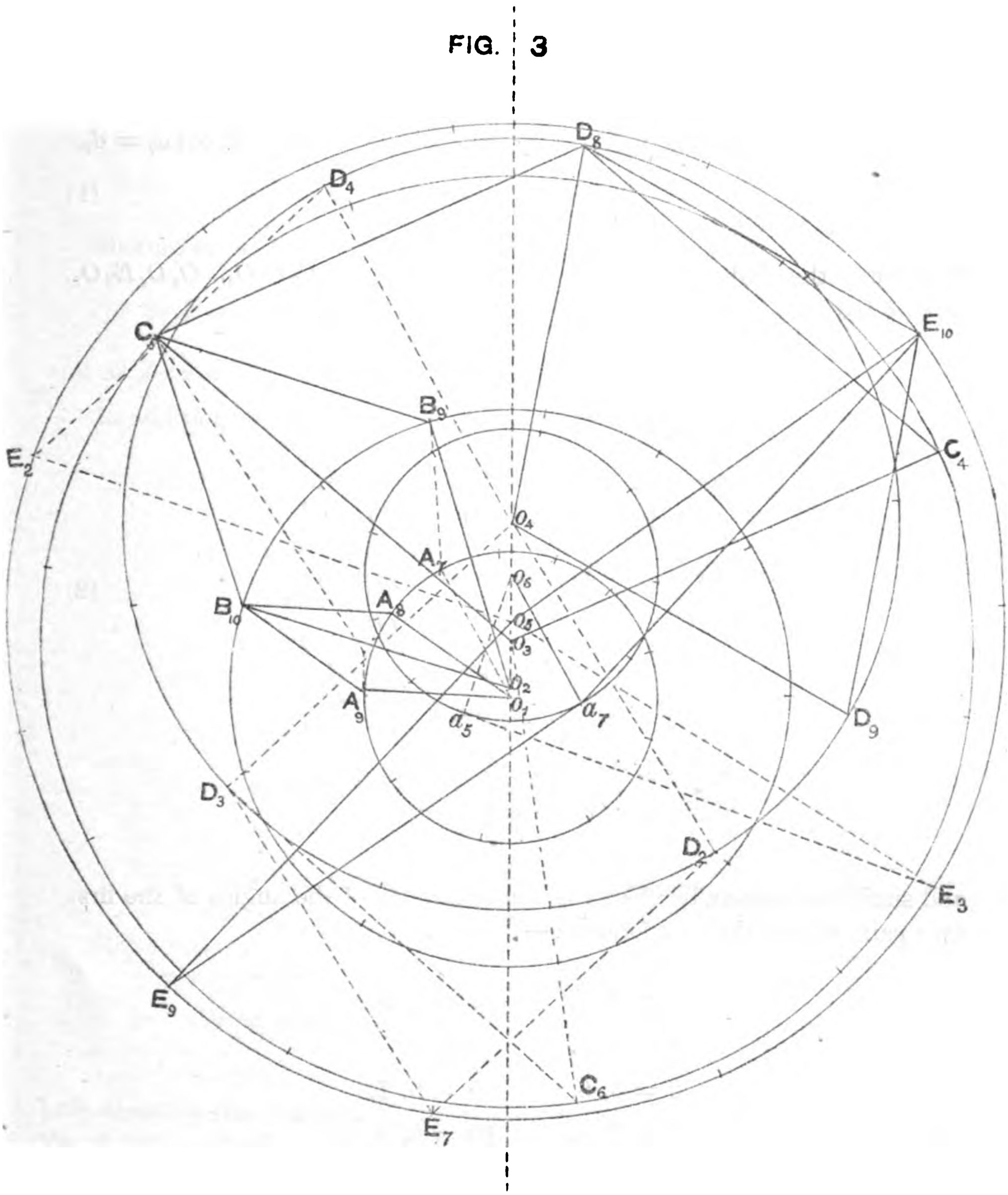
$$\begin{array}{c} A \\ \left.\begin{array}{l}\phi_1+\phi_2\\ \phi_2+\phi_3\\ \phi_3+\phi_4\end{array}\right\} \\ \cdots \end{array} \text{ generated by } \begin{array}{c} B \qquad\quad B \\ \left\{\begin{array}{ll}\phi_1+\phi_3, & \phi_1+\phi_3\\ \phi_2+\phi_4, & \phi_3+\phi_5\\ \phi_3+\phi_5, & \phi_5+\phi_7\end{array}\right\} \\ \cdots \qquad \cdots \end{array} \text{ generated by } \begin{array}{c} C \\ \left\{\begin{array}{l}\phi_1+\phi_5,\\ \phi_3+\phi_7,\\ \phi_5+\phi_9,\end{array}\right. \\ \cdots \end{array}$$

$$\begin{array}{c} C \\ \left.\begin{array}{l}\phi_1+\phi_5\\ \phi_5+\phi_9\\ \phi_9+\phi_2\end{array}\right\} \\ \cdots \end{array} \text{ generated by } \begin{array}{c} D \qquad\quad D \\ \left\{\begin{array}{ll}\phi_1+\phi_9, & \phi_1+\phi_9\\ \phi_5+\phi_2, & \phi_9+\phi_6\\ \phi_9+\phi_6, & \phi_6+\phi_3\end{array}\right\} \\ \cdots \qquad \cdots \end{array} \text{ generated by } \begin{array}{c} E \\ \left\{\begin{array}{l}\phi_1+\phi_6, \quad i.e., \ \phi_1+\phi_{17},\\ \phi_9+\phi_3,\\ \phi_6+\phi_{11},\end{array}\right. \\ \cdots \end{array}$$

$$\begin{array}{c} E \\ \left.\begin{array}{l}\phi_1+\phi_6\\ \phi_6+\phi_{11}\\ \phi_{11}+\phi_5\end{array}\right\} \end{array} \text{ generated by } \begin{array}{c} A \\ \left\{\begin{array}{l}\phi_1+\phi_{11}, \quad i.e., \ \phi_1+\phi_{33},\\ \phi_6+\phi_5,\\ \phi_{11}+\phi_{10}.\end{array}\right. \end{array}$$

Denote the radial angles of the C, D, and E polygons by γ, δ, ϵ; the radii of the A, B, C, D, E and a circles in Fig. 3 by r_1, r_2, r_3, r_4, r_5, r_6; and the distances O_1O_2, O_2O_3, O_3O_4, O_4O_5, O_5O_6, O_6O_1 by d_1, d_2, d_3, d_4, d_5,

FIG. 3



d_6 (negative if so directed), then

$$d_6 = d_1 + d_2 + d_3 + d_4 + d_5.$$

In the linkage $O_1 A_9 B_{10} C_5 D_8 E_{10} a_7 O_6$, $A_9 B_{10}$ is inclined to the axis at an angle α_8, $B_{10} C_5$ at β_9, $C_5 D_8$ at γ_4, $D_8 E_{10}$ at δ_9, $E_{10} a_7$ at $\pi + \epsilon_9$. Hence we have

$$r_1 \cos \alpha_9 + r_1 \cos \alpha_8 + r_2 \cos \beta_9 + r_3 \cos \gamma_4 + r_4 \cos \delta_9 - r_5 \cos \epsilon_9 + r_6 \cos \alpha_7 = d_6, \tag{1}$$

and from the linkages $O_1 A_7 B_9 O_2$, $O_2 B_{10} C_5 O_3$, $O_3 C_6 D_3 O_4$, $O_4 D_4 E_2 O_5$, $O_5 E_3 a_5 O_6$,

$$\left.\begin{aligned}
r_2 \cos \beta_9 &= r_1 \cos \alpha_6 + r_1 \cos \alpha_7 - d_1 \\
r_2 \sin \beta_9 &= r_1 \sin \alpha_6 + r_1 \sin \alpha_7 \\
r_3 \cos \gamma_5 &= r_2 \cos \beta_9 + r_2 \cos \beta_{10} - d_2 \\
r_3 \sin \gamma_5 &= r_2 \sin \beta_9 + r_2 \sin \beta_{10} \\
r_4 \cos \delta_3 &= r_3 \cos \gamma_5 + r_3 \cos \gamma_6 - d_3 \\
r_4 \sin \delta_3 &= r_3 \sin \gamma_5 + r_3 \sin \gamma_6 \\
-r_5 \cos \epsilon_2 &= r_4 \cos \delta_3 + r_4 \cos \delta_4 - d_4 \\
-r_5 \sin \epsilon_2 &= r_4 \sin \delta_3 + r_4 \sin \delta_4 \\
r_6 \cos \alpha_5 &= r_5 \cos \epsilon_2 + r_5 \cos \epsilon_3 + d_5 \\
r_6 \sin \alpha_5 &= r_5 \sin \epsilon_2 + r_5 \sin \epsilon_3
\end{aligned}\right\}, \tag{2}$$

and similar equations by changing the subscripts of the angles of the first four pairs of equations as follows :—

1 = 1, 2,	7 = 2, 3,
2 = 3, 4,	8 = 4, 5,
3 = 5, 6,	9 = 6, 7,
4 = 7, 8,	10 = 8, 9,
5 = 9, 10,	11 = 10, 11,
6 = 11, 1,	

and those of the fifth pair according to the sequence

$$
\begin{array}{ll}
11 = 1,\ 2, & 5 = 2,\ 3, \\
10 = 3,\ 4, & 4 = 4,\ 5, \\
9 = 5,\ 6, & 3 = 6,\ 7, \\
8 = 7,\ 8, & 2 = 8,\ 9, \\
7 = 9,\ 10, & 1 = 10,\ 11. \\
6 = 11,\ 1. &
\end{array}
$$

Making in (1) the substitutions contained in (2), we get

$$3r_1 \Sigma \cos a - r_1 \cos a_7 + r_6 \cos a_7 = 16d_1 + 8d_2 + 4d_3 + 2d_4 + d_5,$$

and as $\Sigma \cos a$ is a constant by Weill's theorem, therefore $r_6 = r_1$.

In addition to the above formulæ we have

$$
\left.
\begin{array}{rl}
r_2 \cos \beta_1 + r_1 \cos a_3 + r_4 \cos \delta_{11} & = \text{const.} \\
r_2 \sin \beta_1 + r_1 \sin a_3 + r_4 \sin \delta_{11} & = 0 \\
-r_4 \cos \delta_{11} + r_1 \cos a_4 - r_3 \cos \gamma_1 & = \text{const.} \\
-r_4 \sin \delta_{11} + r_1 \sin a_4 - r_3 \sin \gamma_1 & = 0 \\
r_3 \cos \gamma_1 + r_1 \cos a_5 + r_5 \cos \epsilon_1 & = \text{const.} \\
r_3 \sin \gamma_1 + r_1 \sin a_5 + r_5 \sin \epsilon_1 & = 0 \\
r_5 \cos \epsilon_{11} + r_1 \cos a_7 + r_3 \cos \gamma_{11} & = \text{const.} \\
r_5 \sin \delta_{11} + r_1 \sin a_7 + r_3 \sin \gamma_{11} & = 0 \\
-r_3 \cos \gamma_{11} + r_1 \cos a_8 - r_4 \cos \delta_1 & = \text{const.} \\
-r_3 \sin \gamma_{11} + r_1 \sin a_8 - r_4 \sin \delta_1 & = 0 \\
r_4 \cos \delta_1 + r_1 \cos a_9 + r_2 \cos \beta_{11} & = \text{const.} \\
r_4 \sin \delta_1 + r_1 \sin a_9 + r_2 \sin \beta_{11} & = 0
\end{array}
\right\}, \qquad (3)
$$

and others by the interchange of subscripts in rotation as they occur in Fig. 3 (*not* cyclically). These may easily be verified from equations (2), and indicate that linkages exist on the diagram to correspond. For instance, if a line be drawn through C_5 parallel and equal to D_3E_7 its

extremity will always be at a fixed distance from A_9 in a direction parallel to the axis. Links may be added to Fig. 3 to secure this without interfering with the mobility of the framework. In other words, $r_1 r_2 r_4$ will form a linkage closing in 11 upon a base equal to the constant in the above equation which will be found to be

$$\tfrac{1}{3}(d_1+2d_2+d_3+2d_4+d_5).$$

If all the links were drawn in Fig. 3, it would be too confused to be intelligible. It is, however, worthy of notice that independently of the extra linkages just alluded to it consists of 72 joints and 182 rods, or 41 more than are necessary for rigidity by Maxwell's rule.

Translating the above results into the ϕ notation, we have

$$\left.\begin{aligned}
r_1\cos(\phi_1+\phi_2)+r_1\cos(\phi_2+\phi_3)-r_2\cos(\phi_1+\phi_3) &= \text{const.}\\
r_1\sin(\phi_1+\phi_2)+r_1\sin(\phi_2+\phi_3)-r_2\sin(\phi_1+\phi_3) &= 0\\
r_2\cos(\phi_1+\phi_3)+r_1\cos(\phi_3+\phi_4)-r_4\cos(\phi_1+\phi_4) &= \text{const.}\\
r_2\sin(\phi_1+\phi_3)+r_1\sin(\phi_3+\phi_4)-r_4\sin(\phi_1+\phi_4) &= 0\\
r_4\cos(\phi_1+\phi_4)+r_1\cos(\phi_4+\phi_5)-r_3\cos(\phi_1+\phi_5) &= \text{const.}\\
r_4\sin(\phi_1+\phi_4)+r_1\sin(\phi_4+\phi_5)-r_3\sin(\phi_1+\phi_5) &= 0\\
r_3\cos(\phi_1+\phi_5)+r_1\cos(\phi_5+\phi_6)-r_5\cos(\phi_1+\phi_6) &= \text{const.}\\
r_3\sin(\phi_1+\phi_5)+r_1\sin(\phi_5+\phi_6)-r_5\sin(\phi_1+\phi_6) &= 0\\
r_5\cos(\phi_1+\phi_6)+r_1\cos(\phi_6+\phi_7)-r_5\cos(\phi_1+\phi_7) &= \text{const.}\\
r_5\sin(\phi_1+\phi_6)+r_1\sin(\phi_6+\phi_7)-r_5\sin(\phi_1+\phi_7) &= 0\\
r_5\cos(\phi_1+\phi_7)+r_1\cos(\phi_7+\phi_8)-r_3\cos(\phi_1+\phi_8) &= \text{const.}\\
r_5\sin(\phi_1+\phi_7)+r_1\sin(\phi_7+\phi_8)-r_3\sin(\phi_1+\phi_8) &= 0\\
r_3\cos(\phi_1+\phi_8)+r_1\cos(\phi_8+\phi_9)-r_4\cos(\phi_1+\phi_9) &= \text{const.}\\
r_3\sin(\phi_1+\phi_8)+r_1\sin(\phi_8+\phi_9)-r_4\sin(\phi_1+\phi_9) &= 0\\
r_4\cos(\phi_1+\phi_9)+r_1\cos(\phi_9+\phi_{10})-r_2\cos(\phi_1+\phi_{10}) &= \text{const.}\\
r_4\sin(\phi_1+\phi_9)+r_1\sin(\phi_9+\phi_{10})-r_2\sin(\phi_1+\phi_{10}) &= 0\\
r_2\cos(\phi_1+\phi_{10})+r_1\cos(\phi_{10}+\phi_{11})-r_1\cos(\phi_1+\phi_{11}) &= \text{const.}\\
r_2\sin(\phi_1+\phi_{10})+r_1\sin(\phi_{10}+\phi_{11})-r_1\sin(\phi_1+\phi_{11}) &= 0
\end{aligned}\right\}, \qquad (4)$$

which, when added together, give

$$r_1 \Sigma \cos \alpha = \text{const.}, \qquad r_1 \Sigma \sin \alpha = 0.$$

Multiply the first pair by

$$\begin{matrix}\cos \\ \sin\end{matrix} (\phi_1+\phi_2+\phi_3) \quad \text{and} \quad \begin{matrix}\sin \\ \cos\end{matrix} (\phi_1+\phi_2+\phi_3),$$

respectively, and $\begin{matrix}\text{add} \\ \text{subtract}\end{matrix}$; treat the second pair similarly with

$$\begin{matrix}\cos \\ \sin\end{matrix} (\phi_1+\phi_3+\phi_4),$$

the third with $$\begin{matrix}\cos \\ \sin\end{matrix} (\phi_1+\phi_4+\phi_5),$$

and so on, and we obtain

$$\cos(\phi_1+\phi_2+\phi_3) = F_1 \cos\phi_1 + F_2 \cos\phi_2 + F_3 \cos\phi_3,$$
$$\sin(\phi_1+\phi_2+\phi_3) = F_1 \sin\phi_1 + F_2 \sin\phi_2 + F_3 \sin\phi_3,$$
$$\cos(\phi_1+\phi_3+\phi_4) = G_1 \cos\phi_1 + G_3 \cos\phi_3 + G_4 \cos\phi_4,$$
$$\sin(\phi_1+\phi_3+\phi_4) = G_1 \sin\phi_1 + G_3 \sin\phi_3 + G_4 \sin\phi_4,$$
$$\cos(\phi_1+\phi_4+\phi_5) = H_1 \cos\phi_1 + H_4 \cos\phi_4 + H_5 \cos\phi_5,$$
$$\sin(\phi_1+\phi_4+\phi_5) = H_1 \sin\phi_1 + H_4 \sin\phi_4 + H_5 \sin\phi_5,$$
$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

the series ending with $\phi_1+\phi_9+\phi_{10}$ for $\phi_1+\phi_{10}+\phi_{11}$ is formed by cyclical changes in $\phi_1+\phi_2+\phi_3$.* These are the formulæ upon which Prof. A. C. Dixon bases his demonstration that the centroid of the S polygon describes a circle whose radial angle is $\Sigma(2\phi)$, and from which he has derived a very large number of valuable relations between the ϕ angles. Among the most important of them are those expressing the fact that the cosine or

* If we start with the chain $BCDEA$ instead of the chain $ABCDE$, we should arrive at the series $\phi_1+\phi_3+\phi_5$, $\phi_1+\phi_5+\phi_7$, ... ; and if with the chain $CDEAB$ at the series $\phi_1+\phi_5+\phi_9$, $\phi_1+\phi_9+\phi_2$, ..., and so on; and taking into consideration the cyclical variations of these angles, we shall have formulæ for the sum of *any* three ϕ angles.

sine of the sum of any odd number of ϕ angles can be expressed as multiples of the cosines or sines of the angles themselves.

A_2A_3 (Figs. 1 and 2) envelopes a conic which is the polar reciprocal of the S circle with reference to the A circle. O_1 will be one focus of it.* If we form its equation† we shall find that its other focus is at O'. m the mid-point of A_2A_3 describes a circle‡ which must be the auxiliary circle of this conic, for it is the foot of the perpendicular O_1B_7' from the focus on the tangent. If h be the point where $O'B_2'$ cuts A_3A_4, h will be the point at which A_3A_4 touches its envelope, for from the symmetry of the rhombus $O_1A_3B_2'A_4$, O_1h and $O'h$ are equally inclined to A_3A_4. It is also true that $B_7'B_2'$ touches its envelope at p, the point where O_1A_3 cuts it, although $O'B_2'A_3B_7'$ is not a rhombus; for denoting the links O_1A_3, A_3B_2', $B_2'O'$, $O'O_1$ by a, b, c, d, and $O'A_3$ by z, then

$$z^2 = a^2+2ad\cos\alpha_3+d^2,$$

$$\sin A_3O'O_1 = \frac{a\sin\alpha_3}{z}, \qquad \cos A_3O'O_1 = \frac{a\cos\alpha_3+d}{z}.$$

The perpendicular from O' on $B_7'B_2' =$

$$c\cos A_3O'B_7' = \frac{z^2+c^2-b^2}{2z} = \frac{a^2-b^2+c^2+d^2+2ad\cos\alpha_3}{2z};$$

therefore the perpendicular from O_1 on $B_7'B_2'$ is

$$\frac{a^2-b^2+c^2+d^2+2ad\cos\alpha_3}{2z} - \frac{d(a\cos\alpha_3+d)}{z} = \frac{a^2-b^2+c^2-d^2}{2z}.$$

Therefore the equation to $B_7'B_2'$ is

$$x(a\cos\alpha_3+d)+y\,a\sin\alpha_3 = \tfrac{1}{2}(a^2-b^2+c^2-d^2),$$

the origin being at O_1. Differentiating with respect to α_3 gives

$$y = x\tan\alpha_3,$$

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* Salmon's *Conic Sections*, 1900, § 308.

† *Ib.*, § 319, or see below.

‡ Weill, "Sur les Polygons, &c.," *Journal de Mathématique* (Liouville), 3rd Series, Tome IV, 1878.

and the elimination of α_3 gives the envelope

$$r = \frac{a^2-b^2+c^2-d^2}{2(a \pm d\cos\theta)},$$

a conic whose eccentricity is d/a. But the coordinates of p are given by the substitution of $y = x\tan\alpha_3$ in the equation to $B_7'B_2'$. Hence p is on the envelope. Since the perpendicular from O_1 on $B_7'B_2'$ multiplied by z is a constant, the locus of v is the inverse of the locus of A, *i.e.*, it is a circle, which must be the auxiliary circle of the conic. Hence O_1 is a focus.

If O_1A_4 cuts $O'B_2'$ in k,

$$O_1kO' = A_4O_1x - B_2'O'O_1 = \alpha_4 - \beta_2 = \phi_4 - \phi_3,$$

$$A_3O_1h = A_3B_2'k = B_2'kA_4 = O_1kO'.$$

Therefore $$hO_1x = \alpha_3 + \phi_4 - \phi_3 = 2\phi_4.$$

Therefore $\phi_3+\phi_5$ and $2\phi_4$ are the focal angles of the point h, and the eccentricity of the conic is $\dfrac{\sin\frac{1}{2}(\phi_3-2\phi_4+\phi_5)}{\sin\frac{1}{2}(\phi_3+2\phi_4+\phi_5)}$, which is therefore a constant.

Weill shows* that the perpendicular from A_3 upon A_2A_4 passes through the focus O' of the envelope of A_2A_3, and that if w be the foot of this perpendicular, w describes a circle. As $B_2'B_7'$ envelopes a conic† so does B_2B_7, and therefore w which is on the line $A_2B_2B_7A_4$ (Fig. 1) must describe the auxiliary circle of the envelope of B_2B_7, and O' must be one focus of it. v and w must therefore be corresponding points on the auxiliary circles of two similar and similarly situated conics, and the intersection of vw with the axis will be the centre of similitude. As the triangles $B_2'A_3B_7'$ and $A_4O_1A_2$ are equal and similarly situated, the rectangle qA_3 is equidistant between v and O_1 and O_1A_3 and vw are equally inclined to O_1v. Therefore A_3O_1x and vO_7O_1 are the focal angles of the point where $B_2'B_7'$ touches its envelope, and from a property of a conic, vw passes through the centre of the two conics which is therefore the centre of similitude.

Since $$vO_1A_3 = \alpha_3 - \tfrac{1}{2}(\alpha_2+\alpha_4) = O_1vw,$$

the inclination of vw is

$$vO_1x - O_1vw = \alpha_2 - \alpha_3 + \alpha_4 = \phi_2 + \phi_5,$$

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* Liouville, *loc. cit.*

† *Proc. London Math. Soc.*, *loc. cit.*

and vw is perpendicular to S_2S_5. Hence the focal angles of the point where $B_2'B_7'$ touches its envelope are $\phi_2+\phi_5$ and $\phi_3+\phi_4$, and we have

$$\frac{\sin\frac{1}{2}(\phi_2-\phi_3-\phi_4+\phi_5)}{\sin\frac{1}{2}(\phi_2+\phi_3+\phi_4+\phi_5)} = \text{a constant},$$

the eccentricity of this envelope.

Dealing with the focal angles of the envelopes of the remaining polygons, we get the following results for the eccentricities of the conics:—

Polygon.	Focal Angles.	Eccentricity.
A_1A_2	$\phi_1+\phi_3,\quad 2\phi_2$	$\dfrac{\sin\frac{1}{2}(\phi_1-2\phi_2+\phi_3)}{\sin\frac{1}{2}(\phi_1+2\phi_2+\phi_3)}$
B_1B_2	$\phi_1+\phi_5,\quad 2\phi_3$	$\dfrac{\sin\frac{1}{2}(\phi_1-2\phi_3+\phi_5)}{\sin\frac{1}{2}(\phi_1+2\phi_3+\phi_5)}$
C_1C_2	$\phi_1+\phi_9,\quad 2\phi_5$	$\dfrac{\sin\frac{1}{2}(\phi_1-2\phi_5+\phi_9)}{\sin\frac{1}{2}(\phi_1+2\phi_5+\phi_9)}$
D_1D_2	$\phi_1+\phi_{17},$* $\quad 2\phi_9$	$\dfrac{\sin\frac{1}{2}(\phi_1-2\phi_9+\phi_{17})}{\sin\frac{1}{2}(\phi_1+2\phi_9+\phi_{17})}$
E_1E_2	$\phi_1+\phi_{33},$* $\quad 2\phi_{17}$	$\dfrac{\sin\frac{1}{2}(\phi_1-2\phi_{17}+\phi_{33})}{\sin\frac{1}{2}(\phi_1+2\phi_{17}+\phi_{33})}$

(6)

Polygon.	Focal Angles.	Eccentricity.
A_1A_3 or B_1B_7	$\phi_1+\phi_4,\quad \phi_2+\phi_3$	$\dfrac{\sin\frac{1}{2}(\phi_1-\phi_2-\phi_3+\phi_4)}{\sin\frac{1}{2}(\phi_1+\phi_2+\phi_3+\phi_4)}$
B_1B_3 or C_1C_7	$\phi_1+\phi_7,\quad \phi_3+\phi_5$	$\dfrac{\sin\frac{1}{2}(\phi_1-\phi_3-\phi_5+\phi_7)}{\sin\frac{1}{2}(\phi_1+\phi_3+\phi_5+\phi_7)}$
C_1C_3 or D_1D_7	$\phi_1+\phi_{13},\quad \phi_5+\phi_9$	$\dfrac{\sin\frac{1}{2}(\phi_1-\phi_5-\phi_9+\phi_{13})}{\sin\frac{1}{2}(\phi_1+\phi_5+\phi_9+\phi_{13})}$
D_1D_3 or E_1E_7	$\phi_1+\phi_{25},\quad \phi_9+\phi_{17}$	$\dfrac{\sin\frac{1}{2}(\phi_1-\phi_9-\phi_{17}+\phi_{25})}{\sin\frac{1}{2}(\phi_1+\phi_9+\phi_{17}+\phi_{25})}$
E_1E_3 or $A_{11}A_5$	$\phi_1+\phi_{49},\quad \phi_{17}+\phi_{33}$	$\dfrac{\sin\frac{1}{2}(\phi_1-\phi_{17}-\phi_{33}+\phi_{49})}{\sin\frac{1}{2}(\phi_1+\phi_{17}+\phi_{33}+\phi_{49})}$

(7)

* It is best to leave these subscripts as they stand because ϕ_{17} and ϕ_6 are not interchangeable without the precaution that the vertices are taken in such an order that $\phi_{17} = \phi_6 + r\pi$, where r is even. All the subscripts are, of course, cyclically interchangeable.

It has also been shown by Weill* that $O'w$, $O'A_8$ is constant. If A_3O' (Fig. 2) be produced until it again meets the A circle in a_3, A_3O', $O'a_3$ is a constant. Therefore $O'a_3/O'w$ is a constant, and the a polygon is similar to the w polygon. The vectorial angle of w with reference to the centre of its circle is $\phi_2+\phi_5$, which is that of the point D_7. Hence the A polygon inverted with reference to the focus of its envelope, which is the centre of the polygon which generates it, gives rise to the D polygon. If B'_7O_1 be produced till it again meets the B' circle in b'_7, the b' polygon is similar to the S polygon; † *i.e.*, the inverse of the B' polygon with reference to the focus of its envelope which is the centre of the polygon it generates is the S polygon. As all the polygons A, B, C, D, E and a (Fig. 3) are connected in an endless chain of linkages, they are all equally generating and generated polygons. All of them invert into one of the others with reference to one focus and into the S polygon with reference to the other.

$$\left.\begin{array}{lcl} A & \text{inverts into} & D \\ B & ,, & E \\ C & ,, & A \\ D & ,, & B \\ E & ,, & C \end{array}\right\}, \tag{8}$$

$$\begin{array}{lcl} A_1 & \text{inverts into} & S_{17},\ i.e.,\ S_6, \\ B_1 & ,, & S_2, \\ C_1 & ,, & S_3, \\ D_1 & ,, & S_5, \\ E_1 & ,, & S_9. \end{array}$$

The polygons are therefore also connected together in an endless chain of inversions

$$A,\ D,\ B,\ E,\ C,\ A,\ \dots .$$

If we take a polygon A and its inverse D (Fig. 2), of which A_4 and D_8 are corresponding vertices, O' being the centre of inversion, and complete the rhombus $O_1A_4D_3e$ and join $O'e$, then these lines together with O_1O' will form a Peaucellier cell of which the side O_1O' is fixed. e will describe a

* Liouville, *loc. cit.*

† *Proc. London Math. Soc.*, *loc. cit.*

circle about O', and g, which is the mid-point of both $O_1 e$ and $A_4 D_3$, will describe a circle about the mid-point of $O_1 O'$. If $xO_1A_4 = \alpha_4$ and $x'O'D_3 = \delta_3$, $O_1 A_4 e = \alpha_4 - \delta_3$, and $O_1 O'e = \alpha_4 + \delta_3$. Therefore the radial angle of g is $\alpha_4 + \delta_3$. g is the centroid of A_4 and D_3. The centroids of the A and D polygons are both fixed points in the axis by Weill's theorem. Therefore the centroid of the polygon formed by the various points g will also be a fixed point in the axis. Applying this result to the various polygons A, B, C, D, and E, we have the following series of angles, the cyclical sum of whose cosines is constant, and the cyclical sum of whose sines is zero

$$\left.\begin{array}{lcl} \alpha_2+\delta_1 & \text{or} & \phi_1+\phi_2+\phi_3+\phi_4 \\ \beta_2+\epsilon_{11} & ,, & \phi_1+\phi_3+\phi_5+\phi_7 \\ \delta_{10}+\beta_{11} & ,, & \phi_1+\phi_4+\phi_7+\phi_{10} \\ \gamma_2+\alpha_1 & ,, & \phi_1+\phi_5+\phi_9+\phi_2 \\ \epsilon_2+\gamma_1 & ,, & \phi_1+\phi_6+\phi_{11}+\phi_5 \end{array}\right\}. \tag{10}$$

If we take the polygons and their inverse S polygons, we obtain in a similar manner the series

$$\left.\begin{array}{l} \phi_1+2\phi_2+\phi_3 \\ \phi_1+2\phi_3+\phi_5 \\ \phi_1+2\phi_5+\phi_9 \\ \phi_1+2\phi_9+\phi_6 \\ \phi_1+2\phi_6+\phi_{11} \end{array}\right\}. \tag{11}$$

If the locus of the centroid of the S points is a circle of which the radial angle is $\Sigma(2\phi)$, and the centroid of its inverse polygon is a fixed point in the axis, the centroid of the combination of the two will be a circle of half the radius situated half way between the fixed point and the centre of the locus of the S centroid. The radial angle of the combined centroid will be the same as before. Hence the cyclical summation of the cosines of the angles in (11) will be of the form $A+B\cos\Sigma(2\phi)$, and the cyclical sum of their sines of the form $B\sin\Sigma(2\phi)$, where A and B are constants.

As regards these constants, we have

$$\begin{aligned} \sin(\phi_1+2\phi_2+\phi_3) &= 2\sin\tfrac{1}{2}(\phi_1+2\phi_2+\phi_3)\cos\tfrac{1}{2}(\phi_1+2\phi_2+\phi_3) \\ &= \frac{2}{\eta}\sin\tfrac{1}{2}(\phi_1-2\phi_2+\phi_3)\cos\tfrac{1}{2}(\phi_1+2\phi_2+\phi_3), \end{aligned}$$

from (6), where η is the eccentricity of the envelope of A_1A_2, in this case d/c; therefore

$$\sin(\phi_1+2\phi_2+\phi_3) = \frac{1}{\eta}\{\sin(\phi_1+\phi_3)-\sin 2\phi_2\},$$

and as

$$\Sigma \sin(\phi_1+\phi_3) = 0,$$

by Weill's theorem; therefore

$$\Sigma \sin(\phi_1+2\phi_2+\phi_3) = -\frac{1}{\eta}\Sigma \sin 2\phi.$$

Similarly

$$\Sigma \cos(\phi_1+2\phi_2+\phi_3) = \Sigma\{1-2\sin^2\tfrac{1}{2}(\phi_1+2\phi_2+\phi_3)\}$$

$$= \Sigma\left\{1+\frac{1}{\eta}[\cos(\phi_1+\phi_3)-\cos 2\phi]\right\},$$

and if the constant summation of $\cos(\phi_1+\phi_3)$ be denoted by L, and $\Sigma \cos 2\phi$ by $H+K\cos\Sigma 2\phi$,

$$\Sigma \cos(\phi_1+2\phi_2+\phi_3) = \frac{1}{\eta}(n\eta+L-H)-\frac{1}{\eta}K\cos\Sigma 2\phi,$$

$$\Sigma \sin(\phi_1+2\phi_2+\phi_3) = \qquad -\frac{1}{\eta}K\sin\Sigma 2\phi.$$

If we regard the link $O'B_2'$ of the quadrilateral $O'B_2'A_3O_1$ as fixed instead of the link O_1O', we have a new linkage of which the a link is b, the b link a, the c link d, and the d link c. This linkage will be of the fifth class, and since

$$a^2+c^2 = b^2+d^2+\frac{4r}{1-s}bd,$$

and

$$ac = \frac{1+s}{1-s}bd,$$

in virtue of the closing of the original linkage,* therefore

$$b^2+d^2 = a^2+c^2-\frac{4r}{1+s}ac,$$

and

$$bd = \frac{1-s}{1+s}ac.$$

These two last conditions show that this fifth class linkage closes in $2n$* (a fifth class linkage never closing in an odd number of cells).

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* *Proc. London Math. Soc., loc. cit.*

Now since $$O_1kO' = A_4O_1x - B_2'O'O_1,$$
and $$O_1kO' = A_3B_2'O';$$
therefore $$A_3B_2'O' = \alpha_4 - \beta_2 = \phi_4 + \phi_5 - \phi_3 - \phi_5 = \phi_4 - \phi_3.$$

This linkage will therefore give rise to a generated polygon of twenty-two vertices whose vectorial angles are

$$\begin{array}{ccccc} -\phi_1+\phi_2, & \phi_6-\phi_7, & -\phi_{11}+\phi_1, & \phi_5-\phi_6, & -\phi_{10}+\phi_{11}, \\ \phi_2-\phi_3, & -\phi_7+\phi_8, & \phi_1-\phi_2, & -\phi_6+\phi_7, & \phi_{11}-\phi_1, \\ -\phi_3+\phi_4, & \phi_8-\phi_9, & -\phi_2+\phi_3, & \phi_7-\phi_8, & \\ \phi_4-\phi_5, & -\phi_9+\phi_{10}, & \phi_3-\phi_4, & -\phi_8+\phi_9, & \\ -\phi_5+\phi_6, & \phi_{10}-\phi_{11}, & -\phi_4+\phi_5, & \phi_9-\phi_{10}, & \end{array}$$

which is symmetrical with reference to the axis, being formed by the first eleven vertices and their reflections in $O'B_2'$. It will be generated by a polygon whose vertices are the various positions of the end O_1 of the now moving arm $O'O_1$, which will make with the fixed line $O'B_2'$ the supplements of the angles β. The turning of the triangle $B_2'A_3O_1$ about the diagonal $B_2'O_1$ will cause the point A_3 to assume the position A_4, of which the radial angle $O'B_2'A_4$ is negative. The subsequent turning of the triangle $O'A_4O_1$ about $O'A_4$ will bring the arm $O'O_1$ to $O'e$ with a similarly negative angle, the supplement of $B_2'O'e$ or of $B_3'O'O_1$, namely, $-(\pi-\beta_3)$. While, therefore, the generated polygon, which we will call the A' polygon has the radial angles $-\phi_1+\phi_2$, $\phi_2-\phi_3$, $-\phi_3+\phi_4$, ..., the generating polygon will have the angles $-(\phi_1+\phi_3)$, $+(\phi_2+\phi_4)$, $-(\phi_3+\phi_5)$, ... As the a and b links of this linkage are equal, the tangents at the vertices of the A' polygon will intersect on a circle, and following the same method as in the former case we see that the new S polygon formed by these points of intersection will have the radial angles

$$2\phi_1,\ -2\phi_2,\ 2\phi_3,\ \ldots,\ -2\phi_{10},\ 2\phi_{11},\ -2\phi_1,\ 2\phi_2,\ -2\phi_3,\ \ldots,\ 2\phi_{10},\ -2\phi_{11},$$

that is to say, the new S points will be those of the original together with their reflections in the axis, the latter being indicated by negative subscripts in Fig. 1. The lines which touch the circle circumscribing the A' polygon are

$$S_1S_{-2},\ S_{-2}S_3,\ \ldots,\ S_{11}S_{-1},\ S_{-1}S_2,\ S_2S_{-3},\ \ldots,\ S_{-11}S_1.$$

The circle they touch has a large radius and a portion of it is indicated in Fig. 1 at A'.

The same reasoning applies to this new linkage, and, *mutatis mutandis*, the same results may be obtained. Thus if every vertex in the double S polygon is joined to every other one, all the lines so drawn will envelope fixed coaxial circles. There will thus arise ten polygons inscribed in coaxial circles, half of which are interior and half completely exterior to the S circle.

It is convenient for the moment to replace the series

$$2\phi_1,\ -2\phi_2,\ 2\phi_3,\ \ldots,\ \text{by}\ 2\phi_1',\ 2\phi_2',\ 2\phi_3',\ \ldots,\ 2\phi_{22}',$$

where, of course,

$$\begin{array}{llll} \phi_1' = & \phi_1, & \phi_{12}' = & -\phi_1, \\ \phi_2' = & -\phi_2, & \phi_{13}' = & \phi_2, \\ \phi_3' = & \phi_3, & \phi_{14}' = & -\phi_3, \\ \ldots & \ldots & \ldots & \ldots \\ \phi_{10}' = & -\phi_{10}, & \phi_{21}' = & \phi_{10}, \\ \phi_{11}' = & \phi_{11}, & \phi_{22}' = & -\phi_{11}. \end{array}$$

The ten polygons will then be

$$\begin{array}{ccccc} A' & B' & W' & C' & X' \\ \phi_1'+\phi_2', & \phi_1'+\phi_3', & \phi_1'+\phi_4', & \phi_1'+\phi_5', & \phi_1'+\phi_6', \\ \phi_2'+\phi_3', & \phi_3'+\phi_5', & \phi_4'+\phi_7', & \phi_5'+\phi_9', & \phi_6'+\phi_{11}', \\ \phi_3'+\phi_4', & \phi_5'+\phi_7', & \phi_7'+\phi_{10}', & \phi_9'+\phi_{13}', & \phi_{11}'+\phi_{16}', \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ E' & Y' & D' & Z' & F' \\ \phi_1'+\phi_7', & \phi_1'+\phi_8', & \phi_1'+\phi_9', & \phi_1'+\phi_{10}', & \phi_1'+\phi_{11}', \\ \phi_7'+\phi_{13}', & \phi_8'+\phi_{15}', & \phi_9'+\phi_{17}', & \phi_{10}'+\phi_{19}', & \phi_{11}'+\phi_{21}', \\ \phi_{13}'+\phi_{19}', & \phi_{15}'+\phi_{22}', & \phi_{17}'+\phi_3', & \phi_{19}'+\phi_6', & \phi_{21}'+\phi_9. \end{array}$$

It will easily be seen that the polygon whose leading angle is $\phi_1'+\phi_3'$ closes in eleven, for the twelfth angle is $\phi_1'+\phi_3'$; but there still remain the angles $\phi_2'+\phi_4'$, $\phi_4'+\phi_6'$, ..., which are in another order

$$\phi_{12}'+\phi_{14}',\ \phi_{14}'+\phi_{16}',\ \ldots,\ \text{or}\ -(\phi_1'+\phi_3'),\ -(\phi_3'+\phi_5'),\ \ldots,$$

so that this series forms two independent polygons each closing in eleven, one being the reflection of the other. The same applies to all the polygons whose leading angles are of the type $\phi_1'+\phi_{1+s}'$, where s is even; but those of the type where s is odd close in twenty-two and are unipartite.

We have seen that a polygon whose angles are $\phi_1'+\phi_{1+s}'$, $\phi_{1+s}'+\phi_{1+2s}'$ is generated by one whose angles are $\phi_1'+\phi_{1+2s}'$, $\phi_{1+s}'+\phi_{1+3s}'$; or we may put it that a polygon $\phi_1'+\phi_{1+s}'$, $\phi_{1+\frac{1}{2}s}'+\phi_{1+\frac{3}{2}s}'$ will generate one $\phi_1'+\phi_{1+\frac{1}{2}s}'$, $\phi_{1+\frac{1}{2}s}'+\phi_{1+s}'$, if s is even; otherwise not. But the polygon whose leading angle is $\phi_1'+\phi_{1+s}'$ is the same polygon as the one whose leading angle is $\phi_1'+\phi_{1+n+s}'$; therefore we may extend the above and say that a polygon $\phi_1'+\phi_{1+n+s}'$, $\phi_{1+\frac{1}{2}(n+s)}'+\phi_{1+\frac{3}{2}(n+s)}'$ will generate one whose angles are $\phi_1'+\phi_{1+\frac{1}{2}(n+s)}'$, $\phi_{1+\frac{1}{2}(n+s)}'+\phi_{1+n+s}'$, provided these subscripts are whole numbers. Now when n is odd, if $1+\frac{1}{2}s$, $1+\frac{3}{2}s$ are not whole numbers by reason of s being also odd, $1+\frac{1}{2}(n+s)$, $1+\frac{3}{2}(n+s)$ are whole numbers, and therefore one way or the other, whether s is odd or even, all the polygons are generators, but when n is even and s is odd, they are not generators either way. Further, when n is odd no polygon can generate in more than way, for if s is odd $1+\frac{1}{2}s$, $1+\frac{3}{2}s$ are fractional, and when s is even $1+\frac{1}{2}(n+s)$, $1+\frac{3}{2}(n+s)$ are fractional. But when n is even and s is even both series are whole numbers, and so such a polygon can be a generator in two ways.

This fifth class linkage therefore gives rise to an endless chain of linkages as in the former case, viz.,

$$\begin{array}{ccccccccccc}
B' & \leftarrow & C' & \leftarrow & D' & \leftarrow & E' & \leftarrow & F' & \leftarrow & B' \\
\phi_1'+\phi_3' & & \phi_1'+\phi_5' & & \phi_1'+\phi_9' & & \phi_1'+\phi_7' & & \phi_1'+\phi_{11}' & & \phi_1+\phi_{21} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
A' & & Z' & & Y' & & W' & & X' & & \\
\phi_1'+\phi_2' & & \phi_1+\phi_{10}' & & \phi_1'+\phi_8' & & \phi_1'+\phi_4' & & \phi_1'+\phi_6' & &
\end{array}$$

but in addition each of the upper row generates one of the lower row. In consequence of these considerations we have all the results of the former case except one applicable to all the ten polygons, the exception being that the A', W', X', Y', and Z' polygons do not invert into the S polygon.

Taking firstly the chain $B'C'D'E'F'B'$, it will be seen that in reality this consists of two separate and distinct chains, one being the reflection of the other, and working quite independently of it. The A' polygon is excluded from it, but in its place appears an F' polygon, which is the original A polygon and its reflection, while the others are the original B, C, D, and E, and their reflections. We therefore get no new result so far. But if we take the generation of A' by B' we shall find that the A' polygon inverted with reference to the focus of its envelope, which is the centre of the B' polygon, produces the W' polygon. Similarly Z' so

treated will give rise to X', X' to Y', W' to Z', Y' to A', so that we have a new chain of inversions

$$A'W'Z'X'Y'A,$$

from which we deduce that

$$\left.\begin{array}{lcl} \phi_1'+\phi_2'+\phi_3'+\phi_4', & & \phi_1-\phi_2+\phi_3-\phi_4 \\ \phi_1'+\phi_4'+\phi_7'+\phi_{10}', & & \phi_1-\phi_4+\phi_7-\phi_{10} \\ \phi_1'+\phi_6'+\phi_{11}'+\phi_{16}', & \text{that is to say} & \phi_1-\phi_6+\phi_{11}-\phi_5 \\ \phi_1'+\phi_8'+\phi_{15}'+\phi_{22}', & & \phi_1-\phi_8+\phi_4-\phi_{11} \\ \phi_1'+\phi_{10}'+\phi_{19}'+\phi_6', & & \phi_1-\phi_{10}+\phi_8-\phi_6 \end{array}\right\}, \tag{12}$$

form a series of angles whose cyclical cosine summation is constant, and whose cyclical sine summation is zero.

Now, as B' can generate both F' and A', $\phi_1'+\phi_3'$ not only inverts into $2\phi_2'$, but its equivalent $\phi_1'+\phi_{25}'$ inverts into $2\phi_{18}'$, which is $-2\phi_2'$. Hence, in addition to the angle $\phi_1+2\phi_2+\phi_3$, we have also the angle $\phi_1-2\phi_2+\phi_3$ whose cyclical cosine summation is $A\cos\Sigma\, 2\phi+B$, and whose cyclical sine summation is $A\sin\Sigma\, 2\phi$. Similarly for the other polygons. Hence the series

$$\left.\begin{array}{lcl} \phi_1-2\phi_2+\phi_3, & & \phi_1-2\phi_2+\phi_3 \\ \phi_1-2\phi_3+\phi_5, & & \phi_1-2\phi_3+\phi_5 \\ \phi_1-2\phi_5+\phi_9, & \text{or} & \phi_1-2\phi_5+\phi_9 \\ \phi_1-2\phi_9+\phi_{17}, & & \phi_1-2\phi_9+\phi_6 \\ \phi_1-2\phi_{17}+\phi_{33}, & & \phi_1-2\phi_6+\phi_{11} \\ \phi_1-2\phi_{33}+\phi_{65}, & & \phi_1-2\phi_{11}+\phi_{10} \end{array}\right\}, \tag{13}$$

which have these cyclical summations.

It follows from this that while the eccentricity of the envelope of A' is $\dfrac{\sin\frac{1}{2}(\phi_1+2\phi_2+\phi_3)}{\sin\frac{1}{2}(\phi_1-2\phi_2+\phi_3)}$, that of F' (*i.e.*, A) is $\dfrac{\sin\frac{1}{2}(\phi_1-2\phi_2+\phi_3)}{\sin\frac{1}{2}(\phi_1+2\phi_2+\phi_3)}$, so that if one envelope is an ellipse tho other is a hyperbola of reciprocal eccentricity. The same thing applies to the envelopes of B' and Z', &c.

The circles circumscribing the A', X', Y', Z', W' polygons have been shown in part in Fig. 1. They tend to shrink up into one of the Landen points, and on the other hand to expand to the common radical axis of all the circles in the diagram. Since the chord S_5S_6 is equal to the chord $S_{-5}S_{-6}$, the angles subtended at S_{11} by the E circle and the X' circle are equal. This applies to any S point, and therefore to the extremities of

the vertical diameter which are the centres of similitude of the two circles. The C and Y', the B and Z', ... circles have the same properties, each pair having the same generator.

It can be shown geometrically without trouble that, if a point A_4 moving on a circle (Fig. 2) inverts into another point D_3 with reference to a fixed point O' within the circle, there is another fixed point t, outside the circle on the same diameter, such that

$$O_1O'.O_1t = O_1A_4^2,$$

through which A_4 will invert into D_3' the reflection of D_3 in the diameter. The mid-point G' of A_4D_3' will likewise describe a circle. Its radial angle will be $\alpha_4-\delta_3$, *i.e.*, $-\phi_3+\phi_4+\phi_5-\phi_6$. Hence we have also the following series of angles

$$\left.\begin{array}{l} \phi_1-\phi_2-\phi_3+\phi_4 \\ \phi_1-\phi_3-\phi_5+\phi_7 \\ \phi_1-\phi_4-\phi_7+\phi_{10} \\ \phi_1-\phi_5-\phi_9+\phi_2 \\ \phi_1-\phi_6-\phi_{11}+\phi_5 \end{array}\right\}, \tag{14}$$

$$\left.\begin{array}{l} \phi_1+\phi_2-\phi_3-\phi_4 \\ \phi_1+\phi_4-\phi_7-\phi_{10} \\ \phi_1+\phi_6-\phi_{11}-\phi_5 \\ \phi_1+\phi_8-\phi_4-\phi_{11} \\ \phi_1+\phi_{10}-\phi_8-\phi_6 \end{array}\right\}, \tag{15}$$

whose cosine summation is constant, and whose sine summation is zero.

It is naturally suggested by the above that this point t is the centre of *a* generator of A. As a matter of fact it is. For, if we consider the link O_1A_4 as fixed, the A polygon and its reflection will be generated by one having the radial angles $O_1A_2B_1'$, $-O_1A_3B_7'$, $O_1A_4B_2'$, ..., that is to say, $\pi-(\phi_4-\phi_2)$, $-\pi+(\phi_5-\phi_3)$, $\pi-(\phi_6-\phi_4)$; and comparing these angles with those in the table on p. 215, we shall recognise them as those of the Z' polygon. But as the links AB' and $B'O'$ are not equal, this generation is not of the sort considered in this article, that is to say Z' does not invert into S. In this restricted sense also A' generates E'; W', C'; Y', B'; and X', D'. It may be noticed, however, that if we enlarge the radius of this double A polygon until it becomes equal to the A circle, the fixed point A_4 will be removed to t, where

$$O_1t.O_1O' = (O_1A_4)^2.$$

As $\dfrac{O_1 O'}{O_1 A_4} = \dfrac{d}{a}$, the eccentricity of the envelope of $B_1 B_7$, $\dfrac{O_1 t}{O_1 A_4} = \dfrac{a}{d}$ which is the eccentricity of the envelope of Z'.

Further, since the inclination of the bisector of the angle $S_1 S_2 S_3$ is $\frac{1}{2}(\phi_1 + 2\phi_2 + \phi_3)$, and that of the external bisector of $S_1 S_{-2} S_3$ is $-\frac{1}{2}(\phi_1 - 2\phi_2 + \phi_3)$, the distance of the centre of A or F' from the centre of S is

$$R \sin 2\phi_2 \cot \tfrac{1}{2}(\phi_1 + 2\phi_2 + \phi_3) - R \cos 2\phi_3,$$

or

$$-R \frac{\sin \frac{1}{2}(\phi_1 - 2\phi_2 + \phi_3)}{\sin \frac{1}{2}(\phi_1 + 2\phi_2 + \phi_3)} = -\eta R,$$

while that of A' is

$$-R \sin 2\phi_3 \cot \tfrac{1}{2}(\phi_1 - 2\phi_2 + \phi_3) - R \cos 2\phi_2,$$

or

$$-R \frac{\sin \frac{1}{2}(\phi_1 + 2\phi_2 + \phi_3)}{\sin \frac{1}{2}(\phi_1 - 2\phi_2 + \phi_3)} = -\frac{1}{\eta} R,$$

and similarly for the other pairs.

FORMULÆ FOR THE SPHERICAL HARMONIC $P_n^{-m}(\mu)$, WHEN $1-\mu$ IS A SMALL QUANTITY

By H. M. MACDONALD.

[Received February 6th, 1914.—Read February 12th, 1914.]

AN expression for $P_n^{-m}(\mu)$, when $1-\mu$ is a small quantity, has been obtained in the form*

$$P_n^{-m}(\mu) = (n\cos\tfrac{1}{2}\theta)^{-m}\Big[J_m(x)-\sin\tfrac{1}{2}\theta J_{m+1}(x) + \sin^2\tfrac{1}{2}\theta\left\{\frac{x}{6}J_{m+3}(x)-\tfrac{1}{2}J_{m+2}(x)\right\}-\ldots\Big],$$

where
$$x = 2n\sin\tfrac{1}{2}\theta,$$

and this result was used to determine the values of n for which $P_n^{-m}(\mu)$ vanishes, when $1-\mu$ is a small quantity. The object of this note is to obtain other formulæ, which are more suitable when the approximate value of $P_n^{-m}(\mu)$ is required in the summation of a series. The above formula was obtained by expressing $\Gamma(n+r+1)/\Gamma(n-r+1)$ in powers of n in the relation

$$P_n^{-m}(\mu) = \frac{1}{\Gamma(m+1)}\left(\frac{1-\mu}{1+m}\right)^{\frac{1}{2}m} F[-n,\ n+1,\ 1+m,\ \tfrac{1}{2}(1-m)],$$

and rearranging the series. The expression $\Gamma(n+r+1)/\Gamma(n-r+1)$ can also be expressed in terms of powers of $(n+\frac{1}{2})$, the result being

$$\Gamma(n+r+1)/\Gamma(n-r+1) = (n+\tfrac{1}{2})^{2r}-A(n+\tfrac{1}{2})^{2r-2}+B(n+\tfrac{1}{2})^{2r-4}-\ldots,$$

where
$$A = \frac{1}{3}\frac{\Gamma(r+1)}{\Gamma(r-2)}+\frac{\Gamma(r+1)}{\Gamma(r-1)}+\frac{1}{4}\frac{\Gamma(r+1)}{\Gamma(r)},$$

$$B = \frac{1}{18}\frac{\Gamma(r+1)}{\Gamma(r-5)}+\frac{11}{15}\frac{\Gamma(r+1)}{\Gamma(r-4)}+\frac{31}{12}\frac{\Gamma(r+1)}{\Gamma(r-3)}+\frac{29}{12}\frac{\Gamma(r+1)}{\Gamma(r-2)}+\frac{9}{32}\frac{\Gamma(r+1)}{\Gamma(r-1)}.$$

Substituting this expression for $\Gamma(n+r+1)/\Gamma(n-r+1)$ in $P_n^{-m}(\mu)$, writing

* *Proc. London Math. Soc.*, Vol. XXXI, p. 269, 1899.

$x = (2n+1)\sin\frac{1}{2}\theta$, and rearranging the series in a similar way, the result takes the form

$$P_n^{-m}(\mu)$$
$$= \{(n+\tfrac{1}{2})\cos\tfrac{1}{2}\theta\}^{-m}\Big[J_m(x)+\sin^2\tfrac{1}{2}\theta\left\{\frac{x}{6}J_{m+3}(x)-J_{m+2}(x)+\frac{1}{2x}J_{m+1}(x)\right\}$$
$$+\sin^4\tfrac{1}{2}\theta\left\{\frac{x^2}{72}J_{m+6}(x)-\frac{11x}{30}J_{m+5}(x)+\frac{31}{12}J_{m+4}(x)\right.$$
$$\left.-\frac{29}{6x}J_{m+3}(x)+\frac{9}{8x^2}J_{m+2}(x)\right\}+\dots\Big].$$

Another form can be obtained from the relation

$$P_n^{-m}(\mu) = \frac{(1-\mu^2)^{\frac{1}{2}m}}{2^m\,\Gamma(m+1)}F[n+m+1,\ m-n,\ 1+m,\ \tfrac{1}{2}(1-m)],$$

by expressing

$$(n-m+r+1)(n-m+r+2)\dots(n-m)(n+m+1)\dots(n+m+r),$$

in powers of $(n+\frac{1}{2})$, and the result is

$$P_n^{-m}(\mu) = (n+\tfrac{1}{2})^{-m}(\cos\tfrac{1}{2}\theta)^m\Big[J_m(x)+\sin^2\tfrac{1}{2}\theta\left\{\frac{x}{6}J_{m+3}(x)-(m+1)J_{m+2}(x)\right.$$
$$\left.+\frac{2(m+\frac{1}{2})^2}{x}J_{m+1}(x)\right\}+\dots\Big].$$

When $m=0$, both these formulæ become

$$P_n(\mu) = J_0(x)+\sin^2\tfrac{1}{2}\theta\left\{\frac{x}{6}J_3(x)-J_2(x)+\frac{1}{2x}J_1(x)\right\}+\dots.^*$$

These expressions, proceeding according to powers of $\sin^2\frac{1}{2}\theta$, are more convenient for calculation than the former one, and can be used to calculate $P_n^{-m}(\mu)$, or for determining the values of μ near to 1 for which it vanishes, when n and m are given, as well as for determining the zeros of $P_n^{-m}(\mu)$ when m and μ are given.

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* The first term of this series has been used as an approximation for $P_n(\mu)$ by the writer, *Phil. Trans.*, A, Vol. 210, p. 117, 1909.

ON THE THEORY OF FOUCAULT'S PENDULUM, AND OF THE GYROSTATIC PENDULUM

By T. J. I'A. BROMWICH.

[Read June 12th, 1913.—Received September 20th, 1913.]

1. *The effective Lagrangian function for the motion of a heavy particle near a marked point close to the surface of the earth.*

We take the origin at the marked point and the axis of z as vertically upwards; the direction of the vertical being that of apparent gravity, as defined by observations with a plumb-line or with the surface of a mercury trough. The axes of x, y will be taken at first in southerly and easterly directions, respectively, in the horizontal plane through the origin.

Let λ denote the latitude, that is, the angle between the axis of z and the equatorial plane of the earth; λ will be regarded as positive for a point in the northern hemisphere, as usual. Further, let p denote the distance of our origin from the axis of rotation of the earth, and let ω be the earth's angular velocity. Then referred to an origin on the earth's axis the coordinates of (x, y, z) are

$$\begin{aligned} \xi &= p + x\sin\lambda + z\cos\lambda, \\ \eta &= y, \\ \zeta &= \quad -x\cos\lambda + z\sin\lambda, \end{aligned}$$

the axis of ζ being the earth's axis, and the axis of ξ passing through our marked point. Thus, if we neglect the earth's motion in space, the velocities of our particle will be

$$\begin{aligned} \dot\xi - \omega\eta &= \dot x\sin\lambda + \dot z\cos\lambda - \omega y, \\ \dot\eta + \omega\xi &= \dot y + \omega(p + x\sin\lambda + z\cos\lambda), \\ \dot\zeta \qquad &= -\dot x\cos\lambda + \dot z\sin\lambda. \end{aligned}$$

And so the square of the resultant velocity is equal to

$$\begin{aligned} 2T = \dot x^2 + \dot y^2 + \dot z^2 + 2\omega\sin\lambda(x\dot y - \dot x y) + 2\omega\cos\lambda(z\dot y - \dot z y) + 2\omega p\dot y \\ + \omega^2\{y^2 + (p + x\sin\lambda + z\cos\lambda)^2\}. \quad (1) \end{aligned}$$

Next let V, V' denote respectively the potential energies of gravity and of apparent gravity respectively: then, as is well known, we can write

$$V - V' = \tfrac{1}{2}\omega^2(\xi^2+\eta^2)+\text{const.}, \tag{2}$$

because the components of centrifugal force due to the earth's rotation are $\omega^2\xi$, $\omega^2\eta$ parallel to the axes of ξ, η.

Consequently we have the Lagrangian function expressed in the form

$$2(T-V) = \dot{x}^2+\dot{y}^2+\dot{z}^2+2\omega\sin\lambda(x\dot{y}-\dot{x}y)-2V' \\ +2\omega p\dot{y}+2\omega\cos\lambda(z\dot{y}-\dot{z}y). \tag{3}$$

The expression (3) is exact, except for the omission of the effect due to the earth's motion in space; but we shall now proceed to simplify (3) by introducing certain approximations. In the first place it follows from our choice of axes that, at the origin,

$$\frac{\partial V'}{\partial x} = 0, \quad \frac{\partial V'}{\partial y} = 0, \quad \frac{\partial V'}{\partial z} = g,$$

where g is the value of apparent gravity at the origin; and so, under the circumstances of any actual experiment, we can write

$$V' = gz+\text{const.} \tag{4}$$

Secondly, the term $2\omega p\dot{y}$ in (3) may be omitted, as it cannot affect the equations of motion; and, finally, we shall see that the effect of the term

$$2\omega\cos\lambda(z\dot{y}-\dot{z}y) \tag{5}$$

is small compared with that of the other term

$$2\omega\sin\lambda(x\dot{y}-\dot{x}y). \tag{6}$$

Since the effect of (6) will be found to account for Foucault's experiment, and so is small compared with the effect of gravity, it is clear that the effect of (5) must be quite negligible, and we shall accordingly use for the effective Lagrangian function

$$2L = \dot{x}^2+\dot{y}^2+\dot{z}^2+2\omega\sin\lambda(x\dot{y}-\dot{x}y)-2gz. \tag{7}$$

It should be noticed that in (7) *it is no longer necessary to adhere to the axes of x, y originally defined*:* they may be any perpendicular axes in the horizontal plane, such that the direction of rotation from x to y is in the sense NWSE. This follows from the fact that the values of

* Of course this remark will not apply in any problem for which the effect of the term (5) has to be considered, such as the discussion of Foucault's gyroscope.

$\dot{x}^2+\dot{y}^2$ and $(x\dot{y}-\dot{x}y)$ are independent of the axes selected; provided (as regards the second) that the sense of rotation of the axes is unchanged.

Proof that in the theory of Foucault's pendulum the effect of the term (5) *is small compared with that of* (6).

The truth of this statement is not hard to deduce by a direct application of Lagrange's equations; but in the calculation it is necessary to take account of the fact that x, y, z are not independent coordinates, and are in fact connected by the relation

$$x^2+y^2+(l-z)^2 = l^2, \tag{8}$$

where l is the length of the pendulum and the origin is taken at the equilibrium position of the bob.

To avoid the labour of substituting from (8), we may apply the Lagrangian equations in their variational form, so as to obviate the necessity for using independent coordinates, when forming the equations of motion. The variational equation is, in fact,

$$\Sigma\, \delta x \left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} \right\} = 0,$$

and the parts of this which are derived from (5) and (6) are respectively

$$2\omega \cos\lambda\, (\dot{z}\,\delta y - \dot{y}\,\delta z), \tag{9}$$

and

$$2\omega \sin\lambda\, (\dot{x}\,\delta y - \dot{y}\,\delta x). \tag{10}$$

If we now make use of (8), it will be seen that

$$\left.\begin{aligned} (l-z)\,\delta z &= x\,\delta x + y\,\delta y \\ (l-z)\,\dot{z} &= x\dot{x}+y\dot{y} \end{aligned}\right\}.$$

Thus, eliminating $\dot{z}$ and δz, (9) can be written in the form

$$\frac{2\omega\cos\lambda}{l-z}\left\{(x\dot{x}+y\dot{y})\,\delta y - \dot{y}\,(x\,\delta x + y\,\delta y)\right\} = \frac{2\omega\cos\lambda}{l-z}\, x\,(\dot{x}\,\delta y - \dot{y}\,\delta x).$$

Consequently the quotient of (9) by (10) is equal to

$$(x\cot\lambda)/(l-z),$$

and this (unless λ is small) may be taken to be of the order of magnitude of 1/100 to 3/100. For instance, with the dimensions of Foucault's experiment at the Pantheon (see p. 226 below) the greatest value of x/l is about 3/67, and $\cot\lambda = \cdot 874$; and so the maximum value of the quotient is about 1/25, with an average value of about 1/50.

2. *First approximation to the path of the bob of the pendulum in Foucault's experiment.*

To get the first approximation we solve (8) approximately and get the equation

$$2lz = x^2+y^2.$$

Substitute from this in (7) and we find the approximate Lagrangian function

$$2L = \dot{x}^2+\dot{y}^2+2n\,(x\dot{y}-\dot{x}y)-p^2\,(x^2+y^2), \tag{11}$$

where

$$n = \omega\sin\lambda, \quad p^2 = g/l.$$

The equations of motion then become

$$\left.\begin{aligned}\ddot{x}-2n\dot{y}+p^2x &= 0\\ \ddot{y}+2n\dot{x}+p^2y &= 0\end{aligned}\right\}. \tag{12}$$

Thus, if we write $x+\iota y = \zeta$, we find

$$\ddot{\zeta}+2\iota n\dot{\zeta}+p^2\zeta = 0,$$

the solution of which is $\zeta = e^{-\iota nt}(Ae^{\iota qt}+Be^{-\iota qt})$
where $q^2 = p^2+n^2$ } . (13)

In Foucault's actual experiments, the bob was let go from relative rest in a displaced position. And, in view of the fact (pointed out at the end of § 1, p. 223) that the axis of x may have any horizontal direction, we may take the plane of zx to pass through the initial position of the bob. Thus initially we may suppose that

$$x = a, \quad y = 0, \quad \dot{x} = 0, \quad \dot{y} = 0\,;$$

or, in terms of ζ, the initial conditions are

$$\zeta = a, \quad \dot{\zeta} = 0,$$

where, of course, a is real. These conditions determine the constants in (13), and we find that

$$A = \frac{a}{2q}(q+n), \quad B = \frac{a}{2q}(q-n).$$

Thus, in *Foucault's case the path of the bob* (*as seen by an observer*) *is given by the equation*

$$\zeta = x+\iota y = \frac{a}{2q}\left\{(q+n)\,e^{\iota(q-n)t}+(q-n)\,e^{-\iota(q+n)t}\right\}, \tag{14}$$

which represents a hypocycloid.

To see that (14) does represent a hypocycloid, we need only remark that each term on the right-hand side represents a uniform circular motion, and that the two rotations are in opposite senses: thus the path has a hypotrochoidal character, and, since the bob comes to rest when $\dot{\zeta} = 0$, the path is hypocycloidal.

The hypocyloid is contained between the two circles

$$|\zeta| = a, \qquad |\zeta| = an/q,$$

corresponding to values of t of the types given by

$$qt = m\pi, \qquad qt = (m+\tfrac{1}{2})\,\pi,$$

where m is any integer. Further, since $\dot{\zeta} = 0$, when $qt = m\pi$, it

follows that these points give the cusps, which accordingly occur when

$$\zeta = (-1)^m\, a\, e^{-\iota(m n\pi/q)}.$$

Hence in a complete circuit of the larger circle the number of cusps is equal to q/n, if this is an integer; and is the integer next greater than q/n, in general.

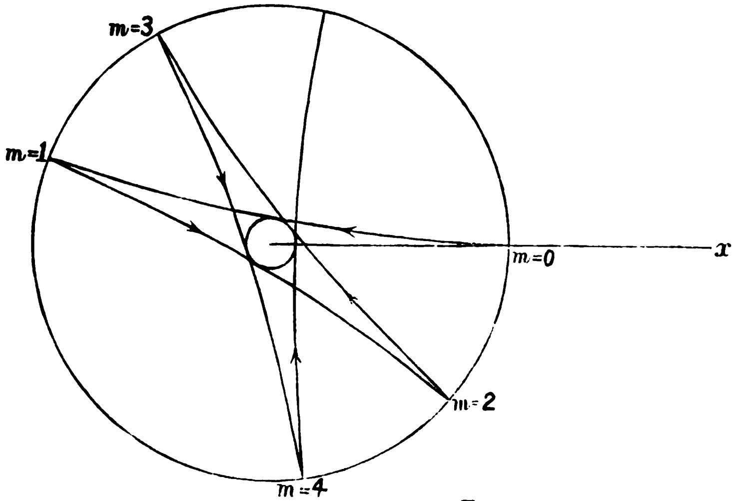

FIG. 1.

The adjoining diagram indicates the general character of the first four swings of the bob; but in the actual experiments, the radius of the inner circle is so small that it is impossible to attempt to draw the diagram to scale. For instance, in Foucault's own experiments* the period $2\pi/n$ was about 32 hours, or more accurately

31 h. 52 m. 47s. (sidereal time) = 31 h. 47 m. 15s. (mean time);

and the period of swing $2\pi/q$ was about 16·42s. Thus q/n was about 7000, and so, although the radius a was 3 metres, the inner radius an/q was less than half a millimetre; and the distance between adjacent cusps on the outer circle was about 2·6 mm.

In theoretical accounts of Foucault's experiment, it has been usual to describe the path of the bob as a *revolving ellipse*: that (14) can be so described is true,† but the description hardly conveys a very clear idea of a hypocycloid.

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* *Comptes Rendus*, t. 32, p. 135 (February 5th, 1851).

† To prove this, write $x' + \iota y' = \zeta' = \zeta e^{\iota n t}$; then the axes of x', y' rotate with angular velocity n with respect to the axes of x, y. Then (14) gives at once

$$x' = a \cos qt, \quad y' = (an/q) \sin qt,$$

so with respect to the axes of x', y' the path of the bob is an ellipse of semi-axes a and (an/q).

In some books a different form of the experiment is described: the bob is supposed to be projected *exactly* from its lowest point. It does not appear that any experimenter has ever succeeded in performing the experiment in this way: but the theory is simple. Suppose the bob projected with small velocity v along any horizontal direction through the origin; this direction may be taken as the axis of x, and the initial conditions are then

$$\zeta = 0, \qquad \dot{\zeta} = v.$$

Thus we find, from (13),

$$\zeta = \{(v/q)\sin qt\}\, e^{-int}. \tag{15}$$

Or with polar coordinates r, θ in the plane of x, y, the path (15) may be written*

$$\left.\begin{aligned} r &= (v/q)\sin qt, \quad \theta = -nt \\ \text{that is} \quad r &= -(v/q)\sin(q\theta/n) \end{aligned}\right\}, \tag{16}$$

and this represents a kind of rosette as in the diagram (which again is not drawn to scale).

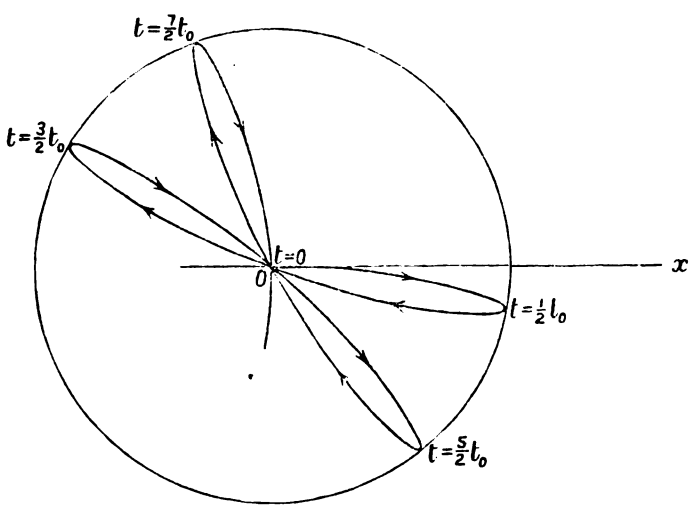

FIG. 2. $t_0 = \pi/q$ = half the period of swing of the pendulum.

3. *Second approximation to the apsidal angle.*

Foucault's experiment was repeated by various experimenters in England and elsewhere: and it was found that however carefully the

* This path (16) has sometimes been described as a *revolving line*: for, with the axes x', y' defined in the last footnote (15) reduces to $x' = (v/q)\sin qt$, $y' = 0$.

pendulum was let go from rest, sooner or later the radius of the inner circle became appreciable, and that a correction had to be made on this account to the angles observed.

We shall now proceed to calculate the apsidal angle for a pendulum oscillating in Foucault's manner between two circles of radii a, b, of which b is the smaller; and both a, b will be small compared with l. The angle between the points in which the bob meets the circle a is what is usually observed; and this is the angle which we here call the apsidal angle.

Introduce spherical polar coordinates in (7) by writing

$$x = l\sin\theta\cos\psi, \quad y = l\sin\theta\sin\psi, \quad z = l(1-\cos\theta),$$

so that ψ is measured round the vertical in the sense NWSE. Then (7) takes the shape

$$2L = l^2\{(\dot\theta^2+\sin^2\theta\,\dot\psi^2)+2n\sin^2\theta\,\dot\psi-2p^2(1-\cos\theta)\}, \tag{17}$$

where, as before, $n = \omega\sin\lambda;\quad p^2 = g/l.$

Since (17) does not contain ψ or t explicitly, Lagrange's equations, derived from L, will have two immediate first integrals

$$\frac{\partial L}{\partial\dot\psi} = \text{const.} \quad\text{and}\quad \dot\theta\frac{\partial L}{\partial\dot\theta}+\dot\psi\frac{\partial L}{\partial\dot\psi}-L = \text{const.}$$

Hence we find

$$\sin^2\theta(\dot\psi+n) = \text{const.}, \tag{18}$$

and

$$\dot\theta^2+\sin^2\theta\,\dot\psi^2+2p^2(1-\cos\theta) = \text{const.} \tag{19}$$

Thus, if we write $1-\cos\theta = x$, (18), (19) become

$$x(2-x)(\dot\psi+n) = nh, \text{ say}, \tag{18a}$$

and

$$\frac{\dot x^2}{x(2-x)}+x(2-x)\,\dot\psi^2+2p^2x = 2p^2k, \text{ say}, \tag{19a}$$

and so we deduce

$$\dot x^2 = 2p^2x(2-x)(k-x)-n^2\{h-x(2-x)\}^2. \tag{20}$$

There are two values of x for which $\dot x$ is zero, corresponding to the two apsidal circles of radii a, b: if these values of x are denoted by α, β, respectively we shall have

$$\alpha(2-\alpha) = a^2/l^2, \quad \beta(2-\beta) = b^2/l^2. \tag{21}$$

We can then bring (20) to the form

$$\left.\begin{aligned}\dot x^2 &= \frac{n^2h^2}{\alpha\beta}(\alpha-x)(x-\beta)(1-\gamma x+\delta x^2)\\ \text{where}\qquad h^2\delta &= \alpha\beta\end{aligned}\right\}. \tag{22}$$

Before proceeding to the calculation of the apsidal angle which is wanted, it is necessary to express the constants h, γ, δ in terms of α, β; this depends on perfectly elementary algebra, but a few of the steps may be useful as a check.

First write $x = \alpha$, $x = \beta$ in (20), and we obtain

$$\frac{2p^2}{n^2}(k-\alpha) = \frac{h^2}{\alpha(2-\alpha)} - 2h + \alpha(2-\alpha),$$

and a similar formula with β in place of α; take the difference and divide by $(\alpha-\beta)$, we then obtain a result without the constant k: and this relation is

$$\frac{2p^2}{n^2} = (2-\alpha-\beta)\left\{\frac{h^2}{\alpha\beta(2-\alpha)(2-\beta)} - 1\right\}. \tag{23}$$

Next write $x = 2$ in (20) and (22): then we get

$$1 - 2\gamma + 4\delta = \frac{\alpha\beta}{(2-\alpha)(2-\beta)}. \tag{24}$$

Thus from (22), (23), (24) we obtain h, γ, δ in the forms

$$\frac{1}{\delta} = \frac{h^2}{\alpha\beta} = (2-\alpha)(2-\beta)\left(\frac{2p^2/n^2}{2-\alpha-\beta}+1\right), \tag{25}$$

and

$$\gamma = \tfrac{1}{2} + 2\delta - \tfrac{1}{2}\frac{\alpha\beta}{(2-\alpha)(2-\beta)}. \tag{26}$$

In the actual experiments α is of the order of magnitude $\frac{1}{2}(3/67)^2$, or say $1/1000$; and β is much smaller. Also p/n is of the order 7000; and so, with relative errors of order $1/10^6$ at most, we can write (25), (26) in the forms

$$\delta = \frac{\alpha\beta}{h^2} = \frac{n^2}{4p^2}, \tag{25a}$$

$$\gamma = \tfrac{1}{2}. \tag{26a}$$

Now the required apsidal angle is the angle by which ψ is increased as x varies from α to β and back to α; and this angle is

$$\Psi = 2\int_\beta^\alpha dx\left(\frac{\dot\psi}{\dot x}\right),$$

or if we substitute $\dot x$, $\dot\psi$ from (22) and (18a) respectively, we find the formula

$$\Psi = \pm\, 2\sqrt{(\alpha\beta)}\int_\beta^\alpha \frac{dx\,(1-\gamma x+\delta x^2)^{-\frac{1}{2}}}{\sqrt{\{(\alpha-x)(x-\beta)\}}}\left\{\frac{1}{x(2-x)} - \frac{1}{h}\right\}, \tag{27}$$

where the ambiguous sign is the same as that of h.

But, remembering that x is less than α, we may write, in (27),

$$(1-\gamma x+\delta x^2)^{-\frac{1}{2}} = 1+\tfrac{1}{2}\gamma x, \qquad 1/(2-x) = \tfrac{1}{2}+\tfrac{1}{4}x,$$

with an error in each case less than $1/10^6$; and then Ψ takes the form

$$\Psi = \pm 2 \int_\beta^\alpha \frac{\sqrt{(\alpha\beta)}\, dx}{\sqrt{\{(\alpha - x)(x-\beta)\}}} \left(\frac{1}{2x} + \frac{1+\gamma}{4} - \frac{1}{h}\right).$$

These integrals are easily found by elementary methods, and we obtain

$$\Psi = \pm \pi \left\{ 1 + 2\sqrt{(\alpha\beta)} \left(\frac{1+\gamma}{4} - \frac{1}{h}\right)\right\}. \tag{28}$$

We can now substitute for h and γ their values found from (25a) and (26a) respectively: and then (28) takes the form

$$\Psi = \pm \pi \{1 + \tfrac{3}{4}\sqrt{(\alpha\beta)}\} - (n\pi/p). \tag{29}$$

If we now substitute for α and β from (21), we get a formula, which is as accurate as (29),

$$\Psi = \pm \pi \left(1 + \tfrac{3}{8}\frac{ab}{l^2}\right) - \frac{n\pi}{p}. \tag{29a}$$

What is most easily observed is the advance of the apse along the outer circle: and in a *complete* swing this advance will be given by the angle

$$2\Psi \mp 2\pi.$$

Hence *the angular advance of the apse, in each complete swing, is given by the formula*

$$2\pi\left(\pm \tfrac{3}{8}\frac{ab}{l^2} - \frac{n}{p}\right), \tag{30}$$

the positive direction of advance being in the sense NWSE.

The path of the bob in the two cases is indicated roughly in the adjoining diagrams.* The angle given by (30) is the angle subtended by two adjacent apses on the outer circle (such as 1, 2) at the centre.

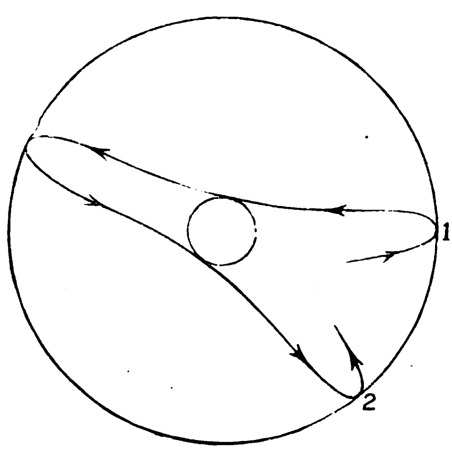

FIG. 3.—h positive.

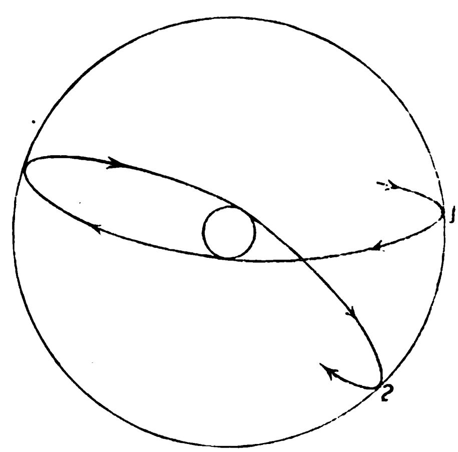

FIG. 4.—h negative.

* For a general explanation, see p. 234 below.

The formula (30) was the formula used (in effect) for the comparison of theory with experiment by the English experimenters* who followed Foucault; but I cannot find that the theory has ever been fully worked out. The method adopted was to superpose the correcting term $\frac{3}{8}(ab/l^2)$, worked out† by neglecting ω, upon the first approximation in which the path of the bob is treated as plane (as in § 2).

Since both terms in (30) are small, it is plausible to suppose that each may be calculated separately, and the results superposed; but this is not entirely convincing. And on account of the fundamental importance of the experiment, it seemed desirable to give a more complete theoretical discussion, as has been carried out here.

4. *A comparison of the theory of Foucault's pendulum with that of the gyrostatic pendulum.*

It is well known that in the first approximation to the oscillations of a gyrostatic pendulum (swinging near to the downward vertical) the motion of a marked point on the axis is given by the equations

$$\left.\begin{aligned} A\ddot{x} - C\omega\dot{y} + Mglx &= 0 \\ A\ddot{y} + C\omega\dot{x} + Mgly &= 0 \end{aligned}\right\}, \tag{31}$$

where A, C are the constants of inertia of the pendulum (at the point of support), M is the mass and l the distance of the centre of inertia from the fixed point. It should be observed that the signs of the terms in $C\omega$ are opposite to those usually adopted: this implies that *here* the positive direction for ω is taken from y towards x.

The equations (31) reduce at once to (12) if we write

$$n = \tfrac{1}{2}\frac{C\omega}{A}, \quad p^2 = \frac{Mgl}{A}, \tag{32}$$

and so, *to the first approximation*, the path of the end of the axis is of the same general character as for Foucault's pendulum. In particular, if the gyrostatic pendulum is let go with its axis at rest, the path will be

* A number of notes of these experiments were published in the *Philosophical Magazine* for 1851; in Vol. 1 (pp. 552–572) will be found some general descriptions; more details were given in Vol. 2 by Bunt (at Bristol), Galbraith and Haughton (at Dublin), Lampry and Schaw (in Ceylon), and Gerard. Other experiments are also described in the immediately following volumes.

† This correction seems to have been first obtained by Bravais (Lagrange, *Mécanique Analytique*, t. II, p. 352, note VII); a good many investigations of the formula were published in connexion with these experiments. In Vol. 2 of the *Phil. Mag.* for 1851 proofs were given by Galbraith and Haughton, Airy, Thacker, Coombe, Tebay, and Anstice.

hypocycloidal; and if it is projected so that the axis passes through the lowest point (the origin) the path will consist of a series of loops (of the type sketched on p. 227 above).

It is not difficult to adjust the constants p, n so as to produce hypocycloids of any desired number of cusps. The number of cusps* for the simplest type of hypocycloid which is contained between two circles of radii a, b is N, where

$$N = a/\{\tfrac{1}{2}(a-b)\},$$

this being supposed an integer. Thus from the fact that $b = an/q$, we see that

$$N = 2q/(q-n),$$

and so
$$n/q = (N-2)/N.$$

Thus
$$\frac{p^2}{n^2} = \left(\frac{N}{N-2}\right)^2 - 1 = \frac{4(N-1)}{(N-2)}.$$

Thus to obtain a three-cusped hypocycloid we must suppose $p^2/n^2 = 8$, and to obtain a four-cusped hypocycloid $p^2/n^2 = 3$.

Illustrations of curves reproduced from actual tracings (the constants having been adjusted to correspond to $N = 3, 4$ as just explained) will be found in Webster's *Dynamics* (Leipzig, 1904, pp. 293 and 294).

But, although the first approximations to the paths are identical for Foucault's pendulum and for the gyrostatic pendulum, yet the analysis for the second approximations is somewhat different in these two problems. With the gyrostatic pendulum, when the Lagrangian function is modified in Routh's way (so as to take account of the constant spin), we obtain the function, corresponding to (17),

$$2R = A\,\{(\dot\theta^2+\sin^2\theta\,\dot\psi^2)-4n\dot\psi\cos\theta-2p^2(1-\cos\theta)\}. \tag{33}$$

Two first integrals follow from (33) as from (17), and these are

$$\sin^2\theta\,\dot\psi-2n\cos\theta = \text{const.} = 2n(h-1), \tag{34}$$

$$\dot\theta^2+\sin^2\theta\,\dot\psi^2+2p^2(1-\cos\theta) = \text{const.} = 2p^2k, \tag{35}$$

which are, respectively, the integrals of angular momentum (round the vertical) and of energy.

Thus with $x = 1-\cos\theta$, as before, we find the equation corresponding to (20),

$$\dot x^2 = 2p^2x(2-x)(k-x)-4n^2(h-x)^2, \tag{36}$$

leading to
$$\dot x^2 = 4\frac{n^2h^2}{\alpha\beta}(\alpha-x)(x-\beta)(1-\lambda x), \tag{37}$$

* The cusps are supposed to be *real*: we are not concerned with the imaginary cusps which are needed in order to account for the Plücker's numbers of the higher hypocycloids.

$$\left.\begin{aligned} &\text{where} && p^2\alpha\beta = 2\lambda n^2 h^2 \\ &\text{and} && \frac{h^2}{\alpha\beta}(2-\alpha)(2-\beta)(1-2\lambda) = (2-h)^2 \end{aligned}\right\}. \tag{38}$$

Hence we find
$$\frac{h^2}{\alpha\beta} - \left(\frac{p}{n}\right)^2 = \frac{(2-h)^2}{(2-\alpha)(2-\beta)}, \tag{39}$$

and when α, β are known (as well as p/n) we can determine h from (39).

Under ordinary conditions of spinning the gyrostat,* the value of n/p can hardly exceed 2; thus we may usually regard n and p as of the same order of magnitude. But with slower spins it may be better to regard n/p as small. With either assumption, however, we may replace (38) and (39) by the approximations

$$\frac{h^2}{\alpha\beta} = \frac{p^2+n^2}{n^2}, \qquad 2\lambda = \frac{p^2}{p^2+n^2}. \tag{40}$$

And, if n/p is small, (40) may be further reduced to the form

$$\frac{h^2}{\alpha\beta} = \frac{p^2}{n^2}, \qquad \lambda = \tfrac{1}{2}. \tag{40a}$$

The apsidal angle in the present problem is

$$\Psi = 2\int_\beta^\alpha dx\left(\frac{\dot\psi}{\dot x}\right) = \pm 2\sqrt{(\alpha\beta)}\int_\beta^\alpha \frac{dx\,(1-\lambda x)^{-\frac{1}{2}}}{\sqrt{\{(\alpha-x)(x-\beta)\}}}\,\frac{1-x/h}{x(2-x)}, \tag{41}$$

where the ambiguous sign is the same as that of h. Thus, as before, we expand in the forms

$$(1-\lambda x)^{-\frac{1}{2}} = 1+\tfrac{1}{2}\lambda x, \qquad 1/(2-x) = \tfrac{1}{2}+\tfrac{1}{4}x,$$

and then evaluate the integrals; the result is

$$\Psi = \pm\pi\left\{1+\sqrt{(\alpha\beta)}\left(\frac{1+\lambda}{2}-\frac{1}{h}\right)-\frac{1+\lambda}{4}\,\frac{\sqrt{(\alpha\beta)}}{h}(\alpha+\beta)\right\}, \tag{42}$$

which corresponds to (28). But it will be seen from (40) that this can be written in the shape

$$\Psi = \{\pm\pi - n\pi/\sqrt{(p^2+n^2)}\} + \text{terms which tend to zero with } \alpha, \beta. \tag{43}$$

Thus the angle between two adjacent apses on the circle α is now in general *not small:* and so, without sensible error, we may omit the terms which tend to zero with α and β. In the Foucault problem, these terms are comparable in practice with the second term in (43). When the small

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* See the numerical calculations given below (p. 235).

terms in (43) are omitted, we have precisely the same result as would follow from the first approximation (31).

However, in the case which really corresponds to Foucault's pendulum (namely, when n/p is small), the equation (42) by the aid of (40a) reduces to*

$$\Psi = \pm\,\pi\left\{1+\sqrt{(\alpha\beta)}\left(\tfrac{3}{4}-\frac{1}{h}\right)\right\} = \pm\,\pi\left\{1+\tfrac{3}{4}\sqrt{(\alpha\beta)}\right\}-n\pi/p,$$

which is the same result as found in (29).

Relations between the Four Diagrams above.

[*Added March 19th*, 1914.]—A question has been asked as to the mutual relations between these types of curves; to explain the connexion, it will be convenient to regard Fig. 1 as the standard type, and to suppose the deviations from this standard to be due to the introduction of a small value of $\dot\psi$ at the outer circle, in place of the zero value, which indicates the presence of cusps.

It will be seen that, according to (18) and (25a), we have the following approximate equation

$$x\,(n+\dot\psi) = \pm\,p\sqrt{(\alpha\beta)}.$$

Thus if the value of $\dot\psi$ is positive at the outer circle (where x is greatest, and equal to α), it will be positive everywhere, and h will be positive; thus we can regard Fig. 3 as derived from Fig. 1, by rounding off the cusps in such a way as to give a positive value of $\dot\psi$ at the outer circle, the inner circle being enlarged to correspond.

On the other hand, if $\dot\psi$ is negative at the outer circle, it might happen that h would still be positive. The path would then be derived from Fig. 1 by rounding off the cusps in the other sense, which would introduce small loops in place of the cusps; this type of curve has not actually been drawn, for a reason which will appear in a moment. If we now think of these small loops as gradually opening out, the inner circle at the same time shrinking, we arrive at Fig. 2 when the inner circle has shrunk to a point. Continuing the same process, by letting the loops pass over the centre, we arrive at curves of the type in Fig. 4.

It is clear, however, that curves of the looped type, intermediate between Figs. 1 and 2, can be of no interest in the Foucault problem: for if $\dot\psi$ is negative at the outer circle, but is numerically less than n (so as to make h positive), we should have $\sqrt{(\beta/\alpha)}$ less than n/p. That is, the radius of the inner circle would be less than about $\frac{1}{7000}$ of the radius of the outer, and would accordingly be too small to be observed. It seemed therefore unnecessary to draw any such curves, although they form a convenient stage in describing the passage from Fig. 1 to Fig. 4. On the other hand, with a gyrostatic pendulum, curves of this type may easily appear; and one (Fig. 103d) will be found amongst the reproductions given in Webster's *Dynamics* (see p. 232 above).

Numerical Estimate of the Value of n/p for the Gyrostatic Pendulum.

To obtain some idea of the relative magnitudes of n and p under ordinary conditions of spinning, we may consider first the apparatus used by Sir George Greenhill. This essentially consists of a bicycle-wheel mounted on a stalk whose length is approximately equal to the

* It will be seen that this agrees with a formula found by elliptic-function methods by Sir George Greenhill (*Elliptic Functions*, p. 340). To make the identity clear we should note that in the article quoted $G/(Cr)$ is equal to the constant $(h-1)$ defined in (34) above, so that

$$\tfrac{3}{4}-\frac{1}{h} = \tfrac{3}{4}-\frac{Cr}{G+Cr} = \tfrac{1}{4}+\tfrac{1}{2}\,\frac{G-Cr}{G+Cr},$$

and that the product denoted there by $(\cot\tfrac{1}{2}\alpha\cot\tfrac{1}{2}\beta)$ is equal to $\sqrt{(\tfrac{1}{2}\alpha)}\sqrt{(\tfrac{1}{2}\beta)} = \tfrac{1}{2}\sqrt{(\alpha\beta)}$ in the present notation.

radius of the wheel. We shall get some idea of the constants of the apparatus, by treating the mass as concentrated round the rim of the wheel: if the radius is then supposed to be equal to l, we find

$$A = \tfrac{3}{2}Ml^2, \qquad C = Ml^2,$$

and so

$$n = \tfrac{1}{3}\omega, \qquad p^2 = \tfrac{2}{3}(g/l).$$

Hence

$$\left(\frac{n}{p}\right)^2 = \frac{1}{6}\,\frac{l\omega^2}{g} = \frac{1}{6}\,\frac{v^2}{gl},$$

where $v = l\omega$ is the velocity of a point on the rim of the wheel. If, on the other hand, the wheel is regarded as a uniform circular disc, the values are

$$A = \tfrac{5}{4}Ml^2, \qquad C = \tfrac{1}{2}Ml^2,$$

and then

$$n = \tfrac{1}{5}\omega, \qquad p^2 = \tfrac{4}{5}(g/l),$$

giving

$$\left(\frac{n}{p}\right)^2 = \frac{1}{20}\,\frac{v^2}{gl}.$$

The true value of n/p is intermediate between these two estimates; and it is probably closer to the former on account of the lightness of the spokes in the wheel.

Thus to make $n = p$, we must suppose on the first hypothesis $v^2 = 6gl$ and on the second $v^2 = 20gl$. Or, taking the diameter of the wheel to be 30 inches, we put $l = 1{\cdot}25$, $g = 32$, in foot-second units; and so $v = 15{\cdot}5$ or $28{\cdot}3$, respectively. That is, in round figures, the assumption $n = p$ leads to a rim-velocity of from 10 to 20 miles an hour, according to the assumption as to the distribution of the mass in the wheel.

Thus, with ordinary means of spinning the wheel, it is hardly likely that the value of n/p can exceed 2, since n/p is directly proportional to the rim-velocity of the wheel.

Secondly, we may consider the Gray-Burnside motor gyrostat* which is capable of producing very high angular velocities. In this apparatus M is about 6·5 lb. and the spinning part is a nearly uniform disc of mass 4 lb. and radius about 3 inches. On the other hand, l is about 3 feet, so that A may be taken as practically equal to Ml^2. Then we find

$$\left(\frac{n}{p}\right)^2 = \frac{1}{4}\left(\frac{C\omega}{A}\right)^2 \frac{A}{Mgl} = \frac{C^2\omega^2}{4M^2gl^3}.$$

Now with the given dimensions we can take approximately

$$\frac{C}{M} = \frac{4}{6{\cdot}5}\,\frac{1}{2}\left(\frac{1}{4}\right)^2 = \frac{1}{50}, \text{ say,}$$

when the foot is unit.

Thus the condition $n = p$ is found to give (when l is 3 feet)

$$\omega = 2900;$$

that is, a speed of say 450 revolutions per second, or 27000 revolutions per minute: and this is rather more than the highest speed attained in the gyrostat. On the other hand, if l were reduced to 2 feet, the value of ω corresponding to $n = p$ would be about 1600; and so, when working at the highest speed, the gyrostat might reach the value $n/p = 2$.

Thus with either Sir George Greenhill's apparatus or the motor-gyrostat, the value 2 may be taken as an upper limit to n/p.

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* *Proc. Roy. Soc., Edin.*, Vol. 32, 1912, p. 324.

THE GREEN'S FUNCTION FOR THE EQUATION $\nabla^2 u + k^2 u = 0$.

By H. S. CARSLAW.

[Received April 28th, 1913.—Read May 8th, 1913.]

1. About a year ago I made a preliminary communication* to the Society regarding an application of the theory of integral equations to the equation $\nabla^2 u + k^2 u = 0$. I pointed out that if $K(x, y, z : x', y', z')$ is the solution of this equation, which vanishes at the boundary of the closed surface S, and is finite and continuous, as also its first and second differential coefficients inside S, except at the point (x', y', z') where it becomes infinite as $e^{-ikr}/4\pi r$, when $r \to 0$, then $K(x, y, z : x', y', z')$ is the kernel of a homogeneous integral equation.†

Indeed if ψ‡ is a solution of

$$\nabla^2 \psi + (\lambda + k^2)\psi = 0,$$

which vanishes at the surface of S, and is finite and continuous, as also its first and second differential coefficients, right through the interior of S, we find

$$\psi(x', y', z') = \lambda \iiint K(x, y, z : x', y', z')\, \psi(x, y, z)\, dx\, dy\, dz, \tag{1}$$

the integration being taken through the interior of S.

This result follows immediately from Green's theorem, applied to the region between S and a small sphere Σ enclosing the point (x', y', z'), the

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* *Records of Proceedings at Meetings* (April 11th, 1912).

† The usual modification has to be made for the two-dimensional problem; the function is to be infinite as $-\frac{1}{2\pi}\log r$, when $r \to 0$.

‡ The value $\lambda = 0$ is excluded since we assume that k^2 does not belong to one of the normal modes of vibration.

radius of Σ tending to zero: for we have

$$\iiint (u\nabla^2\psi - \psi\nabla^2 u)\, dx\, dy\, dz$$
$$= \iint \left(u\frac{\partial\psi}{\partial n} - \psi\frac{\partial u}{\partial n}\right) dS - \iint \left(u\frac{\partial\psi}{\partial n} - \psi\frac{\partial u}{\partial n}\right) d\Sigma.$$

Denoting the point (x, y, z) by (0) and the point (x', y', z') by (1), equation (1) may be written

$$\psi(1) = \lambda \iiint K(0, 1)\, \psi(0)\, dv. \tag{2}$$

With this notation $K(0, 1)$ is the Green's function for the region bounded by S, for this surface condition $(u = 0)$; and it is a symmetrical function of (x, y, z) and (x', y', z').*

In other words $\qquad K(0, 1) = K(1, 0),$

and the kernel of the integral equation is a symmetrical kernel.

Now, if we assume that the Green's function can be expanded in an infinite series

$$\Sigma A_m \psi_m(0),$$

and that this series can be integrated term by term, we have

$$A_m = \iiint K(0, 1)\, \psi_m(0)\, dv,$$

provided the functions ψ_m are normalised: that is, provided

$$\iiint \psi_m^2(0)\, dv = 1.$$

Thus, from (2), we would have

$$K(0, 1) = \Sigma \frac{\psi_m(0)\, \psi_m(1)}{\lambda_m}, \tag{3}$$

the summation extending over all the values of λ.

Further, if it can be proved that the series

$$\Sigma \frac{\psi_m(0)\, \psi_m(1)}{\lambda_m}$$

is uniformly convergent within S, a fundamental theorem in the theory of integral equations allows us to equate $K(0, 1)$ to the sum of the series.

From a paper by Sommerfeld† which has just appeared, I learn that he had called attention to the presence of this Green's function as the

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* Cf. Kneser, *Die Integralgleichungen und ihre Anwendungen in der Mathematischen Physik*, § 31.

† *Jahresbericht der Deutschen Mathematiker-Vereinigung*, Bd. 21, p. 309 (1913).

kernel of an integral equation in a lecture in 1910. In the first part of this paper he discusses the case of

(i) a straight line of length l,

(ii) a circular cylinder of radius a,

and (iii) a rectangular parallelopiped.

In this paper I propose to give the Green's functions for several other cases. I shall then show that each of them reduces to the form

$$\Sigma \frac{\psi_m(0)\, \psi_m(1)}{\lambda_m},$$

suggested by the integral equation (2), so that the assumptions mentioned above need not be made.

As I have proved elsewhere,* the characteristic numbers λ for this Green's function are such that $\lambda + k^2 > 0$. The function has therefore the character of "the functions of positive type" discussed by Mercer,† except that it becomes infinite as $e^{-ikr}/4\pi r$, where $r \to 0$, at a point within the region. It is thus of some interest to establish the expansion of the Green's function in the form desired,‡ even if the proof does not contain a discussion of the uniform convergence of the series, the condition attached to the theorem in the theory of integral equations. It seems most likely that Mercer's theorem holds for functions of positive type, extended as above.

It will be noticed that the existence of the Green's function is established analytically in this paper and not assumed.

Some Theorems regarding Bessel's Functions and Spherical Harmonics.

2. THEOREM I.—*If n is a given positive number, and ρ is any positive root of the equation $J_n(\rho a) = 0$, while k^2 is not equal to ρ^2, then*

$$\sum_\rho \frac{J_n(\rho r)\, J_n(\rho r')}{(\rho^2 - k^2)\int_0^a r J_n^2(\rho r)\, dr} = e^{\frac{1}{2}n i \pi} \frac{J_n(kr')}{J_n(ka)} \left[J_n(ka)\, K_n(ikr) - J_n(kr)\, K_n(ika)\right],$$

for $0 < r' < r < a$.

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* *Messenger of Mathematics*, Vol. 42, p. 135 (1913).

† *Phil. Trans.* (A), Vol. 209, p. 415 (1909).

‡ Reference may also be made to a paper by Abraham ("Über einige, bei Schwingungsproblemen auftretende, Differentialgleichungen"), *Math. Ann.*, Bd. LII, p. 81 (1901), where he makes use of the hypothesis that every possible type of vibration can be expressed as a sum of the normal functions of the regions concerned.

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THEOREM IV.—*If m is zero or a positive integer*

$$(1-\mu^2)\left[P_n^{-m}(\mu)\frac{d}{d\mu}P_n^{-m}(-\mu)-P_n^{-m}(-\mu)\frac{d}{d\mu}P_n^{-m}(\mu)\right]$$
$$=\frac{2}{\Pi(m+n)\Pi(m-n-1)}.$$

It follows from the differential equation

$$\frac{d}{d\mu}(1-\mu^2)\frac{du}{d\mu}+\left(n(n+1)-\frac{m^2}{1-\mu^2}\right)u=0,$$

that the left hand side is independent of μ.

Also it is known* that

$$P_n^{-m}(-\mu)=P_n^{-m}(\mu)\cos(n-m)\pi-\frac{2}{\pi}\sin(n-m)\pi\, Q_n^{-m}(\mu). \tag{6}$$

Thus we have

$$(1-\mu^2)\left\{P_n^{-m}(\mu)\frac{d}{d\mu}P_n^{-m}(-\mu)-P_n^{-m}(-\mu)\frac{d}{d\mu}P_n^{-m}(\mu)\right\}$$
$$=\frac{2}{\pi}\sin(n-m)\pi(1-\mu^2)\left\{Q_n^{-m}(\mu)\frac{d}{d\mu}P_n^{-m}(\mu)-P_n^{-m}(\mu)\frac{d}{d\mu}Q_n^{-m}(\mu)\right\}. \tag{7}$$

But when μ is real and lies between -1 and $+1$, and m is zero or a positive integer, we have

$$Q_n^{-m}(\mu)=\frac{\Pi(n-m)}{\Pi(n+m)}\cos m\pi\, Q_n^m(\mu).$$

If now we use the expression found by Macdonald for $Q_n^m(\mu)$,† and take the limit of (7) when $\mu\to 1$, it will be seen that the limit of the right-hand side is

$$\frac{2}{\Pi(m+n)\Pi(m-n-1)}.$$

The case $m=0$ can be deduced from the above, or evaluated independently. The expression for $Q_n(\mu)$ is not a complicated one, and the limit follows more easily than in the case of m a positive integer.

Since $$\Pi(n)\Pi(-n-1)=-\frac{\pi}{\sin n\pi},$$

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* Cf. Hobson, *loc. cit.*, p. 473, equation (33).

† *Proc. London Math. Soc.*, Ser. 1, Vol. XXXI, p. 274 (1899).

we have

$$(1-\mu^2)\left\{P_n(\mu)\frac{d}{d\mu}P_n(-\mu)-P_n(-\mu)\frac{d}{d\mu}P_n(\mu)\right\} = -\frac{2\sin n\pi}{\pi}.*$$

The Green's Function for the Region Bounded by the Cylinder $r = a$ and the Planes $\theta = 0$, $\theta = \alpha$.

Line Source at (r', θ').

We require the solution of the equation

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+k^2 u = 0, \tag{8}$$

described in § 1, which vanishes at $r = a$, $\theta = 0$ and $\theta = \alpha$, and is infinite as $-\frac{1}{2\pi}\log x$, when $x \to 0$, at the point (r', θ').

We know† that when $r > r'$,

$$K_0(ikR) = K_0(ikr)\,J_0(kr')+2\sum_1^\infty e^{\frac{1}{2}ni\pi}\,K_n(ikr)\,J_n(kr')\cos n(\theta-\theta'),$$

and that when $r < r'$,

$$K_0(ikR) = K_0(ikr')\,J_0(kr)+2\sum_1^\infty e^{\frac{1}{2}ni\pi}\,K_n(ikr')\,J_n(kr)\cos n(\theta-\theta'),$$

where $$R = \sqrt{[r^2+r'^2-2rr'\cos(\theta-\theta')]}.$$

Also when $x \to 0$, we know‡ that $K_0(ix)$ tends to $-\log x$, so that

$$\frac{1}{2\pi}K_0(ikR)$$

has the proper value at (r', θ').

---

* Dougall [*Proc. Edin. Math. Soc.*, Vol. XVIII, p. 49 (1900)] proves Theorem IV, but his argument holds only for m not integral. In that case there is no need to introduce the function $Q_n^m(\mu)$, and the equation

$$\Pi(n-m)\sin(n-m)\pi P_n^m(\mu) = \Pi(n+m)\left\{\sin n\pi\, P_n^{-m}(\mu)-\sin m\pi\, P_n^{-m}(-\mu)\right\},$$

is used instead of (6).

† Macdonald, *Proc. London Math. Soc.*, Ser. 1, Vol. XXXII, p. 157 (1900).

‡ Cf. Nielsen, *Handbuch der Theorie der Cylinderfunktionen*, p. 12 (9), and p. 16 (1).

For the complete cylinder it is clear that the solution is given by

$$u = \frac{1}{2\pi} \frac{J_0(kr')}{J_0(ka)} [J_0(ka)\, K_0(ikr) - J_0(kr)\, K_0(ika)]$$

$$+ \frac{1}{\pi} \Sigma^{\frac{1}{2}n i\pi} . e \frac{J_n(kr')}{J_n(ka)} [J_n(ka)\, K_n(ikr) - J_n(kr)\, K_n(ika)] \cos n(\theta - \theta'),$$

when $r > r'$. Interchange r, r' for $r < r'$.

The terms added to $\frac{1}{2\pi} K_0(ikr)$ form a convergent series right through the cylinder and cause the function to vanish at $r = a$.

As Sommerfeld points out in his paper referred to above,* this result reduces by the aid of § 2, Theorem I, which he proves for n a positive integer, to

$$u = \sum_\rho \sum_n \frac{J_n(\rho r)\, J_n(\rho r')}{\pi(\rho^2 - k^2) \int_0^a r J_n^2(\rho r)\, dr} \cos n(\theta - \theta').$$

In this summation n is to be zero or any positive integer: and ρ is to be any positive root of

$$J_n(\rho a) = 0.$$

For $n = 0$, we have to replace π by 2π.

This is, in fact, the series [cf. § 1 (3)]

$$\Sigma \frac{\psi(1)\, \psi(0)}{\lambda},$$

where the ψ's are the normalised characteristic functions for this region.

3. The method which is followed in this paper consists in expressing $K_0(ikR)$, or the corresponding solution for three dimensions, as a contour integral, and then adding to this integral terms which satisfy the differential equation

$$\nabla^2 u + k^2 u = 0,$$

and the surface condition, while they introduce no new singularity. Finally the result is obtained in the form of an infinite series by the use of Cauchy's theorem of residues.

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* *Loc. cit.*, p. 320.

Consider the integral

$$\frac{1}{2i\pi}\int \frac{\cos n(\pi-\theta+\theta')}{\sin n\pi}\, e^{\frac{1}{2}ni\pi} K_n(ikr)\, J_n(kr')\, dn$$

$$(r > r',\ 0 < \theta' < \theta < 2\pi),$$

over the path A given in Fig. 1 in the n-plane.

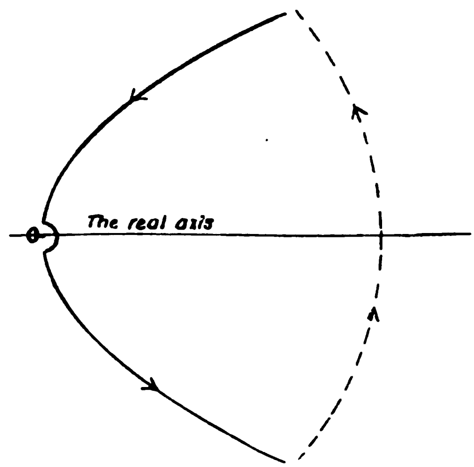

FIG. 1.—The path A in the n-plane.

When $|n|$ is large and its real part positive, the approximate value of $e^{\frac{1}{2}ni\pi} K_n(ikr)\, J_n(kr')$ is easily seen to be* $\frac{1}{2n}\left(\frac{r'}{r}\right)^n$. Using this result and the exponential forms for the sine and cosine, it will be found that the integral over the dotted portion of the path in Fig. 1 vanishes, this part being supposed to represent an arc of a circle of infinite radius joining up the ends of the remainder of the circuit.

Further, the semicircle at the origin in Fig. 1 contributes

$$-\frac{1}{2\pi} K_0(ikr)\, J_0(kr'),$$

to the integral in the limit.

In the path A' of Fig. 2, the dotted portion of the path A of Fig. 1 and the small semicircle at the origin are omitted, the path just stopping short of the origin on each side.

* Cf. Nielsen, *loc. cit.*, p. 7 (3), and p. 11 (8).

It follows from what has been said above that

$$\frac{1}{2i\pi}\int_{A'} \frac{\cos n(\pi-\theta+\theta')}{\sin n\pi} e^{\frac{1}{2}ni\pi} K_n(ikr)\, J_n(kr')\, dn$$
$$(r > r',\ 0 < \theta' < \theta < 2\pi),$$

over the path A',

$$= \frac{1}{2\pi}\left(K_0(ikr)\, J_0(kr') + 2\sum_1^\infty e^{\frac{1}{2}ni\pi} K_n(ikr)\, J_n(kr') \cos n(\theta-\theta')\right),$$

using the theorem of residues.

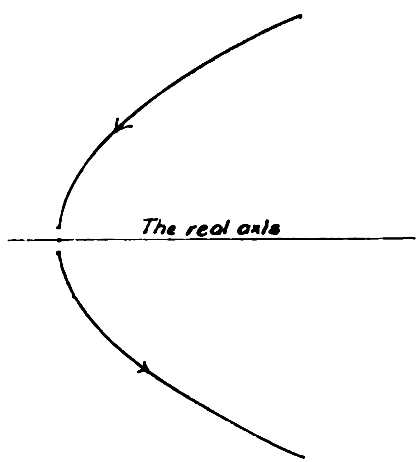

FIG. 2.—The path A' in the n-plane.

When $\theta < \theta'$ or $r < r'$, we have to interchange (θ, θ') or (r, r') in the above.

We have thus found an expression for $K_0(ikR)$ as a contour integral in the n-plane, which holds except when both $r-r'$ and $\theta-\theta'$ vanish.

Writing u_0 for $\frac{1}{2\pi} K_0(ikR)$, we have

$$u_0 = \frac{1}{2i\pi}\int_{A'} \frac{\cos n(\pi-\theta+\theta')}{\sin n\pi} e^{\frac{1}{2}ni\pi} K_n(ikr)\, J_n(kr')\, dn$$
$$(r > r',\ 0 < \theta' < \theta < 2\pi),$$

the integral being taken over the path A' of Fig. 2.*

4. We shall now show that by adding certain terms to this expression, we obtain the required Green's function in the form of a contour integral.

* Cf. Carslaw, *Phil. Mag.* (6), Vol. v, p. 377 (1903).

Consider the following integral over the path A':—

$$u = \frac{1}{2i\pi}$$

$$\times \int_{A'} \frac{\cos n(\pi-\theta+\theta') - \cos n(\pi-a+\theta')\dfrac{\sin n\theta}{\sin na} - \cos n(\pi-\theta')\dfrac{\sin n(a-\theta)}{\sin na}}{\sin n\pi}$$

$$\times e^{\frac{1}{2}ni\pi}\frac{J_n(kr')}{J_n(ka)}\left[J_n(ka)\,K_n(ikr) - J_n(kr)\,K_n(ika)\right] dn$$

for $$r > r', \quad 0 < \theta' < \theta < a < 2\pi. \tag{9}$$

It will be found that the integrals added to u_0 are each convergent, so that no new singularity enters, and that this convergence holds even when $r-r'$ and $\theta-\theta'$ vanish.

Also the elements of these integrals all satisfy the differential equation (8), so that the integrals themselves satisfy it.

Further, it is clear that the terms which have been introduced have been chosen so as to cause u to vanish at $r = a$, $\theta = 0$, and $\theta = a$. For the case $\theta = 0$, it must be remembered that we have to choose the form for u in which $\theta < \theta'$.

It follows that the expression given in (9) is the Green's function of the problem.

The trigonometrical terms can be simplified, and we obtain, finally,

$$u = \frac{i}{\pi}\int_{A'} \frac{\sin n\theta' \sin n(a-\theta)}{\sin na}\, e^{\frac{1}{2}ni\pi}\frac{J_n(kr')}{J_n(ka)}\left[J_n(ka)\,K_n(ikr) - J_n(kr)K_n(ika)\right] dn$$

for $$\theta > \theta', \quad r > r'. \tag{10}$$

The origin is no longer a pole of the integrand and the path A' can now be extended to the origin, forming a complete curve, instead of a curve broken at that point.

5. The solution given in (10) will now be expressed as an infinite series.

In Fig. 3 the path A' is completed at infinity by the dotted portion of the figure, and it is also taken right through the origin as described above.

The integral of (10) vanishes over the dotted portion of this figure, as has been already pointed out, when the proper form of the expression is taken, according as $\theta \gtrless \theta'$ and $r \gtrless r'$.

The poles inside the complete circuit of Fig. 3 are given by

$$\frac{\pi}{a},\quad \frac{2\pi}{a},\quad \frac{3\pi}{a},\quad \ldots,$$

from sin na, and by the zeroes of $J_n(ka)$, regarded as a function of n.

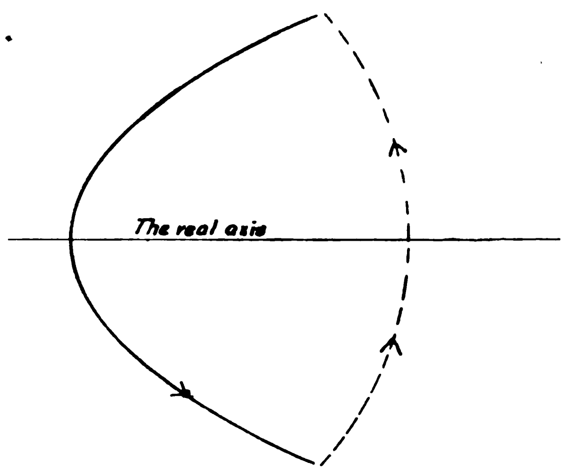

FIG. 3.

By § 2, Theorem II, it is known that these zeroes are finite in number and that they are all real and simple.

We can thus omit the terms which correspond to these values of n from our Green's function, since they vanish at the surface and their sum is finite within the region with which we are dealing.

It follows that, omitting these terms, the expression for u in (10) can be replaced by

$$u = -\frac{2}{a}\sum_1^\infty \frac{\sin\frac{n\pi}{a}\theta' \sin\frac{n\pi}{a}(a-\theta)}{\cos n\pi} e^{\frac{1}{2}(ni\pi^2/a)} \frac{J_{n\pi/a}(kr')}{J_{n\pi/a}(ka)}$$

$$\times\left[J_{n\pi/a}(ka)\, K_{n\pi/a}(ikr) - J_{n\pi/a}(kr)\, K_{n\pi/a}(ika)\right] \quad (r > r',\ \theta > \theta')$$

$$= \frac{2}{a}\sum_1^\infty \sin\frac{n\pi}{a}\theta \sin\frac{n\pi}{a}\theta'\, e^{\frac{1}{2}(ni\pi^2/a)} \frac{J_{n\pi/a}(kr')}{J_{n\pi/a}(ka)}$$

$$\times\left[J_{n\pi/a}(ka)\, K_{n\pi/a}(ikr) - J_{n\pi/a}(kr)\, K_{n\pi/a}(ika)\right], \tag{11}$$

and this is the Green's function for this region.

It will be noticed that in this form the series is symmetrical with regard to θ and θ', and that the only alteration necessary is to interchange r and r', when $r < r'$.

6. The normalised characteristic functions for this region are given by

$$\psi = A_{\rho n} J_{n\pi/a}(\rho r) \sin \frac{n\pi}{a} \theta,$$

where
$$A_{\rho n}^2 = \frac{2}{a} \frac{1}{\int_0^a r J_{n\pi/a}^2 (\rho r)\, dr},$$

n is any positive integer, and ρ is any positive root of

$$J_{n\pi/a}(\rho a) = 0.$$

Thus the series
$$\Sigma \frac{\psi(1)\,\psi(0)}{\lambda}$$

in this case becomes

$$\frac{2}{a} \sum_{\rho} \sum_{n} \frac{J_{n\pi/a}(\rho r)\, J_{n\pi/a}(\rho r')}{(\rho^2 - k^2) \int_0^a r J_{n\pi/a}^2 (\rho r)\, dr} \sin \frac{n\pi}{a} \theta \sin \frac{n\pi}{a} \theta'.$$

If we sum first with regard to ρ, this series reduces to the Green's function given in (11) by the help of § 2, Theorem I.

The Green's Function for the Sphere $r = a$.

Source at (r', θ', ϕ').

7. The equation $\nabla^2 u + k^2 u = 0$ in spherical polar coordinates becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \mu} (1-\mu^2) \frac{\partial u}{\partial \mu} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} + k^2 u = 0. \qquad (12)$$

If R stands for the distance between the points (r, θ, ϕ) and (r', θ', ϕ'), we know* that with the usual notation

$$\frac{e^{-ikR}}{R} = \frac{2}{\sqrt{(rr')}} e^{\frac{1}{2}i\pi} \sum_0^\infty e^{\frac{1}{2}ni\pi} (n+\tfrac{1}{2})\, K_{n+\frac{1}{2}}(ikr)\, J_{n+\frac{1}{2}}(kr')\, P_n(\cos \gamma) \quad (r > r')$$

$$= \frac{2}{\sqrt{(rr')}} e^{\frac{1}{2}i\pi} \sum_0^\infty e^{\frac{1}{2}ni\pi} (n+\tfrac{1}{2})\, K_{n+\frac{1}{2}}(ikr')\, J_{n+\frac{1}{2}}(kr)\, P_n(\cos \gamma) \quad (r < r').$$

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* Macdonald, *Proc. London Math. Soc.*, Ser. 1, Vol. XXXII, p. 157 (1900).

It follows at once that the Green's function for the sphere $r = a$ is given by

$$u = \frac{1}{2\pi\sqrt{(rr')}} e^{\frac{1}{2}i\pi} \sum_0^\infty e^{\frac{1}{2}ni\pi} (n+\tfrac{1}{2}) \frac{J_{n+\frac{1}{2}}(kr')}{J_{n+\frac{1}{2}}(ka)}$$

$$\times \left[J_{n+\frac{1}{2}}(ka)\, K_{n+\frac{1}{2}}(ikr) - J_{n+\frac{1}{2}}(kr)\, K_{n+\frac{1}{2}}(ika)\right] P_n(\cos\gamma) \quad (r > r'),$$

since the terms added to $\dfrac{e^{-ikR}}{4\pi R}$ satisfy equation (12), and they form a series which converges right through the sphere.

Interchange (r, r') for $0 < r < r'$.

8. It will be seen that the normalised characteristic functions for the case of the sphere are given by

$$\psi_{\rho mn} = A_{\rho mn} \frac{J_{n+\frac{1}{2}}(\rho r)}{\sqrt{r}} P_n^{-m}(\mu) \begin{matrix}\cos\\ \sin\end{matrix} m\phi,$$

where m, n are positive integers or zero, and ρ is any positive root of

$$J_{n+\frac{1}{2}}(\rho a) = 0.$$

Also
$$A_{\rho mn} = \frac{1}{\pi \int_0^a r J^2_{n+\frac{1}{2}}(\rho r)\, dr \int_{-1}^1 [P_n^{-m}(\mu)]^2\, d\mu},$$

except when $m = 0$, in which case 2π takes the place of π.

Thus the series
$$\Sigma \frac{\psi(1)\,\psi(0)}{\lambda}$$

becomes
$$\frac{1}{2\pi\sqrt{(rr')}} \sum_\rho \sum_n \frac{J_{n+\frac{1}{2}}(\rho r)\, J_{n+\frac{1}{2}}(\rho r')}{\rho^2 - k^2} P_n(\cos\gamma).$$

Summing with regard to ρ, it follows from § 2, Theorem I, that this expression is the same as that found in § 7 for the Green's function.

The Green's Function for the Region bounded by the Sphere $r = a$ and the Cone $\theta = \theta_0$.

Source at a Point on the Axis.

9. When the source is at a point $(r', 0)$ on the axis of the cone, the disturbance is symmetrical about that axis and u does not involve ϕ.

We start, as in the case of the sphere, with

$$u_0 = \frac{1}{4\pi}\frac{e^{-ikR}}{R} \quad (\text{where } R^2 = r^2 + r'^2 - 2rr'\cos\theta)$$

$$= \frac{1}{2\pi\sqrt{(rr')}} e^{\frac{1}{4}i\pi} \sum_0^\infty e^{\frac{1}{2}ni\pi}(n+\tfrac{1}{2}) K_{n+\frac{1}{2}}(ikr) J_{n+\frac{1}{2}}(kr') P_n(\mu) \quad (r > r')$$

$$= \frac{1}{4\pi\sqrt{(rr')}} e^{-\frac{1}{4}i\pi} \int_B e^{\frac{1}{2}ni\pi} \frac{(n+\frac{1}{2}) K_{n+\frac{1}{2}}(ikr) J_{n+\frac{1}{2}}(kr') P_n(-\mu)}{\sin n\pi} dn, \quad (13)$$

over the path B of Fig. 4 in the n-plane.

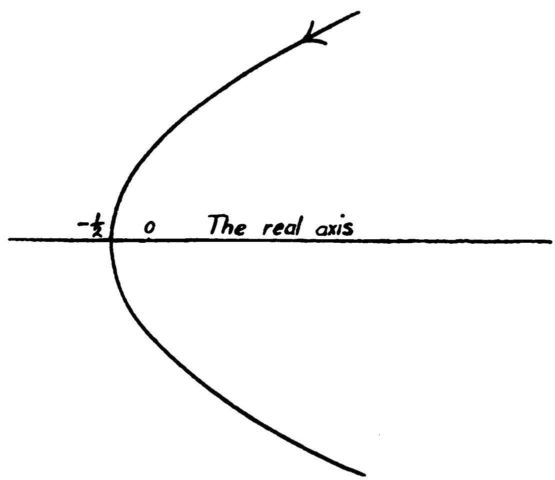

FIG. 4.—The path B in the n-plane.

This transformation follows in the same way as that of § 3 for $K_0(ikR)$. It will be found that the path can be completed at infinity without adding to the integral, and the poles are given by $n = 0, 1, 2, \ldots$. It has to be noticed that the real part of n must not be less than $-\frac{1}{2}$. Also for n a positive integer

$$P_n(-\mu) = (-1)^n P_n(\mu).$$

10. The Green's function is now obtained by adding suitable terms to this expression for u_0, and we are brought to the solution

$$u = \frac{1}{4\pi\sqrt{(rr')}} e^{-\frac{1}{4}i\pi} \int_B e^{\frac{1}{2}ni\pi} \frac{(n+\frac{1}{2}) J_{n+\frac{1}{2}}(kr')}{\sin n\pi J_{n+\frac{1}{2}}(ka)}$$
$$\times [K_{n+\frac{1}{2}}(ikr) J_{n+\frac{1}{2}}(ka) - J_{n+\frac{1}{2}}(kr) K_{n+\frac{1}{2}}(ika)]$$
$$\times \frac{P_n(-\mu) P_n(\mu_0) - P_n(-\mu_0) P_n(\mu)}{P_n(\mu_0)} dn \quad (r > r'). \quad (14)$$

taken over the path B of Fig. 4.

To prove this, the following remarks are sufficient :—

From the approximate values for the Bessel's functions and spherical harmonics when $|n|$ is very great, it will be seen that the integrals in u, which have been added to u_0, are convergent, so that no new singularity enters; and the convergence holds also at (r', θ').

The elements of these integrals all satisfy the differential equation (12), so that the integrals themselves satisfy it.

Further, it is clear that the terms have been introduced so as to cause it to vanish at $r = a$ and $\theta = \theta_0$.

11. Now the path B of Fig. 4 can be completed at infinity and the new portion contributes nothing to the integral.

We are thus able to change the expression of (14) into an infinite series by the theorem of residues.

But it is known that the zeroes of $P_n(\mu_0)$,* regarded as a function of n, are infinite in number and all real and simple.

Also it is known† that the zeroes of $J_{n+\frac{1}{2}}(ka)$, regarded as a function of n, are all real and finite in number.

These latter can be omitted, as in § 5, from our result, and the Green's function can be written down from (14) in the following form :—

$$u = -\frac{i}{2\pi\sqrt{(rr')}} e^{-\frac{1}{4}i\pi} \sum_n e^{\frac{1}{2}ni\pi} \frac{(n+\frac{1}{2}) J_{n+\frac{1}{2}}(kr')}{\sin n\pi \, J_{n+\frac{1}{2}}(ka)}$$

$$\times [K_{n+\frac{1}{2}}(ikr) J_{n+\frac{1}{2}}(ka) - J_{n+\frac{1}{2}}(kr) K_{n+\frac{1}{2}}(ika)] \frac{P_n(-\mu_0) P_n(\mu)}{\frac{d}{dn} P_n(\mu_0)} \quad (r > r'). \quad (15)$$

The summation is taken over all the zeroes of

$$P_n(\mu_0)$$

which are greater than $-\frac{1}{2}$.‡

Now it is known that

$$(1-\mu_0^2) P_n(-\mu_0) \frac{d}{d\mu_0} P_n(\mu_0) = \frac{2}{\pi} \sin n\pi \quad \text{[Cf. § 2, Theorem IV].}$$

* Macdonald, *Proc. London Math. Soc.*, Ser. 1, Vol. XXXI, p. 264 (1899).

† Cf. § 2, Theorem II.

‡ Since $P_n = P_{-n-1}$, this includes all the zeroes of $P_n(\mu_0)$.

Therefore our result can be written

$$u = -\frac{1}{\pi\sqrt{(rr')}} \sum_n e^{\frac{1}{2}(n+\frac{1}{2})i\pi} (n+\tfrac{1}{2}) \frac{J_{n+\frac{1}{2}}(kr')}{J_{n+\frac{1}{2}}(ka)}$$

$$\times [K_{n+\frac{1}{2}}(ikr)\, J_{n+\frac{1}{2}}(ka) - J_{n+\frac{1}{2}}(kr)\, K_{n+\frac{1}{2}}(ika)]$$

$$\times \frac{P_n(\mu)}{(1-\mu_0^2)\dfrac{d}{d\mu_0}P_n(\mu_0)\dfrac{d}{dn}P_n(\mu_0)} \quad (r > r'). \tag{16}$$

12. The normalised characteristic functions for this case are

$$\psi = A_{\rho n} \frac{J_{n+\frac{1}{2}}(\rho r)}{\sqrt{r}} P_n(\mu),$$

where n is a zero greater than $-\frac{1}{2}$ of $P_n(\mu_0)$, and ρ is a positive root of

$$J_{n+\frac{1}{2}}(\rho a) = 0,$$

while $$A_{\rho n}^2 = \frac{1}{2\pi \int_0^a r J_{n+\frac{1}{2}}^2(\rho r)\, dr \int_{\mu_0}^1 [P_n(\mu)]^2\, d\mu}.$$

Also $$P_n(\mu') = P_n(1) = 1.$$

Thus the series $$\Sigma \frac{\psi(1)\,\psi(0)}{\lambda}$$

becomes

$$\frac{1}{2\pi\sqrt{(rr')}} \sum_\rho \sum_n \frac{J_{n+\frac{1}{2}}(\rho r)\, J_{n+\frac{1}{2}}(\rho r')}{(\rho^2-k^2)\int_0^a r J_{n+\frac{1}{2}}^2(\rho r)\, dr} \frac{P_n(\mu)}{\int_{\mu_0}^1 [P_n(\mu)]^2\, d\mu}.$$

But, by § 2, Theorem III,

$$\int_{\mu_0}^1 [P_n(\mu)]^2\, d\mu = -\frac{(1-\mu_0^2)}{2n+1} \frac{d}{dn} P_n(\mu_0) \frac{d}{d\mu_0} P_n(\mu_0).$$

Therefore the above series can be written

$$-\frac{1}{\pi\sqrt{(rr')}} \sum_\rho \sum_n \frac{(n+\frac{1}{2})\, J_{n+\frac{1}{2}}(\rho r)\, J_{n+\frac{1}{2}}(\rho r')}{(\rho^2-k^2)\int_0^a r J_{n+\frac{1}{2}}^2(\rho r)\, dr} \frac{P_n(\mu)}{(1-\mu_0^2)\dfrac{d}{dn}P_n(\mu_0)\dfrac{d}{d\mu_0}P_n(\mu_0)}.$$

This reduces at once to (16) with the help of § 2, Theorem I.

The Green's Function for the Region Bounded by the Sphere $r = a$ and the Cone $\theta = \theta_0$.

Source at (r', θ', ϕ').

13. In this case we start as before with

$$u_0 = \frac{1}{4\pi} \frac{e^{-ikR}}{R} \quad (\text{where } R^2 = r^2 + r'^2 - 2rr' \cos\gamma)$$

$$= \frac{1}{2\pi\sqrt{(rr')}} e^{\frac{1}{4}i\pi} \sum_0^\infty e^{\frac{1}{2}ni\pi} (n+\tfrac{1}{2}) J_{n+\frac{1}{2}}(kr') K_{n+\frac{1}{2}}(ikr) P_n(\cos\gamma) \quad (r > r')$$

$$= \frac{1}{4\pi\sqrt{(rr')}} e^{-\frac{1}{4}i\pi} \int_B e^{\frac{1}{2}ni\pi} \frac{(n+\frac{1}{2})}{\sin n\pi} J_{n+\frac{1}{2}}(kr') K_{n+\frac{1}{2}}(ikr) P_n(-\cos\gamma)\, dn,$$

the integral being taken over the path B of Fig. 4.

But is is known* that

$$P_n(-\cos\gamma) = P_n(-\mu) P_n(\mu')$$
$$+ 2\sum_1^\infty \frac{\Pi(n+m)\Pi(m-n-1)}{\Pi(n)\Pi(-n-1)} P_n^{-m}(-\mu) P_n^{-m}(\mu') \cos m(\phi-\phi'),$$

for $\theta > \theta'$.

Further, when m is very large,

$$\frac{\Pi(n+m)\Pi(m-n-1)}{\Pi(n)\Pi(-n-1)} P_n^{-m}(-\mu) P_n^{-m}(\mu')$$

---

* Dr. Bromwich has pointed out to me that the addition theorem

$$P_n(\cos\gamma) = P_n(\mu) P_n(\mu') + 2\Sigma(-1)^m \frac{\Pi(n+m)\Pi(m-n-1)}{\Pi(n)\Pi(-n-1)} P_n^{-m}(\mu) P_n^{-m}(\mu') \cos m(\phi-\phi')$$
$$(0 < \theta + \theta' < \pi)$$

can be deduced from the definitions of the functions $P_n(z)$, $P_n^{-m}(z)$ as hypergeometric series, for all values of n, real or imaginary. If we put for θ, $\pi-\theta$, and for $\phi-\phi'$, $\pi-(\phi-\phi')$, the condition $0 < \theta+\theta' < \pi$ becomes $\theta' < \theta$, and we have

$$P_n(-\cos\gamma) = P_n(-\mu) P_n(\mu') + 2\Sigma P_n^{-m}(-\mu) P_n^{-m}(\mu') \frac{\Pi(n+m)\Pi(m-n-1)}{\Pi(n)\Pi(-n-1)} \cos m(\phi-\phi').$$

is approximately
$$-\frac{1}{m}\,\frac{\sin n\pi}{\pi}\left\{\frac{\tan\dfrac{\theta'}{2}}{\tan\dfrac{\theta}{2}}\right\}^m.$$

It follows that we can change the order of summation and integration, and we write

$$u_0=\frac{1}{4\pi\sqrt{(rr')}}\,e^{-\frac{1}{4}i\pi}\sum_0^\infty a_m\cos m(\phi-\phi')\int_B\frac{(n+\frac{1}{2})}{\sin n\pi}\,e^{\frac{1}{2}ni\pi}\,J_{n+\frac{1}{2}}(kr')\,K_{n+\frac{1}{2}}(ikr)$$
$$\times\frac{\Pi(n+m)\,\Pi(m-n-1)}{\Pi(n)\,\Pi(-n-1)}\,P_n^{-m}(-\mu)\,P_n^{-m}(\mu')\,dn\quad(r>r',\ \theta>\theta'),\quad(17)$$

where
$$a_0=1,\quad a_m=2\quad(m\geqslant 1).$$

14. We are thus led to the following expression for the Green's function in this region :—

$$u=\frac{1}{4\pi\sqrt{(rr')}}\,e^{-\frac{1}{4}i\pi}\sum_m a_m\cos m(\phi-\phi')\int_B\frac{(n+\frac{1}{2})}{\sin n\pi}\,e^{\frac{1}{2}ni\pi}RS\,dn$$
$$(r>r',\ \theta>\theta'),\quad(18)$$

where $R=\dfrac{J_{n+\frac{1}{2}}(kr')}{J_{n+\frac{1}{2}}(ka)}\left[J_{n+\frac{1}{2}}(ka)\,K_{n+\frac{1}{2}}(ikr)-J_{n+\frac{1}{2}}(kr)\,K_{n+\frac{1}{2}}(ika)\right],$

and

$$S=\frac{\Pi(n+m)\,\Pi(m-n-1)}{\Pi(n)\,\Pi(-n-1)}\,P_n^{-m}(\mu')\left[\frac{P_n^{-m}(-\mu)\,P_n^{-m}(\mu_0)-P_n^{-m}(-\mu_0)\,P_n^{-m}(\mu)}{P_n^{-m}(\mu_0)}\right].$$

The integral is to be taken over the path B of Fig. 4.

When account is taken of the approximate values of the Bessel's function and spherical harmonics, it will be found that the expression in (18) satisfies all the conditions of the problem. The suitable changes have to be made for $\theta<\theta'$ and $r<r'$.

15. The expression of (18) can be transformed as before into an infinite series: since the path B can be completed at infinity, and the integral over the new portion vanishes.

The terms due to the zeroes of $J_{n+\frac{1}{2}}(ka)$, regarded as a function of n, can be omitted, as in § 5, for they are finite in number and their sum will be finite.

We are thus left with the following series for the Green's function :—

$$u = -\frac{1}{2\sqrt{(rr')}} \sum_m a_m \cos m(\phi-\phi') \sum_n \frac{(n+\frac{1}{2})\, e^{\frac{1}{2}(n+\frac{1}{2})i\pi}}{\sin n\pi} \frac{J_{n+\frac{1}{2}}(kr')}{J_{n+\frac{1}{2}}(ka)}$$
$$\times \left[J_{n+\frac{1}{2}}(ka)\, K_{n+\frac{1}{2}}(ikr) - J_{n+\frac{1}{2}}(kr)\, K_{n+\frac{1}{2}}(ika)\right]$$
$$\times \frac{\Pi(n+m)\,\Pi(m+n-1)}{\Pi(n)\,\Pi(-n-1)} \frac{P_n^{-m}(\mu')\, P_n^{-m}(-\mu_0)\, P_n^{-m}(\mu)}{\frac{d}{dn} P_n^{-m}(\mu_0)}, \tag{19}$$

the summation with regard to n extending over all the zeroes of $P_n^{-m}(\mu_0)$ which are greater than $-\frac{1}{2}$. These are infinite in number and are all real.*

Since $$\Pi(n)\,\Pi(-n-1)\sin n\pi = -\pi,$$

and $$\Pi(n+m)\,\Pi(m-n-1)\, P_n^{-m}(-\mu_0) = -\frac{2}{(1-\mu_0^2)\frac{d}{d\mu_0} P_n^{-m}(\mu_0)}$$

[cf. § 2, Theorem IV],

we have, finally,

$$u = -\frac{1}{\pi\sqrt{(rr')}} \sum_m a_m \cos m(\phi-\phi') \sum_n (n+\tfrac{1}{2})\, e^{\frac{1}{2}(n+\frac{1}{2})i\pi} \frac{J_{n+\frac{1}{2}}(kr')}{J_{n+\frac{1}{2}}(ka)}$$
$$\times \left[J_{n+\frac{1}{2}}(ka)\, K_{n+\frac{1}{2}}(ikr) - J_{n+\frac{1}{2}}(kr)\, K_{n+\frac{1}{2}}(ika)\right]$$
$$\times \frac{P_n^{-m}(\mu)\, P_n^{-m}(\mu')}{(1-\mu_0^2)\frac{d}{dn} P_n^{-m}(\mu_0)\frac{d}{d\mu_0} P_n^{-m}(\mu_0)}, \tag{20}$$

for $r > r'$ and $\theta > \theta'$.

16. The normalised characteristic functions for this region are given by
$$\psi_{\rho mn} = A_{\rho mn} \frac{J_{n+\frac{1}{2}}(\rho r)}{\sqrt{r}} P_n^{-m}(\mu) \begin{matrix}\cos\\ \sin\end{matrix} m\phi,$$

where m is zero or a positive integer : n is any zero greater than $-\frac{1}{2}$ of $P_n^{-m}(\mu_0)$, regarded as a function of n : ρ is any positive root of
$$J_{n+\frac{1}{2}}(\rho a) = 0\,;$$

* Cf. Macdonald, *Proc. London Math. Soc.*, Ser. 1, Vol. XXXI, p. 264 (1899).

and
$$A^2_{pmn} = \frac{1}{\pi \int_0^a r J^2_{n+\frac{1}{2}}(\rho r)\, dr \int_{\mu_0}^1 [P_n^{-m}(\mu)]^2 d\mu}.$$

When $m = 0$, we have to replace π by 2π in the above.

By the aid of the results given in § 2, Theorems I, III, it can easily be shown that
$$\Sigma \frac{\psi(1)\,\psi(0)}{\lambda},$$

in this case, is equal to the Green's function of § 15 (20).

17. We might now proceed to the case of the region bounded by the sphere $r = a$, the cone $\theta = \theta_0$, and two planes $\phi = 0$ and $\phi = \alpha$. The full discussion of this problem involves the theorem that the function $P_n^{-m}(\mu_0)$, regarded as a function of m, either has no zeroes whose real part is positive, when n is imaginary with real part greater than $-\frac{1}{2}$, or has only a finite number of such zeroes. From the fact that
$$\int_{\mu_0}^1 \frac{P_n^{-m}(\mu)\, P_n^{-m'}(\mu)}{1-\mu^2}\, d\mu = 0,$$

it follows that for real values of n the zeroes must be real. And Macdonald's results show that $P_n^{-m}(\mu_0)$ cannot vanish for m real and n imaginary. I am not aware of any discussion of the case when m and n are both imaginary.

The region bounded by the sphere and two planes can be treated as the limiting case of the above when $\theta_0 = \pi$.

The methods of this paper can also be applied to the corresponding problems when two cylinders, planes, spheres, or cones enter. The results suggest a number of theorems regarding the zeroes of the Bessel's functions and spherical harmonics which enter into the argument.

The surface condition $u = 0$, has been adopted throughout this paper. The modifications
$$\frac{\partial u}{\partial n} = 0, \quad \frac{\partial u}{\partial n} + hu = 0$$

admit similar treatment in most cases.

18. Sommerfeld devotes some pages in the second part of his paper*

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* *Loc. cit.*, p. 347.

to a discussion of the Green's function in an infinite region, bounded internally by a cylinder. For such a case the surface conditions

$$u = 0, \quad \frac{\partial u}{\partial n} = 0, \quad \frac{\partial u}{\partial n} + hu = 0$$

are not sufficient to make the problem determinate. Macdonald recognises this in his *Electric Waves*, introducing the condition that there is no reflection at infinity, so that terms in e^{ikr} must not be found in the solution, when r becomes infinite.* Sommerfeld obtains the result for the case named above both in the form of an infinite series and a contour integral. It seems probable that his method can be further developed.

Added November 21st, 1913.—The Green's function for the infinite region bounded by a right circular cone is given by me in a paper about to appear in the *Math. Ann.*, entitled "The Scattering of Sound Waves by a Cone." The result is obtained by methods resembling those employed in the present paper.

* Cf. *Electric Waves*, pp. 33, 90, and 189.

ON THE NUMBER OF PRIMES OF SAME RESIDUACITY

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1. *Introduction.*—This paper is intended to introduce a number of Tables (printed at the end of the paper) in which are shown the *numbers* (μ, m) of primes (obtained by actual counting) for which the congruences

$$y^{(p-1)\div\nu} \equiv +1 \pmod{p}, \quad [\nu \text{ a } max\text{; a factor of } (p-1)], \tag{1a}$$

$$y^{(p-1)\div n} \equiv +1 \pmod{p}, \quad [n \text{ any factor of } \nu], \tag{1b}$$

are satisfied for certain Bases (y), to examine the relations of μ, m to the *number* (M) *of primes* capable of such residuacity (of orders ν, n), and to suggest rules for the same.

2. *Notation.*

p denotes an *odd* prime of form $(n\varpi+1)$.
y is the *Base** in the fundamental congruences (1*a*, *b*).
n, x are *any pair* of reciprocal factors of Fermat's Index $(p-1)$, such that

$$nx = p-1 = \nu\xi; \qquad p = nx+1 = \nu\xi+1. \tag{2}$$

ν, ξ are the *special pair* of reciprocal factors of $(p-1)$, such that

$$\nu \text{ is the max. value of } n, \qquad \xi \text{ is the min. value of } x, \tag{2a}$$

so that ξ, ν are the Haupt-exponent and Max. Residue-Index of y modulo p.

$$(y/p)_\nu = 1, \quad (y/p)_n = 1 \quad \text{are abbreviations for the congruences (1}a, b\text{)}. \tag{3}$$

R denotes any "Range" of successive natural numbers.

N = number of odd primes (p) in any Range (R)
M = number of primes (p) of linear form $p = n\varpi+1$
μ = number of primes (p), such that $(y/p)_\nu = 1$
m = number of primes (p), such that $(y/p)_n = 1$
} taken through the same Range (R).

3. *Upper limit of μ, m.*—It is clear that the only primes (p), which

* The Bases used in this paper are y = 2, 3, 5, 6, 7, 10, 11, 12.

can satisfy the congruences ($1a$, b), are those of the forms specified

$$p = \nu\varpi + 1 \text{ [for } (1a)], \quad p = n\varpi + 1 \text{ [for } (1b)], \tag{2}$$

the whole number of which (in a given "Range") has been denoted by M. Hence

$$M \text{ is an upper limit of } \mu,\ m, \tag{4}$$

and the numbers μ, m of any range will be found among the number M of that range, so that the relations of μ, m to M should be the subject of study, and a table of the numbers M for various Ranges is the first desideratum.

3a. *Table of M.*—The Table M (on p. 264) shows the number (M) of primes of form $p = n\varpi + 1$ in various Ranges for various values of the argument (n) as follows:—

Values of n.	Range of p.
All n from 1 to 30; All even $n > 30$ to 60	1 to 10^4, 10^4 to $2 \cdot 10^4$, $2 \cdot 10^4$ to $3 \cdot 10^4$, &c. ... up to $9 \cdot 10^4$ to 10^5 and 1 to 10^5
Selected $n > 60$.........	1 to 10^4, and 1 to 10^5

3b. *Variation of M.*—On a first glance at the Table the variation of M would seem to be quite irregular, but it is not so. It has, in fact, been shown* (by the author) to follow the rules—

$$M = N \div \phi(n) \textit{ approximately}, \text{ when the Range is very large.} \tag{5}$$

$$M \text{ is usually rather } < N \div \phi(n). \tag{5a}$$

This sufficiently explains the apparent irregularity in passing from one value of ν to another in the same Range, and also the fairly regular decrease of M in passing from Range to Range with the same ν.

4. *Relation of μ to m.*—It is clear, from the definitions of ν, n, μ, m (Art. 2) that ν is a multiple of n, so that

The primes (p), which have $(y/p)_n = 1$, are *all those which have* $(y/p)_\nu = 1$, where $\nu = n, 2n, 3n$, &c., ..., including every multiple of n which occurs within the Range of p, (6)

and that therefore

$$m = \Sigma(\mu), \text{ [the summation being for all the multiples of } n]. \tag{6a}$$

In fact this formula gives the readiest means of computing m when μ is known.

* In the paper quoted in Art. 7 below.

5. *Tables of μ.*—The three Tables B_y, B_2, B_{10} (on pp. 265–269) show the *numbers* (μ) of primes (p) satisfying $(y/p)_\nu = 1$ for various Bases (y) in various ranges for various values of the argument (ν) as follows:—

Tab.	Base (y).	Ranges.	Range.
B_y	2, 3, 5, 6, 7, 10, 11, 12	One	1 to 10^4
B_2	2	Ten	1 to 10^4, 10^4 to $2 \cdot 10^4$, $2 \cdot 10^4$ to $3 \cdot 10^4$, &c.... up to $9 \cdot 10^4$ to 10^5
B_{10}	10	One	1 to 10^5

Each of these Tables is in two Parts :—

PART I (of each) shows the values of μ for every value of $\nu \not> 60$ (as argument), in each range of p as stated above: and the line M at foot gives (for comparison with μ) the value of M for every value of $\nu \not> 30$, and for selected values up to $\nu \not> 60$.

PART II (a continuation of Part I) shows the values of μ for all values of $\nu > 60$ within the Range (omitting, for brevity, all cases of $\mu = 0$).

The totals of μ on the right of Part II are the totals of μ in each line *for all values of* ν within the Range, and are therefore equal to the values of M in that Range [these are given merely as a verification that the numbers μ are *complete*].

5*a*. *Variation of μ.*—A slight examination of the three Tables (B_y, B_2, B_{10}) shows at once that the variation of μ is too irregular, when μ is small, to be worth discussing; and further that, roughly speaking, when μ is large,

(1) In equal Ranges μ decreases *gradually*, but *irregularly*, as p increases [see Tables B_2, B_{10}, Parts I]. (7*a*)

(2) In any one Range μ usually decreases *rapidly*, but *very irregularly*, as ν increases [see Tables B_y, B_2, B_{10}, Parts I]. (7*b*)

5*b*. *Ratio $M \div \mu$.*—It seems evident that great part of the decrease of μ last mentioned (Art. 5*a*), is really due to the decrease of the number M out of which μ is formed, so that the variation of the ratio $M \div \mu$ should perhaps be examined in preference to that of μ itself. The short Table below shows the value of the ratio $M \div \mu$ when $\nu = 1, 2, 3, 4$ for the same Bases (y) and through the same Ranges as in the Tables B_y, B_2, B_{10} of μ itself.

This Table shows at once that—

When $\nu = 1, 2$, the ratio $M \div \mu$ has a nearly constant value, the same for all Bases alike. (8)

This result may be otherwise expressed thus—

In any large Range the number (μ) of primes with a common primitive root (y) is

$$=^* \frac{1}{2.5} \text{ to } \frac{1}{2.8} \text{ of } M. \qquad (8a)$$

In any large Range the number (μ) of primes with a common quadratic root (y) is

$$=^* \frac{1}{3.2} \text{ to } \frac{1}{3.6} M. \qquad (8b)$$

This short Table shows also that when $\nu = 3, 4$ the variation of $M \div \mu$ is *irregular;* in fact the irregularity increases so rapidly as ν increases (*e.g.*, $\nu = 5$ has $\mu = 0$, $M \div \mu = \infty$ in the range 1 to 10^4, see Table B_y) as to preclude further discussion at present.

Value of the Ratio $M \div \mu$ for Various Bases, through Ranges named.

Range.	Base.	$\nu =$ 1	2	3	4
1 to 10^4	2	2·6	3·5	6·9	9·8
	3	2·6	3·2	8·5	9·0
	5	2·5	3·6	7·6	7·3
	6	2·6	3·4	7·9	6·2
	7	2·6	3·4	6·9	6·9
	10	2·6	3·4	7·3	7·3
	11	2·8	3·5	6·6	6·6
	12	2·7	3·3	7·4	9·0

Range of 10000	$\nu =$	Base 2: 1	2	3	4	Base 10: 1	2	3	4
0 to 1		2·6	3·5	6·9	9·8	2·6	3·4	7·3	7·3
1 to 2		2·8	3·6	7·3	9·4	2·6	3·6	7·1	7·4
2 to 3		2·7	3·4	7·5	11·6	2·7	4·0	7·5	5·8
3 to 4		2·6	3·6	7·4	10·8	2·7	3·4	7·3	6·9
4 to 5		2·6	3·6	6·9	10·3	2·5	3·7	7·5	6·4
5 to 6		2·7	3·4	7·3	9·6	2·6	3·5	7·9	6·3
6 to 7		2·7	3·6	7·8	11·1	2·6	3·6	7·0	8·0
7 to 8		2·7	3·5	7·7	9·9	2·7	3·5	8·3	6·4
8 to 9		2·5	3·6	8·5	12·7	2·8	3·6	8·1	6·8
9 to 10		2·7	3·4	7·7	10·1	2·7	3·6	7·7	6·8
0 to 10		2.7	3·5	7·9	10·4	2·7	3·6	7·5	6·7

6. *Tables of m.*—The three Tables C_y, C_2, C_{10} (on pp. 270–272) show the *numbers* (m) of primes (p) satisfying $(y/p)_n = 1$ for various Bases (y) in various Ranges, as stated below, for every value of the argument $n = 1$ to 40.

Tab.	Base (y)	Ranges.	Range.
C_y	2, 3, 5, 6, 7, 10, 11, 12	One	1 to 10^4
C_2	2	Ten	1 to 10^4, 10^4 to $2 \cdot 10^4$, $2 \cdot 10^4$ to $3 \cdot 10^4$, &c., ... up to $9 \cdot 10^4$ to 10^5
C_{10}	10	One	1 to 10^5

The last line of each Table gives the values of $M \div n$ (for comparison with m), in the full Range of the Table (*i.e.*, 1 to 10^4 in Table C_y, and 1 to 10^5 in Tables C_2, C_{10}).

* The reason for these particular fractions has yet to be discovered.

6*a*. *Variation of m.*—A slight examination of the three Tables will show a very considerable regularity in the variation of m: and a comparison of these numbers (m) with the values of $M \div n$ suggests the following approximate rules *when the numbers m available* (for comparison) *are large*—

$$m = \frac{1}{n} M \textit{ approximately} \quad \text{[except as below]}, \tag{9}$$

$$m \text{ is usually } < \frac{1}{n} M \quad \text{[except as below]}. \tag{9a}$$

The only *marked* exceptions to the above two rules are

Base.	2					3	5		12
$n =$	8	16	24	32	40	12	10	20	12
$m, M \div n =$	596, 298	136, 74	94, 49	35, 19	31, 15	49, 25	58, 31	15, 7	51, 25

and these suggest the following additional approximate rules *when the numbers m available* (for comparison) *are large*—

$$\left.\begin{array}{ll} \textit{Base } 2: & m = \dfrac{1}{n}\, 2M \textit{ approximately} \\ & m \text{ is usually } < \dfrac{1}{n}\, 2M \end{array}\right\} \text{ when } n = 8i, \tag{10, 10a}$$

$$\textit{Other Bases}: \quad m = \frac{1}{n}\, 2M \textit{ approximately}, \text{ in some cases when } n = ky. \tag{10b}$$

There are other exceptions, but *not marked ones* (when m is large).

Some such approximate rule as (9) might be expected on the hypothesis that *all possible results were equally probable.* For a prime (p), which is capable of residuacity of order n, being of form $p = n\varpi+1$, must be the product of $\phi(n)$ complex factors, any one of which may be denoted by π. Then

$$y^{(p-1)\div n} \equiv \text{one of } \rho, \rho^2, \rho^3, \ldots, \rho^{n-1} \pmod{\pi}, \tag{11}$$

where

$$\rho \text{ is an imaginary proper root of } \rho^n - 1 = 0,$$

so that $y^{(p-1)\div n}$ is congruent to some one of n residues modulo π, one of which is

$$y^{(p-1)\div n} \equiv 1 \pmod{\pi}, \quad \text{which alone gives } y^{(p-1)\div n} \equiv 1 \pmod{p}. \tag{11a}$$

Hence, it might be expected that, out of a very large number (M) of primes (p) of form ($n\varpi+1$), about $1/n$-th of that number (M) would have the Base y as an n-ic residue, satisfying the congruence (11*a*) (provided all the cases were equally probable). This is the proposed Rule (9).

The causes of the different Rules (10, 10*b*) when $n = 8i$ with Base 2, and (in some cases) when $n = ki$ with other bases have yet to be sought.

The causes of the usually marked deficiency—see Rules (9*a*), (10*a*)—below the fractions $1/n \,.\, M$ and $1/n \,.\, 2M$ of Rules (9), (10) have also yet to be sought.

7. *Data for these Tables.*—It seems well to state the sources of the extensive masses of data (some still unpublished), utilised in preparing these Tables.

Table of M.—Extracted (and adapted), from the Table in the author's paper on the "Number of Primes of given Linear Form" in *Proc. London Math. Soc.*, Ser. 2, Vol. 10, 1911, p. 251.

Tables of μ.—The primes were *counted* from the following Tables of Haupt-exponents (ξ) and Max. residue indices (ν).

Table B_y.—From a Table of ξ and ν of the eight Bases $y = 2, 3, 5, 6, 7, 10, 11, 12$ for all primes $p \not> 10000$, prepared by the author and Mr. H. J. Woodall jointly. [This Table is in print, but not yet published.]

Table B_2.—From a Table of ν of the Base 2 for all primes $p \not> 100000$, prepared by the author and Mr. H. J. Woodall jointly, and published in five papers on "Haupt Exponents of 2," in the *Quarterly Journal of Pure and Applied Mathematics*, Vols. XXXVII, 1905; XLII, 1911; XLIV, 1912–13; and XLV, 1914.

Table B_{10}.—From Kessler's Table of ν of the Base 10 for all primes $p \not> 100000$ in Dr. M. Borck's *Periodische Decimalbrüche*, Berlin, 1895 (with corrections made by the author after collation with previous Tables).

Tables of m.—These were compiled from the tables of μ already made.

7*a*. *Preparation of the Tables.*—The work consisted simply of counting the primes of the kinds required. It is very tedious work: and the risk of error (especially of omitting one or two primes is considerable). The original counting was all done by one assistant (Miss C. Woodward), and was checked throughout by another (Mr. R. F. Woodward), under the author's constant supervision. The Tables B_y, B_2 were prepared independently also by Mr. H. J. Woodall, one of the joint compilers (see above, Art. 7).

Number (*M*) *of Primes* ($p = n\varpi + 1$) *in each* 10000 *Numbers* $\not> 10^5$.

TABLE M.

Range of 10000	n =	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0—1	Value of M in 10000	1228	1228	611	609	306	611	203	295	203	306	125	300	99	203	152	144	78	203	64	152
1—2		1033	1033	513	516	257	513	174	261	169	257	99	255	88	174	123	135	66	169	56	129
2—3		983	983	486	486	245	486	153	240	168	245	99	237	82	153	127	120	66	168	56	123
3—4		958	958	479	474	236	479	168	233	157	236	97	241	80	168	111	122	49	157	48	10[illegible]
4—5		930	930	467	464	230	467	150	228	151	230	87	231	79	150	118	115	61	151	58	112
5—6		924	924	458	469	234	458	157	234	153	234	92	237	77	157	116	116	66	153	49	115
6—7		878	878	429	431	219	429	147	221	138	219	86	212	79	147	111	106	51	138	49	108
7—8		902	902	455	454	220	455	144	223	156	220	90	220	70	144	106	110	61	156	49	110
8—9		876	876	440	445	223	440	147	219	142	223	87	223	73	147	114	109	49	142	46	106
9—10		879	879	446	435	217	446	150	230	155	217	83	218	71	150	111	111	56	155	50	11[illegible]
0—10	M =	9591	9591	4784	4783	2387	4784	1593	2384	1592	2387	945	2374	798	1593	1189	1188	603	1592	525	1181

Range of 10000	n =	21	22	23	24	25	26	27	28	29	30	32	34	35	36	38	40
0—1	Value of M in 10000	98	125	55	143	63	99	70	103	43	152	73	78	48	101	64	71
1—2		87	99	43	124	53	88	59	85	41	123	73	66		83	56	6[illegible]
2—3		82	99	44	118	51	82	53	72	32	127	59	66		85	56	6[illegible]
3—4		86	97	40	116	43	80	50	82	39	111	62	49		78	48	51
4—5		69	87	39	118	47	79	49	76	28	118	60	61		74	58	5[illegible]
5—6		76	92	43	113	52	77	56	77	32	116	53	66		82	49	55
6—7		68	86	41	110	45	79	47	71	30	111	55	51		68	49	5[illegible]
7—8		75	90	38	117	43	70	53	71	35	106	54	61		74	49	5[illegible]
8—9		70	87	43	108	43	73	46	72	29	114	56	49		71	46	5[illegible]
9—10		76	83	43	114	46	71	52	78	36	111	54	56		81	50	6[illegible]
0—10	M =	787	945	429	1181	486	798	535	787	345	1189	599	603	392	797	525	58[illegible]

Range of 10000	n =	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
0—1	Value of M in 10000		98		63	50	55		69		63		50		70	29	50		43		76
1—2			87		45		43		67		53		45		59		43		41		61
2—3			82		48		44		56		51		45		53		38		32		62
3—4			86		50		40		58		43		37		50		34		39		49
4—5			69		39		39		58		47		37		49		34		28		58
5—6			76		43		43		57		52		38		56		39		32		61
6—7			68		40		41		52		45		36		47		38		30		56
7—8			75		47		38		58		43		38		53		34		35		53
8—9			70		46		43		55		43		33		46		36		29		51
9—10			76		45		43		51		46		33		52		39		36		58
0—10	M =		787		466	388	429		581		486		392		535	231	385		345		585

Range of 10000	n =	64	65	70	72	75	80	88	90	96	100	104	105	110	120	130	140	150	210	808
0—1	M =	38	25	48	47	29	35	29	50	35	30	22	23	29	35	25	24	29	23	21
0—10	M =	300	202	392	397	235	293	228	388	292	243	194	192	231	286	202	183	235	192	104

Number (μ) *of Primes* $p < 10^4$ *with same Max. Residue Index* (ν) *for Bases* $y = 2, 3, 5, 6, 7, 10, 11, 12.$

[M = number of primes $p < 10^4$ of form $p = \nu\varpi + 1$,]

TABLE B_y [PART I].

Base	$\nu =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	Value of μ	470	348	88	62	25	62	12	40	2	17	3	8	6	9	5	7	3	6	1	5
3		476	375	72	68	20	39	15	13	12	18	2	34	2	8	4	3	4	2	1	3
5		492	337	80	83	.	55	11	18	10	32	4	13	2	5	.	3	3	9	.	7
6		470	359	77	98	14	59	11	14	11	13	8	9	2	9	5	3	3	3	1	3
7		465	359	89	88	24	54	12	24	7	16	5	12	2	2	2	5	2	7	1	4
10		467	360	84	84	20	61	11	21	8	17	2	14	1	10	3	5	2	9	.	2
11		443	348	93	92	22	64	14	32	10	15	6	11	5	7	3	4	2	7	.	4
12		459	377	83	68	26	39	13	13	11	17	6	31	3	8	5	1	1	4	3	3
	M =	1228	1228	611	609	306	611	203	295	203	306	126	300	99	203	152	144	78	203	64	152

Base	$\nu =$	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
2	Value of μ	.	3	.	10	.	.	3	2	.	2	2	1	.	3	.	.	.	2	.	.
3		1	4	.	4	.	1	3	1	.	2	.	1	.	5	.	2	2	1	1	.
5		2	2	1	4	.	3	1	1	1	5	1	1	1	.	.	2	.	2	2	2
6		.	2	1	5	.	.	.	.	1	3	.	.	2	.	2	2	1	1	.	.
7		2	4	.	2	.	1	.	4	.	3	.	2	.	.	.	3	.	1	1	.
10		3	1	.	2	1	1	1	2	.	1	.	.	1	1	.	1	.	.	.	1
11		1	1	.	3	1	4	.	1	.	2	.	1	.	2	1	1	1	1	.	.
12		.	3	.	8	.	2	.	2	2	2	.	1	.	2	1	4	.	.	1	1
	M =	98	126	55	143	63	99	70	103	43	152		73		78	48	101		64		71

Base	$\nu =$	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
2	Value of μ	.	2	.	.	1	1	.	2	.	1	.	.	.	.	.	1	.	1	.	.
3		.	.	.	.	.	1	.	2	.	2	.	.	.	1	.	.	.	.	1	2
5		.	1	.	1	.	.	.	.	.	1	.	1	.	.	.	1	.	.	.	3
6		.	.	3	.	1	1	1	2	.	2	.	.	2	1	.	1	1	.	1	1
7		.	.	.	.	.	2	.	.	.	.	.	.	.	.	1	1	.	1	.	1
10		.	1	.	2	.	1	1	.	.	.	.	.	.	1	1	1	.	.	.	.
11		.	.	.	3	2	1	.	.	.	2	.	.	1	.	.	.	.	.	.	1
12		1	1	.	1	.	.	.	.	.	.	.	1	.	1	.	1	1	1	.	1
	M =		98		63	50	55		69		63		50		70	29	50		43		76

TABLE B_y [PART II].

Base	ν, μ	Values of μ for each ν.	Total of μ
2	ν	64 70 72 81 96 105 120 129 226 630	
	μ	1 1 2 2 1 1 1 1 1 1	1228
3	ν	66 75 82 84 105 110 111 125 128 156 168 188 270 304 350 358	
	μ	1 1 1 2 1 1 1 1 1 1 2 1 1 2 1 1	1227
5	ν	64 66 70 74 80 81 83 90 92 111 116 120 123 138 154 181 201 240 254 287 330 390 458	
	μ	1 1 1 1 1 1 1 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1	1227
6	ν	62 66 75 78 80 90 116 124 132 140 162 172 214 264 274 339	
	μ	1 2 1 1 1 2 1 1 1 1 1 1 1 1 1 1	1227
7	ν	62 65 72 78 88 99 102 110 118 134 150 156 201 560 676	
	μ	1 1 1 3 1 1 2 1 1 1 1 1 1 1 1	1227
10	ν	62 64 67 68 72 76 78 82 90 92 98 101 110 116 118 140 177 186 399 664 825 909	
	μ	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1	1227
11	ν	62 66 69 70 78 82 84 93 108 120 132 143 406 644 915	
	μ	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1	1227
12	ν	61 72 75 78 83 84 87 92 94 96 100 144 160 216 652 706	
	μ	1 2 1 1 1 1 1 1 1 2 1 1 1 1 1 1	1227

Number (μ) *of Primes* (p) *in each* 10000 *Range with same Max. Residue-Index* (ν).

BASE 2.

M = number of primes $p < 10^5$ of form $p = \nu\varpi + 1$.

TABLE B_2 [PART I].

Range of 10000	ν =	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0—1	Value of μ	470	348	88	62	25	62	12	40	2	17	3	8	6	9	5	7	3	6	1	5
1—2		370	288	70	55	28	61	9	44	11	15	4	3	4	5	2	5	4	5	1	.
2—3		365	287	65	42	16	49	7	37	8	9	4	11	6	5	4	8	3	6	2	1
3—4		365	264	65	44	13	57	10	31	7	11	7	8	2	7	2	10	2	8	.	1
4—5		353	259	68	45	15	43	4	36	8	7	4	9	5	8	5	7	3	3	.	2
5—6		340	271	63	49	19	37	9	36	5	14	3	5	1	7	4	5	1	3	.	2
6—7		326	247	55	39	22	41	8	32	7	14	1	5	4	9	4	10	2	7	.	2
7—8		339	258	59	46	10	48	11	30	10	9	.	8	3	8	3	9	.	4	1	4
8—9		346	246	52	35	9	44	10	31	5	12	4	9	3	7	2	7	1	3	1	1
9—10		329	258	58	43	9	40	10	30	11	11	2	5	3	10	.	7	.	2	2	3
0—10	μ =	3603	2726	643	460	166	482	90	347	74	119	32	71	37	75	31	75	19	47	8	21
0—10	M =	9591	9591	4784	4783	2387	4784	1593	2384	1592	2387	945	2374	798	1593	1189	1188	603	1592	525	1181

Range of 10000	ν =	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
0—1	Value of μ	.	3	.	10	.	.	3	2	.	2	2	1	.	3	.	.	.	2	.	.
1—2		.	4	1	6	2	1	1	.	.	2	1	2	1	1	.	1	2	1	.	3
2—3		1	1	1	8	2	.	.	2	.	3	.	2	.	.	.	.	1	.	.	2
3—4		1	3	.	6	3	2	.	.	.	2	.	3	1	1	.	1	.	.	.	4
4—5		2	2	.	3	1	2	.	1	.	3	.	3	1	1	1	1	.	1	.	2
5—6		1	1	1	3	2	2	.	.	.	4	.	4	2	.	.	1	.	.	.	3
6—7		2	4	.	5	1	.	.	1	.	1	.	.	1	1	.	1	2	1	.	3
7—8		1	2	1	2	.	.	.	.	2	4	.	1	2	.	.	.	.	1	1	2
8—9		.	1	1	5	.	2	2	.	2	2	.	.	.	.	2	3	.	2	1	2
9—10		2	2	1	5	.	1	.	1	.	2	.	1	.	1	.	1	.	.	.	.
0—10	μ =	10	23	6	53	11	10	6	7	4	25	3	17	8	8	3	9	5	8	2	21
0—10	M =	787	945	429	1181	486	798	535	787	345	1189		599		603	392	797		525		582

Range of 10000	ν =	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
0—1	Value of μ	.	2	.	.	1	1	.	2	.	1	.	.	.	.	.	1	.	1	.	.
1—2		1	.	.	.	.	.	.	1	.	.	.	.	.	1	.	1	1	2	.	.
2—3		.	2	.	1	3	.	1	2	.	1	.	1	.	.	.	1	.	.	.	1
3—4		.	1	.	1	.	.	.	.	.	1	.	.	.	.	.	2	1	1	.	1
4—5		.	.	.	.	.	.	.	.	.	1	.	.	.	1	1	2	.	.	.	.
5—6		.	.	.	.	1	2	.	3	1	1	.	.	.	1	.	1	.	.	.	.
6—7		.	.	.	.	.	.	.	1	.	.	1	1	.	.	.	1	.	.	.	.
7—8		.	.	.	.	.	2	.	4	.	.	.	.	.	2	.	2	.	.	.	.
8—9		.	1	1	.	.	1	.	.	1	.	.	.	.	1	.	.	.	1	.	.
9—10		.	1	.	.	1	.	.	.	1	.	.	.	1	.	.	1	.	.	.	.
0—10	μ =	1	7	1	2	6	6	1	13	3	5	1	2	1	6	1	12	2	5	.	2
0—10	M =		787		466	388	429		581		486		392		535	231	385		345		585

TABLE B_2 [PART II].

Range of 10000																		Total of μ
0—1	ν =	64	70	72	81	96	105	120	129	226	630							
	μ =	1	1	2	2	1	1	1	1	1	1							1228
1—2	ν =	62	64	67	72	74	79	82	118	120	130	172	218	326				
	μ =	1	1	1	1	1	1	1	1	1	1	1	1	1				1033
2—3	ν =	64	85	90	110	140	170	206	278	316	387	518						
	μ =	1	1	2	1	1	1	1	1	1	1	1						983
3—4	ν =	70	96	122	164	226	315	329	538	568								
	μ =	1	1	1	1	1	1	1	1	1								958
4—5	ν =	62	72	78	80	82	90	91	92	100	107	112	208	216	240	399	1285	
	μ =	1	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	930
5—6	ν =	64	66	88	96	112	126	129	136	144	198	216	256	300	304	370		
	μ =	1	1	1	1	1	1	1	1	2	1	1	1	1	1	1		924
6—7	ν =	72	80	81	96	103	111	202	213	288	312	584	1310	1542	2048			
	μ =	2	2	1	1	1	1	1	1	1	1	1	1	1	1			878
7—8	ν =	65	72	74	88	125	137	140	184	386	474	508	878					
	μ =	1	2	1	1	1	1	1	1	1	1	1	1					902
8—9	ν =	64	67	72	80	110	118	136	141	160	168	180	210	288	736	1231	1615	
	μ =	1	1	1	2	1	1	1	1	1	1	1	1	1	1	1	1	876
9—10	ν =	64	66	70	72	76	78	80	81	90	104	107	108	134	138	139	152	
	μ =	1	1	1	2	1	1	1	1	1	1	1	1	1	1	1	1	
	ν =	175	178	190	192	198	1472											
	μ =	1	1	1	1	2	1											879

Range 1 to 10^3

ν =	62	64	65	66	67	70	72	74	76	78	79	80	81	82	85	88
μ =	2	6	1	2	2	3	12	2	1	2	1	6	4	2	1	2
ν =	90	91	92	96	100	103	104	105	107	108	110	111	112	118	120	122
μ =	4	1	1	4	1	1	1	1	2	1	2	1	2	2	2	1
ν =	125	126	129	130	134	136	137	138	139	140	141	144	152	160	164	168
μ =	1	1	2	1	1	2	1	1	1	2	1	2	1	1	1	1
ν =	170	172	175	178	180	184	190	192	198	202	206	208	210	213	216	218
μ =	1	1	1	1	1	1	1	1	3	1	1	1	1	1	2	1
ν =	226	240	256	278	288	300	304	312	315	316	326	329	370	386	387	399
μ =	2	1	1	1	2	1	1	1	1	1	1	1	1	1	1	1
ν =	474	508	518	538	568	584	630	736	878	1231	1285	1310	1472	1542	1615	2048
μ =	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Total $\mu = 9591 = M$ from $\nu = 1$ to $\nu = 2048$

Number (μ) *of Primes* (p) *in each* 10000 *Range with same Max. Residue-Index* (ν).

BASE 10.

M = number of primes $p < 10^5$ of form $p = (\nu\varpi + 1)$.

TABLE B_{10} [PART I].

Range	ν =	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0—1	Value of μ	467	360	84	84	20	61	11	21	8	17	2	14	1	10	3	5	2	9	.	2
1—2		398	287	72	71	24	52	7	10	5	11	3	10	3	8	3	5	.	8	1	1
2—3		369	247	65	84	25	46	6	17	8	17	6	15	5	6	4	6	2	11	2	4
3—4		353	286	66	69	17	42	6	15	14	7	3	10	1	8	3	3	.	7	2	1
4—5		360	253	63	73	14	41	13	15	5	11	2	4	2	9	2	6	.	1	.	2
5—6		349	262	58	75	15	44	8	8	6	13	2	9	5	6	6	4	1	4	.	.
6—7		343	244	61	54	18	35	5	13	7	14	4	14	1	7	1	3	.	3	.	2
7—8		336	260	55	71	16	48	9	14	8	15	1	7	.	3	3	4	.	10	1	3
8—9		316	243	54	66	24	53	7	14	5	8	2	12	3	7	5	1	2	5	1	3
9—10		327	242	58	64	19	51	11	9	4	15	4	11	2	6	3	3	2	5	1	2
0—10	μ =	3618	2684	636	711	192	473	83	136	70	128	29	106	23	70	33	40	9	63	8	20
0—10	M =	9590	9590	4784	4783	2387	4784	1593	2384	1592	2387	945	2374	798	1593	1189	1188	603	1592	525	1181

Range	ν =	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
0—1	Value of μ	3	1	.	2	1	1	1	2	.	1	.	.	1	1	.	1	.	.	.	1
1—2		4	.	1	3	3	2	1	3	.	2	.	3	.	3	.	1	1	1	.	4
2—3		4	.	.	1	.	.	.	1	.	5	.	.	.	1	1	1	.	1	.	.
3—4		2	4	.	3	1	4	.	.	1	3	.	1	.	1	1	4	1	1	1	.
4—5		2	1	1	2	1	2	.	.	.	1	1	3	2	.	1	2	.	3	.	2
5—6		2	3	.	2	1	.	2	2	.	4	.	1	.	1	.	1	.	1	.	4
6—7		6	6	1	2	1	4	.	.	.	2	.	1	1	2	.	.	.	1	.	4
7—8		.	1	1	2	.	2	1	.	.	1	.	.	.	1	.	1	.	1	.	1
8—9		.	4	1	1	2	.	.	2	.	2	.	2	.	1	.	4	.	.	.	3
9—10		3	1	.	2	.	2	2	1	.	2	.	2	.	1	1	1	.	.	.	3
0—10	μ =	26	21	5	20	10	17	7	11	1	23	1	13	4	12	4	16	2	9	1	22
0—10	M =	787	945	429	1181	486	798	535	787	345	1189		599		603	392	797		525		582

Ran	ν =	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
0—1	Value of μ	.	1	.	2	.	1	1	.	.	.	.	.	.	1	1	1	.	.	.	.
1—2		.	2	1	.	.	1	.	1	.	.	.	.	.	.	.	1	.	.	.	1
2—3		.	1	.	1	.	.	.	1	.	2	.	.	.	2	.	.	.	1	.	.
3—4		.	1	.	.	.	.	.	.	.	.	.	.	.	2	.	.	.	1	.	1
4—5		.	1	.	2	.	1	.	1	.	3	1	.	.	2	.	.	.	.	.	2
5—6		.	1	.	.	.	.	.	1	.	1	.	.	.	1	.	.	.	.	.	.
6—7		.	1	.	1	.	.	.	1	.	1	.	.	.	.	.	.	.	.	1	.
7—8		.	1	.	.	.	.	1	1	.	1	.	.	.	.	.	.	1	.	.	1
8—9		.	1	.	1	1	.	1	1	1	.	.	.	.	.	.	.	1	.	.	.
9—10		.	1	.	.	.	1	.	2	.	2	.	1	.	.	1	1	.	.	.	.
0—10	μ =	.	11	1	7	1	4	3	9	1	10	1	1	.	8	2	3	2	2	1	5
0—10	M =		787		466	388	429		581		486		392		535	231	385		345		585

TABLE B_{10} [PART II].

Range of 10000																					Total of μ
0—1	$\nu =$	62	64	67	68	72	76	78	82	90	92	98	101	110	116	118	140	177	186	399	
	$\mu =$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
	$\nu =$	664	825	909																	
	$\mu =$	1	1	1																	1227
1—2	$\nu =$	62	70	82	84	124	130	132	148	156	172	196	310	578							
	$\mu =$	2	2	1	1	1	1	1	1	1	1	1	1	1							1033
2—3	$\nu =$	61	78	79	84	104	133	148	157	329	370	374	856	1024	1398	1968					
	$\mu =$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1					983
3—4	$\nu =$	70	78	80	86	102	108	204	225	274	316	335	352								
	$\mu =$	1	1	1	1	1	1	1	1	1	1	1	1								958
4—5	$\nu =$	62	64	68	70	72	78	110	114	148	204	237	240	546	724	1484					
	$\mu =$	1	1	2	1	1	1	1	1	1	1	1	2	1	1	1					930
5—6	$\nu =$	61	62	65	67	70	84	88	89	92	112	117	138	181	186	208	218	264	624	2921	
	$\mu =$	1	2	1	1	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	924
6—7	$\nu =$	72	84	93	94	143	152	417	434	601	672	1144	2138	2183							
	$\mu =$	1	1	1	1	1	1	1	1	1	1	1	1	1							878
7—8	$\nu =$	68	80	82	102	104	122	152	166	168	187	190	197	200	204	211	216	238	349	834	
	$\mu =$	1	1	1	1	1	1	2	1	1	1	1	1	1	1	1	1	1	1	1	902
8—9	$\nu =$	79	82	87	88	90	94	113	133	186	198	205	513	612	665						
	$\mu =$	1	1	1	1	1	1	1	2	2	1	1	1	1	1						876
9—10	$\nu =$	74	97	108	114	120	122	134	172	714	1370										
	$\mu =$	1	1	1	1	1	1	1	1	1	1										879

Range 1 to 10^5	$\nu =$	61	62	64	65	67	68	70	72	74	76	78	79	80	82	84	86	87	88	89	Total $\mu = 9590 = M$ from $\nu = 1$ to $\nu = 2921$
	$\mu =$	2	6	2	1	2	4	6	3	1	1	4	2	2	4	4	1	1	2	1	
	$\nu =$	90	92	93	94	97	98	101	102	104	108	110	112	113	114	116	117	118	120	122	
	$\mu =$	2	2	1	2	1	1	1	2	2	2	2	1	1	2	1	1	1	1	2	
	$\nu =$	124	130	132	133	134	138	140	143	148	152	156	157	166	168	172	177	181	186	187	
	$\mu =$	1	1	1	3	1	1	1	1	3	3	1	1	1	1	2	1	1	4	1	
	$\nu =$	190	196	197	198	200	204	205	208	211	216	218	225	237	238	240	264	274	310	316	
	$\mu =$	1	1	1	1	1	3	1	1	1	1	1	1	1	1	2	1	1	1	1	
	$\nu =$	329	335	349	352	370	374	399	417	434	513	546	578	601	612	624	664	665	672	714	
	$\mu =$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
	$\nu =$	724	825	834	856	909	1024	1144	1370	1398	1484	1968	2138	2183	2921						
	$\mu =$	1	1	1	1	1	1	1	1	1	1	1	1	1	1						

Number (m) *of Primes* $p < 10^4$, *having* $(y/p)_n = 1$,
$[y = 2, 3, 5, 6, 7, 10, 11, 12]$.

M = number of primes $p < 10^4$, of form $p = (n\varpi + 1)$.

TABLE C_y.

$n =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Base 2	1228	603	200	143	60	97	29	66	17	28	6	24	6	16	11	12	6	9	3	6
3	1227	607	191	142	57	95	31	28	21	30	8	49	5	14	11	9	9	6	3	5
5	1227	607	200	146	58	100	24	33	25	58	11	24	9	10	14	7	3	13	2	15
6	1227	607	192	145	49	93	24	27	21	26	16	21	3	11	13	6	3	9	3	6
7	1227	611	193	151	55	90	22	37	19	27	13	20	10	8	7	8	4	11	2	6
10	1227	615	198	143	50	94	31	33	23	24	9	18	3	16	6	6	4	13	2	4
11	1227	618	207	158	56	95	28	41	21	26	13	20	11	12	10	5	4	9	1	6
12	1227	607	202	147	59	99	27	32	24	26	10	51	8	13	9	6	3	13	4	7
$M \div n$	1228	614	204	152	61	102	29	37	23	31	12	25	8	15	10	9	5	11	3	7

$n =$	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
Base 2	4	3	1	16	1	.	5	3	1	4	.	3	.	3	.	.	.	2	.	1
3	6	6	1	8	5	2	5	5	.	5	.	2	1	5	2	2	3	2	2	.
5	3	6	3	6	1	5	2	2	2	14	1	2	3	.	1	2	2	2	3	5
6	.	6	2	8	3	1	2	2	2	6	2	.	6	.	3	2	1	1	1	1
7	2	6	2	3	1	6	.	5	1	5	1	2	1	2	.	4	.	1	5	1
10	5	4	2	3	2	2	2	4	1	2	2	1	2	2	1	2	.	1	1	1
11	2	6	3	4	3	5	1	3	1	4	2	1	2	2	2	2	1	1	1	1
12	2	4	1	14	2	4	2	4	4	3	.	4	.	2	1	8	.	.	2	2
$M \div n$	5	6	2	6	3	4	3	4	2	5	1	2	.	2	1	3	1	2	.	2

Number (m) *of Primes* (p), *in each* 10000 *Range, having* $(2/p)_n = 1$.

M = number of primes $p < 10^5$, of form $p = (n\varpi + 1)$.

TABLE C_2.

Range	n =	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0—1		1228	603	200	143	60	97	29	66	17	28	6	24	6	16	11	12	6	9	3	6
1—2		1033	518	168	125	54	82	15	65	20	22	9	13	6	6	5	9	5	8	3	4
2—3		983	492	166	122	48	84	20	61	20	22	7	22	7	12	13	13	5	8	2	5
3—4		958	477	163	115	40	85	24	58	17	21	12	17	4	11	6	14	3	9	1	6
4—5		930	455	154	120	43	69	21	60	17	19	8	17	10	12	10	14	4	9	2	7
5—6		924	470	142	123	52	65	21	65	16	26	9	17	3	10	10	19	2	10	1	6
6—7		878	438	140	110	50	67	21	61	19	23	6	17	6	11	5	16	4	11	1	7
7—8		902	456	151	114	35	75	23	54	18	20	5	16	5	11	7	14	.	8	2	7
8—9		876	429	136	103	37	73	23	54	17	23	6	21	6	10	6	13	3	10	4	7
9—10		879	445	139	107	31	66	28	52	22	20	7	15	6	14	4	12	1	9	5	4
0—10	m =	9591	4783	1559	1182	450	763	225	596	183	224	75	179	59	113	77	136	33	91	24	59
0—10	$M \div n$	9591	4795	1594	1196	477	797	228	298	177	239	86	198	61	114	79	74	36	88	28	59

Range	n =	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
0—1		4	3	1	16	1	.	5	3	1	4	2	3	.	3	3	2	.	2	.	1
1—2		.	4	1	9	2	2	2	1	2	3	2	3	1	1	.	2	3	1	.	4
2—3		3	3	1	10	3	1	.	4	.	6	.	3	.	1	1	.	2	.	.	2
3—4		3	4	.	7	4	2	.	2	1	3	.	4	1	1	2	1	.	.	.	4
4—5		3	2	1	7	3	4	2	4	.	5	1	3	1	1	1	4	.	1	1	4
5—6		2	4	3	10	4	2	2	2	.	5	.	7	4	1	.	4	1	1	.	3
6—7		2	4	.	11	1	2	1	2	.	1	.	3	1	1	.	4	3	1	1	5
7—8		1	3	4	8	1	.	2	3	2	4	.	1	2	.	1	2	1	1	1	2
8—9		3	2	3	8	.	2	3	1	3	4	.	4	.	1	3	6	.	2	1	5
9—10		3	5	3	8	1	3	2	2	.	3	.	4	3	1	2	4	.	3	1	1
0—10	m =	24	34	17	94	20	18	19	24	9	38	5	35	13	11	13	29	10	12	5	31
0—10	$M \div n$	37	43	19	49	19	31	20	28	12	39	.	19	.	18	11	22	.	14	.	15

Number (m) *of Primes* (p), *in each* 10000 *Range, having* $(10/p)_n = 1$.

M = number of primes $p < 10^5$, of form $p = (n\varpi + 1)$.

TABLE C_{10}.

Range	n =	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0—1	Value of m	1227	615	197	143	50	94	31	33	23	24	9	18	3	16	6	6	4	13	2	4
1—2		1033	506	168	121	52	83	28	27	15	22	4	19	7	17	6	9	4	9	2	6
2—3		983	491	168	136	59	87	22	29	22	29	8	20	7	9	9	9	4	14	4	4
3—4		958	484	164	112	38	77	19	24	29	14	8	20	7	10	8	6	3	14	3	3
4—5		930	459	140	124	43	64	29	33	11	25	8	15	6	13	7	13	4	6	4	8
5—6		924	462	147	114	47	72	23	25	15	24	7	16	9	13	10	9	2	6	1	4
6—7		878	423	139	100	43	61	22	28	11	23	14	20	7	11	3	6	2	4	2	6
7—8		902	465	145	114	44	77	15	29	21	25	3	15	3	6	5	6	6	12	6	7
8—9		876	442	152	112	51	84	21	23	19	17	9	19	3	10	9	4	4	12	6	6
9—10		879	440	149	105	50	79	25	23	13	26	6	18	5	10	6	7	4	7	2	6
0—10	m =	9590	4787	1569	1181	477	778	235	274	179	229	76	180	57	115	69	75	37	97	32	54
0—10	M÷n	9590	4795	1594	1196	477	797	228	298	177	239	86	198	61	114	79	74	36	88	28	59

Range	n =	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
0—1	Value of m	5	4	2	3	2	2	2	4	1	2	2	1	2	2	1	2	.	1	1	1
1—2		7	1	2	4	3	4	1	6	.	3	2	3	1	4	2	1	2	1	1	4
2—3		6	2	.	3	2	2	2	2	1	5	.	1	.	2	1	1	2	1	1	.
3—4		3	5	.	3	2	5	3	.	2	4	.	2	.	3	2	5	1	1	2	1
4—5		4	4	2	6	4	4	2	1	.	5	2	4	2	3	2	3	1	4	2	4
5—6		4	5	3	5	2	2	3	4	.	4	3	1	1	1	2	1	.	1	2	4
6—7		9	8	1	5	2	5	.	2	.	2	2	2	1	2	.	1	1	2	.	4
7—8		2	1	1	5	2	3	2	1	.	2	.	.	.	5	.	2	.	4	.	3
8—9		1	7	1	2	2	.	1	2	1	3	2	2	1	2	1	5	.	.	.	3
9—10		5	1	1	5	2	3	3	2	.	3	.	2	.	2	1	2	1	1	.	4
0—10	m =	46	38	13	41	23	30	19	24	5	33	13	18	8	26	12	23	8	16	9	28
0—10	M÷n	37	43	19	49	19	31	20	28	12	39	.	19	.	18	11	22	.	14	.	15

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LIMITING FORMS OF LONG PERIOD TIDES

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1. As the equations of tidal motion have been solved in only a few cases, which are very restricted when compared with the actual conditions of terrestrial tides, it is of interest to try to obtain some approximation to the long period tides by means of a discussion of the limiting forms of these tides as the period of the disturbing forces tends to become infinite.

When friction is taken into account it is known that these limiting forms are the forms given by the equilibrium theory; but it has been shown by Hough * that for the great oceanic basins, the time required for the frictional forces to produce appreciable damping is long compared with a fortnight or even a half-year. It appears from this that, if any approximation is to be obtained in the limiting forms, they must be calculated on the hypothesis of no friction, and this is what we shall try to do in the following paper.

Sir G. H. Darwin was the first to point out that the long period tides need not approximate to their "equilibrium forms." The dynamical reason for this and the connexion with free steady motions was first given by Prof. Lamb in the second edition of his *Hydrodynamics* (1895).

For an ocean of uniform depth covering the whole globe the limiting forms have been calculated by Lamb,† and later by Hough,‡ who took into account the mutual attraction of the particles of water. The general question has been discussed to some extent by Poincaré,§ who terms the equilibrium forms "marées statiques de la première sorte," and the limiting forms in which there is steady relative motion "marées statiques

* "On the Influence of Viscosity on Waves and Currents," *Proc. London Math. Soc.*, Vol. XXVIII, p. 264 (1896).

† *Hydrodynamics*, 2nd ed., p. 353; 3rd ed., p. 321.

‡ "On the Application of Harmonic Analysis to the Dynamical Theory of the Tides, Part I," *Phil. Trans. Roy. Soc.*, A, Vol. CLXXXIX, p. 201 (1897).

§ *Leçons de Mécanique Céleste*, t. III, c. VIII (1910).

de la deuxième sorte." A discussion of the limiting forms for flat seas of uniform depth, with one or two examples, has been given in a recent paper.*

The question remains, however, whether, in general, these limiting forms uniquely exist, and, if so, whether they always give an approximation to the actual forced tides of long period. In the cases worked out by Lamb and Hough this is so. Prof. Lamb points out that his results differ very little from those previously obtained by Sir G. H. Darwin with the actual value of the period of the fortnightly tide, while Hough shows that the limiting forms he has obtained are very nearly the same as the fortnightly tides he has calculated in the same paper.

The following paper is an attempt to discuss, firstly, the existence, uniqueness and nature of the limiting forms for a general disturbing potential, and, secondly, the possibility of their application as approximations.

The method adopted in the first of these discussions is to use the limiting forms of the general equations of forced harmonic tidal motion as the period tends to become infinite, and to add as conditions any properties of the general motion which are independent of the period, so long as it is finite, but do not follow as consequences of the limiting forms of the equations themselves.

The method adopted in the second is to try to use, in the ordinary variables of tidal theory, results which have been established only for systems with a finite number of degrees of freedom.

General Equations and Properties.

2. We consider the tidal motion of water of uniform density on a sphere of radius a, which is rotating about its axis with uniform angular velocity ω. Let θ denote the co-latitude of any point on the sphere, and ϕ the longitude of the point measured from some meridian fixed on the sphere. When the water is rotating in free relative equilibrium, let its depth at any point on the sphere be denoted by h, and in any other state by $h+\zeta$. Also let Ω be the potential (supposed single valued) of the disturbing forces acting on unit mass of the water, and Π the potential due to the elevated water.

Take
$$P = g\zeta+\Omega+\Pi, \tag{1}$$

* "On some Cases of Tidal Motion of Rotating Sheets of Water," *Proc. London Math. Soc.*, Ser. 2, Vol. 12, p. 453 (1913).

where g is the acceleration due to gravity, and suppose that the time t only enters through the factor $e^{i\sigma t}$. Then the component of velocity of the water in any direction is given by

$$\frac{1}{\sigma^2-4\omega^2\cos^2\theta}\left\{i\sigma\frac{\partial P}{\partial s_1}+2\omega\cos\theta\frac{\partial P}{\partial s_2}\right\}, \tag{2}*$$

where $\partial/\partial s_1$, $\partial/\partial s_2$ denote differentiation along the surface of the sphere, the former in the direction in question, the latter in a direction which is a right angle in advance.

The equation of continuity may be written in the form

$$\frac{\partial\zeta}{\partial t}=-\frac{1}{a\sin\theta}\left\{\frac{\partial}{\partial\theta}(hu\sin\theta)+\frac{\partial}{\partial\phi}(hv)\right\}, \tag{3}$$

where u, v are the components of velocity along the positive directions of the meridian and parallel of latitude respectively.

Substituting from (2) into (3) and using (1), we obtain as the general equation of tidal motion on a sphere

$$\frac{g}{a^2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{h\sin\theta}{\sigma^2-4\omega^2\cos^2\theta}\frac{\partial P}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{h}{\sigma^2-4\omega^2\cos^2\theta}\frac{\partial P}{\partial\phi}\right)\right.$$
$$\left.+\frac{2\omega}{i\sigma}\left\{\frac{\partial}{\partial\theta}\left(\frac{h\cos\theta}{\sigma^2-4\omega^2\cos^2\theta}\right)\frac{\partial P}{\partial\phi}-\frac{\partial}{\partial\phi}\left(\frac{h\cos\theta}{\sigma^2-4\omega^2\cos^2\theta}\right)\frac{\partial P}{\partial\theta}\right\}\right]$$
$$+P=\Omega+\Pi. \tag{4}$$

The boundary condition takes different forms according as the boundary is or is not a vertical wall.

(i) For a vertical wall

$$i\sigma\frac{\partial P}{\partial n}+2\omega\cos\theta\frac{\partial P}{\partial s}=0, \tag{5}$$

where $\partial/\partial n$ denotes differentiation along the outward drawn normal to the bounding line, and $\partial/\partial s$ along the positive direction of its arc.

(ii) For a shelving beach, $h=0$, and the condition to be satisfied is taken to be that P shall be finite there.

The condition for the constancy of total volume of water is

$$\iint\zeta\,dS=0, \tag{6}$$

* See, for instance, Poincaré, *l.c.*, c. VII. The expression (2) may easily be obtained from (79) of the appendix to the present paper.

where the integral is taken over the whole surface of the sea. This can be deduced from the above when $\sigma \neq 0$.

Consider now a closed vertical wall boundary. At each point of it the condition (5) is satisfied. The relative circulation round such a bounding line is given by

$$\int \frac{1}{\sigma^2-4\omega^2\cos^2\theta}\left\{i\sigma\frac{\partial P}{\partial s}-2\omega\cos\theta\frac{\partial P}{\partial n}\right\}ds, \tag{7}$$

where the integral is taken round the bounding line, and $\partial/\partial s$, $\partial/\partial n$ have the same meaning as in (5). By means of (5) this may be written in the form

$$-\frac{1}{i\sigma}\int\frac{\partial P}{\partial s}ds.$$

But P is a single-valued function of position, so that the integral in this vanishes, and we deduce that the *relative circulation vanishes*, provided $\sigma \neq 0$.

Again, consider any number of closed contour-lines of the function $h\sec\theta$, *i.e.* closed members of the family of curves $h\sec\theta = \text{constant}$, bounding a portion S of the area of the sea. Then from continuity, or from (1) and (4), we have

$$i\sigma\iint\zeta\,dS+\Sigma\int\frac{h}{\sigma^2-4\omega^2\cos^2\theta}\left\{i\sigma\frac{\partial P}{\partial n}+2\omega\cos\theta\frac{\partial P}{\partial s}\right\}ds=0,$$

where the surface integral is taken over the portion S of the area, and the line integrals round all the contour-lines in question, positively with regard to the area S. Since in each line integral $h\sec\theta$ is constant, the above may be written in the form

$$i\sigma\iint\zeta\,dS+\frac{i\sigma}{2\omega}\Sigma\,h\sec\theta\int\frac{1}{\sigma^2-4\omega^2\cos^2\theta}\left\{2\omega\cos\theta\frac{\partial P}{\partial n}+\frac{4\omega^2}{i\sigma}\cos^2\theta\frac{\partial P}{\partial s}\right\}ds$$
$$=0. \tag{8}$$

Now $$\int\frac{4\omega^2\cos^2\theta}{\sigma^2-4\omega^2\cos^2\theta}\frac{\partial P}{\partial s}ds=\int\frac{\sigma^2}{\sigma^2-4\omega^2\cos^2\theta}\frac{\partial P}{\partial s}ds,$$

so that (8) may be written

$$i\sigma\iint\zeta\,dS+\frac{i\sigma}{2\omega}\Sigma\,h\sec\theta\int\frac{1}{\sigma^2-4\omega^2\cos^2\theta}\left\{2\omega\cos\theta\frac{\partial P}{\partial n}-i\sigma\frac{\partial P}{\partial s}\right\}ds=0, \tag{9}$$

and then, by (7), this may be written, provided $\sigma \neq 0$,

$$2\omega\iint\zeta\,dS=\Sigma\,h\sec\theta\,(\text{relative circulation}). \tag{10}$$

Now let us make the portion S of area tend to zero. The limiting values of $h \sec \theta$ will be the same for all the contour-lines (assuming h to be a continuous function of position), and the surface integral in (10) will tend to zero. We see, therefore, that the sum of the limiting values of the circulations will be zero. Even if the circulations be made up of parts from equal and opposite elements of circuits, this is not merely a consequence of the continuity of velocity. If for any reason there should be slipping along a contour-line of $h \sec \theta$, then the above condition, being a *dynamical* one, would still have to be satisfied.

Π will usually be small compared with Ω, and we shall often neglect it. If in any case it has been neglected in the direct treatment, its effect may be allowed for by successive approximations.

Types of Regions.

3. We shall find it convenient to divide the area of the surface of the sea into regions of three different types by means of the contour-lines of $h \sec \theta$.*

A region of Type I is such that over it $h \sec \theta$ is not uniform, but that its contour-lines are all broken by coast lines.

A region of Type II is such that over it $h \sec \theta$ is uniform.

A region of Type III is a simply or doubly connected region such that over it $h \sec \theta$ is not uniform, but that its contour-lines are all closed curves.

A region of Type II will have a closed exterior bounding line and any number (including zero) of closed interior bounding lines. These interior bounding lines may be either vertical coasts, partly vertical coasts and boundaries of regions of Type I, or the external boundaries of regions of Type III.

We shall suppose that the equator $\theta = \frac{1}{2}\pi$ strikes the coast line so that it always runs through regions of Type I, and $h \sec \theta$ can never be infinite in regions of the other types.

For a region of Type III the orthogonal trajectories of the contour-lines of $h \sec \theta$ will be the lines of greatest slope of $h \sec \theta$. When the region is doubly connected all the contour-lines will form closed curves round the inner boundary. When the region is simply connected all the contour-lines will form closed curves round an isolated point or finite line.

* This idea was suggested by a remark of E. Fichot in his note entitled, "Sur la production des marées statiques de la deuxième sorte dans un océan répondant à une loi quelconque de profondeur," *Comptes Rendus*, t. CLVI, p. 211 (1913).

At this point, or at the ends of this line, as the case may be, lines of greatest slope of $h \sec \theta$ will intersect, but we suppose that such intersection does not occur elsewhere.

It is evident that all boundary lines of regions of Types II and III will be contour-lines of $h \sec \theta$.

Now let us examine the limiting forms of the equations and conditions in § 2 as $\sigma \to 0$.

From (2) we obtain that the velocity in any direction is given by

$$-\frac{1}{2\omega} \sec \theta \frac{\partial P}{\partial s_2}, \tag{11}$$

$\partial/\partial s_2$ preserving its significance.

From (4) we obtain

$$\frac{\partial}{\partial \theta}(h \sec \theta) \frac{\partial P}{\partial \phi} - \frac{\partial}{\partial \phi}(h \sec \theta) \frac{\partial P}{\partial \theta} = 0, \tag{12}$$

which shows that there is a functional relation between P and $h \sec \theta$. If one is constant over any area, then (12) leaves the other unrestricted over that area.

From (5) we obtain $\partial P/\partial s = 0$, or

$$P = \text{constant}, \tag{13}$$

along a vertical coast.

If the coast is not vertical it will be a contour-line of $h \sec \theta$, for which $h \sec \theta$ is zero. We see therefore that on all boundary lines for all regions the condition (13) holds, though the constant may be different for distinct boundary lines.

As the equilibrium theory gives everywhere $P =$ constant, we see that at any instant the height of the water along a coast line only differs by a constant from its equilibrium value, and in special cases the constant may be zero.*

If we neglect Π we have all along any coast line

$$\zeta = \bar{\zeta} + \text{constant}, \tag{14}$$

where $\bar{\zeta} = -\Omega/g$.

From the results of harmonic analysis ζ could be calculated for a long period tide at different points along the same coast line, and at the same instant of time. Utilizing this, it might be possible to obtain useful information from the relation (14).†

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* An example occurs in § 9 below.

† See § 15.

Although the condition (6) is not a consequence of (12) and (13) it will be satisfied by any tide of finite, however great, period. We therefore prescribe it as a necessary condition for our limiting forms. The same applies to the conditions regarding the relative circulations which were deduced in § 2.

We shall assume that there are no other similarly necessary conditions independent of those which we have laid down above.

4. A region of *Type I* is very easily discussed. Since P is constant round the boundary and along each contour-line, and since each contour-line either cuts the boundary or forms part of it, it follows that P is constant over the whole region. The velocity is then everywhere null and the equilibrium theory holds.

When the whole sea consists of a region of Type I the limiting form exists uniquely, the constant value of P being determined from the condition (6).

It may be noted that in the present case the relation (10) does not necessarily hold, so that the condition for the constancy of total volume of water, *i.e.* (6), is not included in the coast-line conditions.

5. For a region of *Type II* the condition (12) puts no restriction on P. Going back to equation (4) we see that in this case its limiting form is

$$-\frac{gh_0}{4\omega^2 a^2 \sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\tan\theta\,\frac{\partial P}{\partial\theta}\right)+\frac{1}{\sin\theta\cos\theta}\,\frac{\partial^2 P}{\partial\phi^2}\right\}+P=\Omega+\Pi, \quad (15)$$

on writing h_0 for $h \sec\theta$, which is now constant.

We may restrict ourselves to either the Northern or Southern Hemisphere, since the Equator lies entirely in regions of Type I; h_0 and $\sec\theta$ will then have the same constant sign over the whole region.

From (15) we see that, if there is any tide at all, it cannot have exactly the "equilibrium height" over any finite portion of the region.

From (15) again we obtain

$$\iint \zeta\, dS=\frac{h_0}{4\omega^2}\iint\left\{\frac{\partial}{\partial\theta}\left(\tan\theta\,\frac{\partial P}{\partial\theta}\right)+\frac{1}{\sin\theta\cos\theta}\,\frac{\partial^2 P}{\partial\phi^2}\right\}d\theta\, d\phi,$$

which may be transformed to

$$\iint \zeta\, dS=\frac{h_0}{4\omega^2}\,\Sigma\int \sec\theta\,\frac{\partial P}{\partial n}\,ds, \quad (16)$$

the surface integral being taken over the area of the region, and the line

integrals round the boundaries. From this we see that the relation (10) holds in the present instance, though it did not for a region of Type I.

For a closed coast-line the corresponding integral on the right-hand side of (16) must vanish, and when the sea consists entirely of a region of Type II the circulation conditions include that for the constancy of total volume of water, viz., (6).

If we neglect II and write $\beta_0 = 4\omega^2 a^2/gh_0$ (15) becomes

$$\left\{\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\tan\theta\,\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta\cos\theta}\,\frac{\partial^2}{\partial\phi^2}\right\}(\zeta-\bar{\zeta})-\beta_0\zeta = 0. \tag{17}$$

The condition (14) now holds at all the boundaries, while for a closed coast-line the condition for the vanishing of the circulation gives

$$\int \sec\theta\,\frac{\partial}{\partial n}(\zeta-\bar{\zeta})\,ds = 0, \tag{18}$$

the integral being taken round the coast-line.

We now proceed to discuss the existence and uniqueness of a solution of the equations (17) and (14) when the integrals on the left-hand side of (18) take prescribed values for each boundary line.

The uniqueness is very easily disposed of. Suppose if possible that there were two solutions; then their difference ζ' (say) would satisfy the following conditions:—

$$\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\tan\theta\,\frac{\partial\zeta'}{\partial\theta}\right)+\frac{1}{\sin^2\theta\cos\theta}\,\frac{\partial^2\zeta'}{\partial\phi^2}-\beta_0\zeta' = 0, \tag{19}$$

inside the region, and

$$\zeta' = \text{constant}, \qquad \int \sec\theta\,\frac{\partial\zeta'}{\partial n}\,ds = 0,$$

for the boundaries.

Now, if ξ be any solution of the equation (19), we have

$$\begin{aligned}\beta_0\iint \xi^2 dS &= a^2\iint \xi\left\{\frac{\partial}{\partial\theta}\left(\tan\theta\,\frac{\partial\xi}{\partial\theta}\right)+\frac{1}{\sin\theta\cos\theta}\,\frac{\partial^2\xi}{\partial\phi^2}\right\}d\theta\,d\phi \\ &= a^2\iint\left\{\frac{\partial}{\partial\theta}\left(\xi\tan\theta\,\frac{\partial\xi}{\partial\theta}\right)+\frac{1}{\sin\theta\cos\theta}\,\frac{\partial}{\partial\phi}\left(\xi\frac{\partial\xi}{\partial\phi}\right)\right\}d\theta\,d\phi \\ &\quad -a^2\iint\left\{\tan\theta\left(\frac{\partial\xi}{\partial\theta}\right)^2+\frac{1}{\sin\theta\cos\theta}\left(\frac{\partial\xi}{\partial\phi}\right)^2\right\}d\theta\,d\phi,\end{aligned}$$

which may be further transformed to

$$\beta_0 \iint \xi^2 dS + \iint \sec\theta \left\{ \left(\frac{\partial \xi}{\partial \theta}\right)^2 + \left(\frac{1}{\sin\theta} \frac{\partial \xi}{\partial \phi}\right)^2 \right\} dS = a^2 \Sigma \int \xi \sec\theta \frac{\partial \xi}{\partial n} ds, \quad (20)$$

the integral signs having the same significance as in (16).

Now when ζ' is substituted for ξ, the line integrals vanish separately, and since β_0 and $\sec\theta$ have the same constant sign over the whole region, it follows that ζ' must vanish everywhere. This proves the uniqueness of the solution when one exists.

6. To discuss the existence, let us first consider the determination of solutions of (17) and (19) which take prescribed values over the boundaries. For this purpose we may appeal to the theory of integral equations. Using the stereographic projection* from that pole which is on the opposite side of the equator to the region we are considering, the equation (17) may be written in the "elliptic" form, and its coefficients will possess second derivatives.

When the prescribed boundary values are everywhere zero, the equation (19) has been seen to possess no solution other than zero.

We assume that the region is such that the ordinary problem of Dirichlet possesses a solution.

In these circumstances the theory of Fredholm's equation shows that the problem we are considering also possesses a solution.†

We know then that a solution ζ_0 of (17) exists, such that on all the boundaries $\zeta_0 = \bar{\zeta}$.

Let the distinct boundaries be denoted by $s_1, s_2, \ldots, s_n$.

We now show the existence of a function ζ, satisfying (19) inside the region, taking constant values on all the boundaries, and further being such that

$$\int \sec\theta \frac{\partial \zeta_1}{\partial n} ds$$

vanishes for every boundary except s_1.

For this purpose let us take ξ_1 satisfying (19) inside the region, taking the value unity on s_1, and taking the value zero on all the other boundaries. Also let us take ξ_2 satisfying (19) inside the region, taking the

* Any conformal representation on a plane could be used. This is what is done by Poincaré when using integral equations; *Sechs Vorträge*, Teubner, Leipzig (1910); *Leçons de Mécanique Céleste*, t. III, c. X.

† See, for instance, Heywood and Frechet, *L'Equation de Fredholm*, c. III (1912).

value unity on s_2, and taking the value zero on all the other boundaries. Such functions we know to exist.

Then from (20) we see that

$$\int \sec\theta \frac{\partial \xi_2}{\partial n}\, ds_2 \neq 0,$$

since ξ_2 is not everywhere zero. Here $\int ds_2$ denotes the line integral taken round s_2.

From ξ_1, ξ_2 we can construct $\xi_3 = \xi_1 + a_1\xi_2$ (where a_1 is a constant), such that ξ_3 satisfies (19) inside the region, takes on s_1 the value unity, on s_2 the value a_1, and on all the other boundaries the value zero, while a_1 is chosen so as to make

$$\int \sec\theta \frac{\partial \xi_3}{\partial n}\, ds_2 = 0.$$

Similarly we can determine a function ξ_4 satisfying (19) inside the region, taking on s_3 the value unity, on s_2 the (constant) value a_2, and on all the other boundaries the value zero, while a_2 is chosen so as to make

$$\int \sec\theta \frac{\partial \xi_4}{\partial n}\, ds_2 = 0.$$

Then it follows from (20) that

$$\int \sec\theta \frac{\partial \xi_4}{\partial n}\, ds_3 \neq 0.$$

Now from ξ_3, ξ_4 we can construct $\xi_5 = \xi_3 + a_3\xi_4$, such that ξ_5 satisfies (19) inside the region, takes on s_1 the value unity, on s_2, s_3 constant values, on all the other boundaries the value zero, while the constant a_3 is chosen so as to make

$$\int \sec\theta \frac{\partial \xi_5}{\partial n}\, ds_3 = 0.$$

We have also $$\int \sec\theta \frac{\partial \xi_5}{\partial n}\, ds_2 = 0,$$

while (20) shows that $$\int \sec\theta \frac{\partial \xi_5}{\partial n}\, ds_1 \neq 0.$$

The process can obviously be continued until we obtain ζ_1.

Similarly we can find $\zeta_2, \zeta_3, \ldots, \zeta_n$, such that each satisfies (19) inside the region, takes constant values over all the boundaries, and such that

$$\int \sec\theta \frac{\partial \zeta_r}{\partial n}\, ds$$

vanishes for every boundary except s_r.

By multiplying ζ_r by the proper constants, we can make the value of this line integral round s_r unity.

Let the prescribed values of the line integrals of the enunciation be $C_1, C_2, \ldots, C_n$, for the boundaries $s_1, s_2, \ldots, s_n$, respectively. Then the solution we require will be given by

$$\zeta = C_1\zeta_1 + C_2\zeta_2 + \ldots + C_n\zeta_n.$$

If the sea consists entirely of a region of Type II, then

$$C_r = \int \sec\theta \frac{\partial\bar{\zeta}}{\partial n} ds_r \quad (r = 1, 2, \ldots, n),$$

from (18), and we see that the limiting form of the tide exists uniquely.

7. We now consider a region of *Type III*, and shall find it convenient to make a change of coordinates. Let us call the new coordinates ξ, η, and take them such that $\xi =$ constant, gives the contour lines of $h \sec\theta$, and $\eta =$ constant, gives the lines of greatest slope of $h\sec\theta$. This will make $h\sec\theta = f(\xi)$. Along a line $\eta =$ constant, let $ds = A\,d\xi$, and along a line $\xi =$ constant, let $ds = B\,d\eta$. Then we choose the coordinates so that A, B have neither zeros nor infinities except where lines $\eta =$ constant, intersect. At such points, which only occur when the region is simply connected, we shall have $B = 0$.

The equation (4) now takes the form

$$\frac{g}{AB}\left[\frac{\partial}{\partial\xi}\left(\frac{h}{\sigma^2-4\omega^2\cos^2\theta}\,\frac{B}{A}\,\frac{\partial P}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{h}{\sigma^2-4\omega^2\cos^2\theta}\,\frac{A}{B}\,\frac{\partial P}{\partial\eta}\right)\right.$$
$$\left.+\frac{2\omega}{i\sigma}\left\{\frac{\partial}{\partial\xi}\left(\frac{h\cos\theta}{\sigma^2-4\omega^2\cos^2\theta}\right)\frac{\partial P}{\partial\eta}-\frac{\partial}{\partial\eta}\left(\frac{h\cos\theta}{\sigma^2-4\omega^2\cos^2\theta}\right)\frac{\partial P}{\partial\xi}\right\}\right]+P$$
$$= \Omega+\Pi. \tag{21}$$

As $\sigma \to 0$, the limiting form of this equation gives simply $P = F(\xi)$, which of course we already know. To find the form of the function $F(\xi)$ we shall use the next approximation when σ is small.* For this purpose we have, from (21),

$$-\frac{g}{4\omega^2 AB}\left\{\frac{\partial}{\partial\xi}\left(h\sec^2\theta\,\frac{B}{A}\,\frac{\partial F}{\partial\xi}\right)+\frac{2\omega}{i\sigma}f'(\xi)\,\frac{\partial P}{\partial\eta}\right\}+F = \Omega+\Pi.$$

Now, after multiplying by AB, let us integrate this with regard to η,

* This method is taken from Poincaré, *l.c.*

round a whole circuit of the lines $\xi =$ constant. We thus obtain

$$-\frac{g}{4\omega^2}\frac{\partial}{\partial\xi}\left(\overline{h\sec^2\theta\frac{B}{A}}\frac{\partial F}{\partial\xi}\right)+\overline{AB}\,F=\overline{AB(\Omega+\Pi)}, \tag{22}$$

where $$\overline{\psi(\xi)}=\int\psi(\xi,\,\eta)\,d\eta,$$

$\psi(\xi,\,\eta)$ being any function of ξ, η, and the integral being taken round a whole circuit of the lines $\xi =$ constant.

The value of the circulation round a boundary line is given by

$$\frac{1}{2\omega}\int\sec\theta\,\frac{B}{A}\,\frac{\partial F}{\partial\xi}\,d\eta,$$

which may be written $$\frac{1}{2\omega}\left(\overline{\sec\theta\,\frac{B}{A}}\right)\frac{\partial F}{\partial\xi}.$$

A prescription of the circulation is therefore equivalent to a prescription of $\partial F/\partial\xi$ at the boundary, and all along a vertical coast we notice that the velocity of the water will be zero.

Now $$\iint\zeta\,dS=\iint\zeta AB\,d\xi d\eta=\int\overline{AB\zeta}\,d\xi,$$

and this, by (22), is equivalent to

$$\frac{1}{4\omega^2}\int\frac{\partial}{\partial\xi}\left(\overline{h\sec^2\theta\,\frac{B}{A}}\,\frac{\partial F}{\partial\xi}\right)d\xi,$$

so that here again the relation (10) holds, and the condition (6) is included in the coast-line conditions when the sea consists entirely of a region of Type III.

If we neglect Π we have in (22) a linear differential equation of the second order to determine $F(\xi)$. The boundary conditions are that F shall be finite when $h=0$, and that $\partial F/\partial\xi$ shall be prescribed where $h\neq 0$. The discussion of the unique existence of a solution could now be made by the methods of the theory of differential equations, but we may avoid this by observing that the conditions are the same as those of an actual physical problem, as opposed to the limiting case of one.

For a flat sea and a disturbing force such that Ω is a function only of h, equation (21) takes the form

$$\frac{g}{\sigma^2-4\omega^2}\left\{\frac{\partial}{\partial\xi}\left(h\,\frac{B}{A}\,\frac{\partial P}{\partial\xi}\right)\right\}+ABP=AB\Omega.$$

Now we can choose the coordinates, h and $\sigma^2-4\omega^2$ so that this is identical

with (22), when we neglect II. The above-mentioned boundary conditions are the expressions in the present auxiliary case of the condition for $h=0$ and for a prescription of normal velocity where $h\neq 0$, respectively. These problems, we know from physical reasons, will have unique solutions.

When the region forms a complete sea there cannot be resonance in the auxiliary problem, since $\sigma^2-4\omega^2$ is negative. To show this we observe that for a free period $\sigma^2-4\omega^2=\sigma_0^2$, where σ_0 is the "speed" of a free oscillation when there is no rotation, as ω only enters the equation through $\sigma^2-4\omega^2$. Now all the values of σ_0 will be real and consequently $\sigma^2-4\omega^2$ will be positive for a free period.

We therefore conclude that the problem above formulated has a unique solution, and when the sea consists entirely of a region of Type III, the limiting forms exist uniquely.

It is to be remarked that for a region of Type III the limiting form of the tide may have the equilibrium value. In particular this will be the case if $\overline{AB\Omega}=0$, examples of which occur in § 10 below.

8. We now endeavour to combine our solutions so as to apply to a sea which consists of more than one region. Let us first suppose that there are no regions of Type I. We associate with each boundary line which is not a coast-line a certain circulation, and treat for a moment these circulations as variables. In terms of them we can calculate the function ζ all over the sea, when the coast-line conditions are introduced.

Now we have two sets of conditions at the intermediate boundary lines. One is that from one region to another ζ must be continuous, since the pressure must be continuous. For a group of n boundary lines enclosing zero area this will give $n-1$ conditions. The other is the condition concerning circulations deduced in § 2 from (10). This gives *one* condition for such a group of boundary lines. In all we have n conditions, and this is the number of the circulations to be determined.

We know that the relation (10) holds for regions of both Types II and III. On adding these relations, in virtue of the circulation conditions both at coast-lines and intermediate boundary lines, we see that the condition (6) will be satisfied.

If the circulations are determined uniquely by the equal number of conditions involving them linearly, then the limiting form of the tide exists uniquely.

When, however, a region of Type I is present, we may find a difficulty

in satisfying all our conditions, owing to the fact that for such a region the relation (10) does not necessarily hold. Suppose, for example, that we have a simply connected sea, consisting of a region A of Type I, and a region B of Type II. P will be a constant over A and the same constant on the boundary of B. This constant has to be found, but if the region B touches the coast-line along a finite part of its length, we shall have to satisfy two conditions which are not now dependent. These are the condition (6) for the constancy of total volume of water, and the condition for the vanishing of the circulation round the coast-line.

It appears that we have to conclude that when a region of Type I is present, limiting forms of the kind we have been considering do not necessarily exist.

Examples with Flat Seas.

9. We now proceed to some simple illustrations of the above principles, and will first consider examples of seas which are so small that they may be considered as flat. The above theory can easily be shown to apply to this limiting case.

As an example of a region of Type II, let us consider a *rectangular* sea of *uniform depth*, when the disturbing potential is a two-dimensional harmonic of the first order. This is the form of long period disturbing potential which actually occurs for small seas on the earth's surface which are not in the neighbourhood of a pole.*

Using Cartesian coordinates, let the equations of the sides of the rectangle be $x = \pm a$, $y = \pm b$, and the disturbing potential such that $\bar{\zeta} = x$. We have to determine ζ to satisfy the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k^2\right) \zeta = 0, \tag{23}$$

which is the limiting form of (19) if we write $k^2 = 4\omega^2 \cos^2 \theta / gh$, k being regarded as a constant. Also we must have $\zeta = x + \text{constant}$, on the boundary, and the condition (6) satisfied, which in the present instance is equivalent to the condition for the vanishing of the circulation round the boundary.

We make use of the following expansions, which can be established by

* In the neighbourhood of a pole, the potential is of a higher order of smallness. The solution for a rectangle was given in the paper quoted in § 1.

Fourier's theorem or otherwise:

$$x = -\frac{2a}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{n\pi x}{a}, \tag{24}$$

for $-a < x < a$, and

$$1 = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cos \frac{(2n+1)\pi y}{2b}, \tag{25}$$

for $-b < y < b$.

Consider now the series

$$\zeta_1 = -\frac{2a}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{\cosh \mu_n y}{\cosh \mu_n b} \sin \frac{n\pi x}{a}, \tag{26}$$

where
$$\mu_n^2 = \frac{n^2\pi^2}{a^2} + k^2. \tag{27}$$

This satisfies (23), and when $y = \pm b$, by virtue of (24), we have $\zeta_1 = x$, while for $x = \pm a$, we have $\zeta_1 = 0$.

Again, consider the series

$$\zeta_2 = \frac{4a}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \frac{\sinh \nu_n x}{\sinh \nu_n a} \cos \frac{(2n+1)\pi y}{2b}, \tag{28}$$

where
$$\nu_n^2 = \frac{(2n+1)^2 \pi^2}{4b^2} + k^2. \tag{29}$$

This satisfies (23), and when $x = \pm a$, by virtue of (25), we have $\zeta_2 = x$, while for $y = \pm b$, we have $\zeta_2 = 0$.

It is easily seen that all our conditions are satisfied by $\zeta_1 + \zeta_2$, except at the corners, which however need not trouble us. Thus

$$\zeta = \frac{4a}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \frac{\sinh \nu_n x}{\sinh \nu_n a} \cos \frac{(2n+1)\pi y}{2b} - \frac{2a}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{\cosh \mu_n y}{\cosh \mu_n b} \sin \frac{n\pi x}{a}, \tag{30}$$

where μ_n, ν_n are given by (27), (29) respectively.

On the sides the height of the tide is that given by the equilibrium theory, though there is now a tangential velocity. Along the line $x = 0$, there is a node.

By interchanging x, y and a, b, we could obtain the solution for $\bar{\zeta} = y$, and then by superposition, for $\bar{\zeta} = Ax + By$, where A, B are given constants.

10. An example of a flat sea of Type III is considered by Lamb in *Hydrodynamics*, Art. 211.* Here we have, in polar coordinates r, ϕ,

$$h = h_0\left(1-\frac{r^2}{a^2}\right), \tag{31}$$

where h_0, a are constants, and corresponding to

$$\bar{\zeta} = \left(\frac{r}{a}\right)^s e^{i(\sigma t+s\phi)}, \tag{32}$$

we have
$$\zeta = \frac{2sgh_0}{2sgh_0-(\sigma^2-2\omega\sigma)\,a^2}\,\bar{\zeta}, \tag{33}$$

while corresponding to
$$\bar{\zeta} = \left(\frac{r}{a}\right)^2 e^{i\sigma t}, \tag{34}$$

it is easily seen from Lamb's equations that

$$\zeta = \frac{4gh_0}{8gh_0-(\sigma^2-4\omega^2)\,a^2}\left\{2\left(\frac{r}{a}\right)^2-1\right\} e^{i\sigma t}. \tag{35}$$

When we make $\sigma \to 0$, from (33) we get the equilibrium form, but from (35) we do not.

Let us now consider the limiting forms by the methods of the present paper. For the coordinates ξ, η of § 7, we may take simply r, ϕ. Then $A = 1$, $B = r$, and the differential equation becomes, on writing $\zeta' = \zeta-\bar{\zeta}$,

$$-\frac{gh_0}{4\omega^2}\frac{\partial}{\partial r}\left\{r\left(1-\frac{r^2}{a^2}\right)\frac{\partial\zeta'}{\partial r}\right\}+r\zeta' = -\overline{r\bar{\zeta}}. \tag{36}$$

Corresponding to (32), $\bar{\bar{\zeta}} = 0$, so that the equilibrium form follows at once, while corresponding to (34) the limiting form of (35) follows easily.

Suppose that instead of a shelving beach at $r = a$, the boundary consists of a vertical wall at $r = b$ $(< a)$, the law of depth remaining the same. The boundary condition is now $\partial\zeta'/\partial r = 0$, at $r = b$.

The admissible complementary function of (36) is

$$CF\left\{\gamma, \left(\frac{r}{a}\right)^2\right\} \equiv C\left[1+\sum_{n=1}^{\infty}\left\{\prod_{s=0}^{n-1}\frac{s(s+1)+\gamma}{(s+1)^2}\right\}\left(\frac{r}{a}\right)^n\right],$$

* 3rd ed. (1906); it was first given in the 2nd ed., Art. 205 (1895), and is also given by Poincaré, *l.c.*, § 72 (1910).

where $\gamma = \omega^2 a^2/gh_0$, and C is an arbitrary constant. A particular integral corresponding to (34) is

$$-\frac{1}{2+\gamma}\left\{1+\gamma\left(\frac{r}{a}\right)^2\right\},$$

so that, on determining C by means of the boundary condition, we have

$$\zeta' = \frac{1}{2+\gamma}\left[\gamma\frac{F\{\gamma,\,(r/a)^2\}}{F'\{\gamma,\,(b/a)^2\}} - \left\{1+\gamma\left(\frac{r}{a}\right)^2\right\}\right], \tag{37}$$

where
$$F'\{\gamma,\,z\} \equiv \frac{\partial}{\partial z}F\{\gamma,\,z\}.$$

Corresponding to (32) the equilibrium form holds as before.

Let us now consider the more general case of a sea bounded by the *ellipse* $\xi =$ constant, where

$$x = c\cosh\xi\cos\eta, \qquad y = c\sinh\xi\sin\eta,$$

x, y being Cartesians, and $h = f(\xi)$. Then we have

$$A = B = c\sqrt{\tfrac{1}{2}(\cosh 2\xi - \cos 2\eta)},$$

and if we take for $\bar{\zeta}$ a two-dimensional harmonic of the first order

$$P\cosh\xi\cos\eta + Q\sinh\xi\sin\eta,$$

where P, Q are constants, we have $\overline{AB\bar{\zeta}} = 0$, and the limiting form of the tide takes the equilibrium value.

11. As an example of the combination of solutions considered in § 8, let us consider a circular sea in which the depth follows the law (31) from $r = 0$ to $r = b$ ($< a$), but is constant from $r = b$ to $r = c$, having the value $h_1 = h_0(1-b^2/a^2)$, there being a vertical wall boundary at $r = c$. We have here a circular region of Type III, surrounded by an annular region of Type II.

The differential equation which holds for the inner region is that of the last section, and we have, corresponding to (34),

$$\zeta = CF\left\{\gamma,\,\left(\frac{r}{a}\right)^2\right\} + \frac{1}{2+\gamma}\left\{2\left(\frac{r}{a}\right)^2 - 1\right\},$$

where C is a constant which we have to determine.

The differential equation which holds in the annular region is

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right)\zeta'-k^2\zeta=0, \tag{38}$$

where $k^2 = 4\omega^2/gh_1$. For the same value of $\bar{\zeta}$ we obtain

$$\zeta = DI_0(kr)+EK_0(kr)-4/(ka)^2,$$

where $I_0(kr)$ and $K_0(kr)$ are the usual independent solutions of the "modified Bessel's equation of zero order" [*i.e.* (38) when $\zeta'=\zeta$, and $\partial\zeta/\partial\phi=0$], and D, E are constants to be determined.

Now the condition for the vanishing of the circulation round the coast-line gives

$$DkI_0'(kc)+EkK_0'(kc)-2c/a^2=0,$$

while the condition for the continuity of circulation round $r=b$, which in this case is equivalent to the condition for the continuity of tangential velocity, gives

$$2\,\frac{b}{a^2}\,CF'\left\{\gamma,\left(\frac{b}{a}\right)^2\right\}+\frac{4\,(b/a^2)}{2+\gamma}=DkI_0'(kb)+EkK_0'(kb).$$

The condition for the continuity of ζ (or of pressure) gives

$$CF\left\{\gamma,\left(\frac{b}{a}\right)^2\right\}+\frac{1}{2+\gamma}\left\{2\left(\frac{b}{a}\right)^2-1\right\}=DI_0(kb)+EK_0(kb)-\frac{4}{(ka)^2}.$$

We have now three linear equations to determine C, D, E.

In a similar manner we could calculate the limiting form of the tide corresponding to (32).

As another example of the combination of solutions, and one in which a region of Type I may occur, let us consider a circular sea of radius b, in which there is a barrier of negligible area, stretching from the boundary inwards to a point at a distance a from the centre. Let us suppose that for $r<a$ the depth is uniform, so that we have a region of Type II, and that for $r>a$ we have a single region of Type I.

The differential equation which holds for the inner region is (38), if we take $k^2=4\omega^2/gh$, h being the depth. Corresponding to a value of $\bar{\zeta}$ given by (32), we have

$$\zeta'=\left\{DI_s(kr)-\left(\frac{r}{a}\right)^s\right\}e^{is\phi},$$

where D is a constant, while for the outer region we have

$$\zeta'=C,$$

where C is also a constant.

The condition for the continuity of ζ across $r = a$ gives

$$DI_s(ka)-1 = 0, \qquad C = 0,$$

and these also satisfy the other conditions.

The solution is thus

$$\zeta = \frac{I_s(kr)}{I_s(ka)} e^{is\phi},$$

for $r < a$, and

$$\zeta = \left(\frac{r}{a}\right)^s e^{is\phi},$$

for $r > a$, the same as if each region were a separate sea. We notice that the solution involves slipping at $r = a$.

The tide corresponding to (34) could be similarly calculated.

Polar Sea on a Rotating Sphere.

12. As a final example we will consider the limiting forms of the forced tides of long period, for a sea which is bounded by any single parallel of latitude, and is of uniform depth h. Let the sea be given by $0 \leqslant \theta \leqslant \gamma$, where θ denotes, as before, the co-latitude. This is an example of a region of Type III, and since the disturbing potential is independent of longitude, the equation (22) of § 7 takes the form

$$-\frac{gh}{4\omega^2 a^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{\cos^2\theta}\frac{\partial P}{\partial\theta}\right)+P = \Omega+\Pi, \tag{39}$$

which may be obtained directly from (4).

The boundary condition is $\partial P/\partial\theta = 0$ at $\theta = \gamma$.

Neglecting Π and writing

$$\beta = \frac{4\omega^2 a^2}{gh}, \tag{40}$$

the equation (39) becomes

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{\cos^2\theta}\frac{\partial\zeta'}{\partial\theta}\right)-\beta\zeta' = \beta\bar{\zeta}, \tag{41}$$

where again

$$\zeta' = \zeta-\bar{\zeta}.$$

Now the form of $\bar{\zeta}$ which is of interest in connection with terrestrial tides is

$$\bar{\zeta} = H'(\cos^2\theta-\tfrac{1}{3}), \tag{42}$$*

but for our purpose the constant term is immaterial.

* See, for instance, Lamb, *Hydrodynamics*, c. VIII, App.

We shall find it convenient to change the independent variable. Let us write

$$x = \sin^2 \tfrac{1}{2}\theta,$$

so that the differential equation (41) takes the form

$$\frac{d}{dx}\left\{\frac{x-x^2}{(1-2x)^2}\frac{d\zeta'}{dx}\right\} - \beta\zeta' = \beta\bar{\zeta}, \tag{43}$$

while $\bar{\zeta}$ becomes

$$-4H'(x-x^2),$$

on dropping additive constants. In solving the equation we shall also drop for the present the factor $4H'$, since this will obviously occur as a factor in the final expression for ζ' or ζ.

To solve the equation (43) we assume

$$\frac{1}{(1-2x)^2}\frac{d\zeta'}{dx} = \sum_{n=0}^{\infty} C_n x^{a+n}, \tag{44}$$

where a and C_n $(n = 1, 2, \ldots)$ are constants to be determined. Then

$$\frac{x-x^2}{(1-2x)^2}\frac{d\zeta'}{dx} = \sum_{n=1}^{\infty} (C_{n-1} - C_{n-2})\, x^{a+n},$$

provided that we take $C_{-1} = 0$, and therefore

$$\frac{d}{dx}\left\{\frac{x-x^2}{(1-2x)^2}\frac{d\zeta'}{dx}\right\} = \sum_{n=1}^{\infty} (a+n)(C_{n-1} - C_{n-2})\, x^{a+n-1}. \tag{45}$$

Again, from (44),

$$\frac{d\zeta'}{dx} = \sum_{n=0}^{\infty} (C_n - 4C_{n-1} + 4C_{n-2})\, x^{a+n},$$

taking also $C_{-2} = 0$, and consequently

$$\zeta' = C + \sum_{n=0}^{\infty} (C_n - 4C_{n-1} + 4C_{n-2})\frac{x^{a+n+1}}{a+n+1}, \tag{46}$$

where C is a constant.

For the complementary function of (43) we must equate (45) to β times (46). Then we see that $a = 0$, $C_0 = \beta C$, and in general

$$(n+1)(C_n - C_{n-1}) = \frac{\beta}{n}(C_{n-1} - 4C_{n-2} + 4C_{n-3}). \tag{47}$$

From this we see that as $n \to \infty$, $C_n/C_{n-1} \to 1$, so that our series will be valid for $|x| < 1$, as we otherwise know that they must.

The resulting series for ζ' will obviously contain C as a factor. When

C is taken equal to unity let the corresponding value of ζ' be called $R(\beta, x)$.

For a particular integral of (43) we may take $a = 0$, $C = 0$, and (47) will then be valid for $n > 2$, while for $n = 0, 1, 2$, we have

$$C_0 = 0, \quad 2C_1 = -\beta, \quad 3(C_2 - C_1) = \tfrac{1}{2}\beta C_1 + \beta.$$

Let us call the corresponding value of ζ', $F(\beta, x)$.

Then the solution we require is given by

$$\zeta' = F(\beta, x) - \frac{F'(\beta, a)}{R'(\beta, a)} R(\beta, x), \tag{48}$$

where now and in what follows

$$a = \sin^2 \tfrac{1}{2}\gamma,$$

and $$R'(\beta, x) = \frac{\partial}{\partial x} R(\beta, x), \qquad F'(\beta, x) = \frac{\partial}{\partial x} F(\beta, x).$$

On restoring the factor $4H'$, we have

$$\frac{\zeta}{4H'} = -x + x^2 + F(\beta, x) - \frac{F'(\beta, a)}{R'(\beta, a)} R(\beta, x). \tag{49}$$

It is interesting to have for comparison the corresponding equilibrium form ζ_0 (say). It is easily found to be given by

$$\frac{\zeta_0}{4H'} = -x + x^2 + \frac{a}{2} - \frac{a^2}{3}. \tag{50}$$

If a, β are not too large the series in (49) converge rapidly. We find, for example,

$R(10, x)$

$$= 1 + 10x + 10x^2 - 35{\cdot}\dot{5}x^3 - 32{\cdot}\dot{2}x^4 + 59{\cdot}\dot{1}x^5 + 49{\cdot}4x^6 - 40x^7 \ldots,$$

$R(20, x)$

$$= 1 + 20x + 70x^2 - 37{\cdot}\dot{7}x^3 - 342{\cdot}\dot{2}x^4 - 23{\cdot}\dot{1}x^5 + 754{\cdot}6x^6 + 216x^7 - 860x^8 \ldots,$$

$F(10, x)$

$$= -2{\cdot}5x^2 + 3{\cdot}\dot{3}x^3 + 4{\cdot}58\dot{3}x^4 - 5{\cdot}1\dot{6}x^5 - 6{\cdot}3x^6 \ldots,$$

$F(20, x)$

$$= -5x^2 + 1{\cdot}\dot{1}x^3 + 18{\cdot}\dot{8}x^4 + 4{\cdot}\dot{4}x^5 - 40{\cdot}6x^6 - 17x^7 \ldots .$$

Using these series we can calculate the values of $\zeta/4H'$ given in the

following table. Here $a = \cdot 1, \cdot 05$ make $\gamma = 36^\circ 52', 25^\circ 50'$ respectively, while for the size and rotation of the earth $\beta = 10, 20$ correspond roughly to depths of 8 k.m., 4 k.m. respectively. In the last column is given the corresponding equilibrium value. For each case the tide is calculated both at the pole $x = 0$, and at the coast $x = a$.

a	β	x	$\zeta/4H'$	$\zeta_0/4H'$
·1	10	0	+·03549	+·04667
		a	−·03809	−·04333
	20	0	+·02821	+·04667
		a	−·03442	−·04333
·05	10	0	+·02079	+·02417
		a	−·02169	−·02333
	20	0	+·01817	+·02417
		a	−·02035	−·02333

Application of Limiting Forms as Approximations.

13. As stated in § 1 the object of discussing the limiting forms is that they may possibly serve as approximations to the tides of long period. As a preliminary to the discussion of this possibility we shall obtain some relations in the ordinary variables of Tidal Theory which have already been given* for a dynamical system with a finite number of degrees of freedom.

The general equations of the motion described at the beginning of § 2, when not restricted to be harmonic as regards time, are

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - 2\omega v \cos\theta &= -\frac{\partial P}{a\,\partial\theta} \\ \frac{\partial v}{\partial t} + 2\omega u \cos\theta &= -\frac{\partial P}{a \sin\theta\,\partial\phi} \end{aligned}\right\}. \tag{51}†$$

If we multiply the first of these equations by hu, the second by hv, and integrate over any portion S of the area of the surface of the sea, we obtain

$$\frac{\partial}{\partial t}\tfrac{1}{2}\iint h(u^2+v^2)\,dS = -\iint h\left\{u\frac{\partial P}{a\,\partial\theta} + v\frac{\partial P}{a\sin\theta\,\partial\phi}\right\}dS. \tag{52}$$

* See Poincaré, *l.c.*, c. I.

† See Lamb, *l.c.*, p. 313; or the Appendix to the present paper.

On supplying the value of P from (1), the right-hand side of (52) becomes

$$-g\iint h\left\{u\frac{\partial\zeta}{a\partial\theta}+v\frac{\partial\zeta}{a\sin\theta\,\partial\phi}\right\}dS-\iint h\left\{u\frac{\partial\Pi}{a\partial\theta}+v\frac{\partial\Pi}{a\sin\theta\,\partial\phi}\right\}dS$$
$$-\iint h\left\{u\frac{\partial\Omega}{a\partial\theta}+v\frac{\partial\Omega}{a\sin\theta\,\partial\phi}\right\}dS.$$

Transforming the first two integrals in this, using the equation of continuity (3) and a theorem on gravitational potentials, (52) gives

$$\frac{\partial}{\partial t}\left\{\tfrac{1}{2}\iint h(u^2+v^2)\,dS+\tfrac{1}{2}g\iint\zeta^2dS+\tfrac{1}{2}\iint\zeta\Pi\,dS\right\}+\int(g\zeta+\Pi)\,hu_n\,ds$$
$$=-\iint h\left\{u\frac{\partial\Omega}{a\partial\theta}+v\frac{\partial\Omega}{a\sin\theta\,\partial\phi}\right\}dS, \quad (53)$$

u_n denoting the outward normal velocity at the boundary of S, and the line integral being taken round this boundary.

Now when all the functions are real, and the equation is multiplied throughout by ρ, the right-hand side of (53) represents the rate of doing work on the water by the disturbing forces.

The equation is then that of the rate of change of energy for the water whose surface is S. However, since it is a mathematical consequence of the equations (51), we may use it when the variables are complex, on the usual understanding that the real part only of each variable is to be interpreted.

The right-hand side of (53) when transformed gives

$$-\int\Omega\,hu_n\,ds-\iint\Omega\frac{\partial\zeta}{\partial t}\,dS.$$

On taking S to be the whole area of the sea, we have from (53), on using the boundary conditions,

$$\frac{\partial}{\partial t}E(P,\zeta)=-\iint\Omega\frac{\partial\zeta}{\partial t}\,dS, \quad (54)$$

where $$E(P,\zeta)=\tfrac{1}{2}\iint h(u^2+v^2)\,dS+\tfrac{1}{2}g\iint\zeta^2dS+\tfrac{1}{2}\iint\zeta\Pi\,dS, \quad (55)$$

the surface integrals being taken over the whole surface of the sea.

Consider now a motion which is formed by the superposition of two motions, these being given by P_n, ζ_n and P_m, ζ_m respectively. We shall have

$$E(P_n+P_m,\,\zeta_n+\zeta_m)=E(P_n,\,\zeta_n)+2E(P_n,\,P_m;\;\zeta_n,\,\zeta_m)+E(P_m,\,\zeta_m), \quad (56)$$

where

$$E(P_n, P_m;\ \zeta_n, \zeta_m)$$
$$= \tfrac{1}{2}\iint h(u_n u_m + v_n v_m)\,dS + \tfrac{1}{2}g\iint \zeta_n \zeta_m\,dS + \tfrac{1}{4}\iint (\zeta_n \Pi_m + \zeta_m \Pi_n)\,dS. \quad (57)$$

Let now P_n, ζ_n refer to a free motion with speed σ_n, and P_m, ζ_m refer to a free motion with speed σ_m. From (54) we obtain

$$E(P_n, \zeta_n) + 2E(P_n, P_m;\ \zeta_n, \zeta_m) + E(P_m, \zeta_m) = \text{constant},$$

since $\Omega = 0$. Now since $E(P_n, \zeta_n)$ contains the factor $e^{2i\sigma_n t}$, $E(P_m, \zeta_m)$ the factor $e^{2i\sigma_m t}$, and $E(P_n, P_m;\ \zeta_n, \zeta_m)$ the factor $e^{i(\sigma_n+\sigma_m)t}$, we must have

$$\left.\begin{aligned} E(P_n, \zeta_n) = E(P_m, \zeta_m) = 0 \\ E(P_n, P_m;\ \zeta_n, \zeta_m) = 0 \end{aligned}\right\}, \quad (58)$$

and

unless $\sigma_n + \sigma_m = 0$.

When $\sigma_n + \sigma_m = 0$, we suppose P_n taken of such a magnitude that

$$\tfrac{1}{2}E(P_n + P_{-n}, \zeta_n + \zeta_{-n}) = E(P_n, P_{-n};\ \zeta_n, \zeta_{-n}) = 1, \quad (59)$$

P_{-n}, ζ_{-n} being conjugate to P_n, ζ_n respectively.

On writing out in full the first of the relations (58), for example, we have

$$-\tfrac{1}{2}\iint \frac{h}{\sigma_n^2 - 4\omega^2\cos^2\theta}\left\{\left(\frac{\partial P_n}{a\,\partial\theta}\right)^2 + \left(\frac{\partial P_n}{a\sin\theta\,\partial\phi}\right)^2\right\} dS$$
$$+\tfrac{1}{2}g\iint \zeta_n^2\,dS + \tfrac{1}{2}\iint \zeta_n \Pi_n\,dS = 0. \quad (60)$$

Of course these relations may be obtained from the equations of § 2, and in fact we may obtain another in that way.

Thus, writing in (4), $\Omega = 0$, $P = P_n$, $\sigma = \sigma_n$, multiplying by $\tfrac{1}{2}P_{-n}$ and integrating over the whole area of the sea, transforming and using the boundary conditions, we obtain

$$\tfrac{1}{2}\iint \frac{h}{\sigma_n^2 - 4\omega^2\cos^2\theta}\left\{\frac{\partial P_n}{a\,\partial\theta}\,\frac{\partial P_{-n}}{a\,\partial\theta} + \frac{\partial P_n}{a\sin\theta\,\partial\phi}\,\frac{\partial P_{-n}}{a\sin\theta\,\partial\phi}\right\} dS$$
$$-\frac{1}{i\sigma_n}\iint \frac{h\omega\cos\theta}{\sigma_n^2 - 4\omega^2\cos^2\theta}\left\{\frac{\partial P_n}{a\,\partial\theta}\,\frac{\partial P_{-n}}{a\sin\theta\,\partial\phi} - \frac{\partial P_{-n}}{a\,\partial\theta}\,\frac{\partial P_n}{a\sin\theta\,\partial\phi}\right\} dS$$
$$+\tfrac{1}{2}g\iint \zeta_n\zeta_{-n}\,dS + \tfrac{1}{4}\iint (\zeta_n \Pi_{-n} + \zeta_{-n}\Pi_n)\,dS = 0. \quad (61)$$

14. For a gyrostatic system with a finite number of degrees of freedom

a very elegant formula has been given by Poincaré,* by which forced motion of any period can be expressed in terms of the possible free oscillations. If this formula could be extended to Tidal Theory, expressed in the ordinary variables of that theory, it would give us much information, but there are several difficulties in the way of this.

One of these difficulties is that every solution of the general equations we have been considering does not give a possible case of tidal motion, and all the possible oscillations of an ocean are not in accordance with these equations.

Another is that when steady motions are possible the usual coordinates of Tidal Theory are not of the independent Lagrangian type, while those of Poincaré's formula are.

However, guided by the formula referred to, we will assume, with the notation we have already used, the following relation :

$$\zeta = \frac{k_n}{\sigma-\sigma_n}\zeta_n + \frac{k_{-n}}{\sigma+\sigma_n}\zeta_{-n} + G_n, \tag{62}$$

where k_n, k_{-n} do not involve the coordinates of position or σ, and G_n is a function of position and σ which remains finite when $\sigma = \pm\sigma_n$.

We will now determine k_n, k_{-n} by Poincaré's method.

Substitute the above expression for ζ and the derived one for P in the formula

$$\frac{\partial}{\partial t} E(P+P_{-n},\, \zeta+\zeta_{-n}) = -\iint \Omega \frac{\partial}{\partial t}(\zeta+\zeta_{-n})\, dS,$$

which is a particular case of (54), and equate the coefficients of $e^{i(\sigma-\sigma_n)t}$. We have

$$2(\sigma-\sigma_n)\, E(P,\, P_{-n};\; \zeta,\, \zeta_{-n}) = \sigma_n \iint \Omega\zeta_{-n}\, dS,$$

and we may suppose the time factor $e^{i(\sigma-\sigma_n)t}$ omitted. On making $\sigma \to \sigma_n$, we obtain

$$2k_n E(P_n,\, P_{-n};\; \zeta_n,\, \zeta_{-n}) = \sigma_n \iint \Omega\zeta_{-n}\, dS,$$

which, by virtue of (59), gives

$$k_n = \tfrac{1}{2}\sigma_n \iint \Omega\zeta_{-n}\, dS. \tag{63}$$

Similarly we may obtain

$$k_{-n} = -\tfrac{1}{2}\sigma_n \iint \Omega\zeta_n\, dS. \tag{64}$$

In (63) and (64) we are to suppose the time factor omitted.

* *L.c.*, p. 26.

On referring to the preceding section, we see that k_n and k_{-n} will be zero, if the disturbing forces are incapable, at every instant, of doing work on the water as it moves in the free mode of speed σ_n.

An illustration of this occurs in the example of Prof. Lamb already considered in § 1C.

When the expression for ζ contains t and ϕ only in the factor $e^{i(\sigma t+s\phi)}$, Prof. Lamb finds the free periods to be given by

$$\left(\frac{\sigma^2-4\omega^2}{gh_0}\right)a^2-\frac{4\omega s}{\sigma} = n(n-2)-s^2,$$

where $n = s+2j$, j being some positive integer.

Now when
$$\Omega = -gC\left(\frac{r}{a}\right)^s e^{i(\sigma t+s\phi)},$$

C being a constant, it is clear that all values of k_{-n} will be zero, while those values of k_n will also be zero for which s is not the same in ζ_n as in Ω. Also, it is easily found from the differential equation that

$$(n+s)(n-s-2)\int_0^a \zeta_n r^{s+1}\,dr = 0,$$

so that the only values of k_n which do not vanish are those which correspond to $j = 1$.

If we let σ_1, σ_2 be the two speeds corresponding to $j = 1$, the corresponding free oscillations are found by Lamb to be given by

$$\zeta_1 = \sqrt{\frac{2(s+1)}{\pi a g}\frac{\sigma_2}{\sigma_2-\sigma_1}}\left(\frac{r}{a}\right)^s e^{i(\sigma_1 t+s\phi)},$$

$$\zeta_2 = \sqrt{\frac{2(s+1)}{\pi a g}\frac{\sigma_1}{\sigma_1-\sigma_2}}\left(\frac{r}{a}\right)^s e^{i(\sigma_2 t+s\phi)},$$

where, however, we have determined the constant factors in accordance with (59).

We then obtain, from (63),

$$k_1 = -\sigma_1 C\sqrt{\frac{\pi a g}{2(s+1)}\frac{\sigma_2}{\sigma_1-\sigma_2}}, \quad k_2 = -\sigma_2 C\sqrt{\frac{\pi a g}{2(s+1)}\frac{\sigma_1}{\sigma_1-\sigma_2}},$$

and the corresponding terms in ζ are

$$\frac{\sigma_1\sigma_2}{\sigma_1-\sigma_2}\frac{1}{\sigma-\sigma_1}C\left(\frac{r}{a}\right)^s e^{i(\sigma t+s\phi)},$$

and
$$\frac{\sigma_1\sigma_2}{\sigma_2-\sigma_1}\frac{1}{\sigma-\sigma_2}C\left(\frac{r}{a}\right)^s e^{i(\sigma t+s\phi)}.$$

Adding, we obtain
$$\frac{C}{(1-\sigma/\sigma_1)(1-\sigma/\sigma_2)}\left(\frac{r}{a}\right)^s e^{i(\sigma t+s\phi)},$$

which is the whole of the expression found by Prof. Lamb for the forced tides. We notice that the limiting form (which is here the equilibrium form) will be an approximation to the forced tide if the speed σ of the tide is small compared with σ_1 and σ_2.

15. One of the conditions for tidal motion* is that the distance from crest to crest of the surface of the water must be large compared with the depth of the water. This sets an upper limit to the magnitude of the gradient of P.

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* See Lamb, *l.c.*, p. 244.

We then see from (60), that as σ increases indefinitely, the motion ultimately ceases to be tidal in character.

Also, the discussion in the preceding part of the paper shows that in those cases in which limiting forms exist for a general value of Ω, as σ_n decreases indefinitely all the free oscillations ultimately cease to be tidal.

When a particular forced tide does not tend to a limit, and we have seen that (according to the tidal equations) this may occur, we conclude that as σ diminishes indefinitely the forced motion also ceases to be tidal.

From the results of the preceding section we may write

$$\zeta = \tfrac{1}{2}\Sigma\sigma_n\left(\frac{k_n\zeta_n}{\sigma-\sigma_n} - \frac{k_{-n}\zeta_{-n}}{\sigma+\sigma_n}\right) + G, \tag{65}$$

where the summation extends to all the true free tidal oscillations, and G is a function of position and σ which remains finite when σ is equal to any of the speeds of the true free tidal oscillations.

Now, again referring to Poincaré's formula, we may assume G to be constructed in a manner similar to the other part of ζ, but from those oscillations which are not tidal. The constants corresponding to k_n, k_{-n} may be calculated differently, but will be proportional to the rate at which the free oscillations can draw energy from the disturbing forces, at any instant.

When the limiting form exists it will be given by

$$\zeta = -\tfrac{1}{2}\sum_n (k_n\zeta_n + k_{-n}\zeta_{-n}) + \lim_{\sigma\to 0} G. \tag{66}$$

Now in those oscillations which are not tidal, the number of undulations of the surface increases as the speed gets more removed from the speeds of the tidal oscillations. We may therefore reasonably assume that for a simple form of disturbing potential, such as occurs in the long period terrestrial tides, the constants k will decrease with the corresponding free speeds. In applications we might perhaps assume G to consist of a finite number of terms, or even to be zero, but if it is uniformly convergent with respect to σ, then the limiting form of a tide will serve as an approximation to it, if its speed be small compared with the speeds of all free oscillations which can, at any instant, draw energy from the disturbing force.

This is what actually occurs in the cases originally worked out by Lamb and Hough. The slow free oscillations in those cases (called by Hough "Oscillations of the Second Class") have ζ proportional to circular functions of the longitude, while the disturbing potential is independent of the longitude, and consequently the constants k_n, k_{-n} are zero for all

those terms which could prevent the limiting forms from being approximations.

However, in order to see whether any particular observed tide is approximately a limiting form, the first test to make would be to see whether equation (14) were satisfied, and if so to test whether condition (12) were satisfied or not. But before applying the results of observations to these formulæ it is necessary to take account of all extraneous circumstances of effects comparable with those we have been considering. The only one we need consider is that of the yielding of the earth itself to tidal forces.

It is shown in the Appendix to this paper that for elastico-viscous yielding, due principally to the direct action of the disturbing forces, (14) will remain true if we take

$$\bar{\zeta} = -\frac{K}{g} e^{ik}\Omega, \tag{67}$$

where K, k are constants depending on the physical properties of the earth, k being zero for perfect elasticity, and K being unity for perfect rigidity.

If ζ_1, ζ_2 are the observed heights of a tide at two stations on the same continuous coast, then we obtain from (14) and (67),

$$\zeta_1 - \zeta_2 = -\frac{K}{g} e^{ik}(\Omega_1 - \Omega_2), \tag{68}$$

and the best method of applying the first of the above mentioned tests would probably be to use this formula for consecutive pairs of stations on the same continuous coast.

Both K and k would be derived from any such pair of observations, and these constants should be the same for each pair, and also be possible values for the physical constants they denote, or else the theory will not apply.

For the Lunar Fortnightly Tide we have*

$$\Omega = -gH'(\cos^2\theta - \tfrac{1}{3})\, e^{i(\sigma t - 2\xi)}, \tag{69}$$

where H', ξ are known and may be regarded as constant over a few months.

If then $$\zeta_1 = A_1 e^{i(\sigma t - \epsilon_1)}, \quad \zeta_2 = A_2 e^{i(\sigma t - \epsilon_2)}, \tag{70}$$

* See Thomson and Tait, *Natural Philosophy*, Art. 848.

we shall have from (68), (69) and (70)

$$A_1 e^{-\iota\epsilon_1} - A_2 e^{-\iota\epsilon_2} = KH' e^{i(k-2\xi)} (\cos^2\theta_1 - \cos^2\theta_2), \tag{71}$$

which gives $$\tan(k-2\xi) = -\frac{A_1 \sin\epsilon_1 - A_2 \sin\epsilon_2}{A_1 \cos\epsilon_1 - A_2 \cos\epsilon_2}, \tag{72}$$

and $$K^2 H'^2 (\cos^2\theta_1 - \cos^2\theta_2)^2 = A_1^2 + A_2^2 - 2A_1 A_2 \cos(\epsilon_1 - \epsilon_2). \tag{73}$$

Using the tables for the Indian ports given by Sir G. H. Darwin in Thomson and Tait's *Natural Philosophy* (2nd ed., pp. 456–458), and taking as pairs of stations, Bombay and Karwar, Karwar and Beypore, Madras and Vizagapatam, we soon see that the theory does not apply to the Lunar Fortnightly Tide in the Indian Ocean.

This may be due to the possibility, in this ocean, of free tidal oscillations of periods comparable with a fortnight which are capable of drawing energy from the moon's disturbing force; but it may perhaps be due to the inapplicability of (67).

As regards the determination of the yielding of the earth, perhaps the following suggestion may not be entirely out of the question.

The Lake Victoria Nyanza in Africa is in the neighbourhood of the equator, so that it has no effective rotation. Consequently the limiting forms of the tides will exist and be given by the equilibrium theory, while the slowest free period must be small compared with a day. Also the diurnal disturbing force is greatest at the equator, while the orientation of the lake is such as to enable this force to produce a maximum effect. In fact, if the moon's declination be about 15° the Lunar Diurnal Tide on the northern and southern shore (calculated on the equilibrium theory) appears to have an amplitude of about a centimetre. As this is about the same as the observed amplitude of the Oceanic Lunar Fortnightly Tide, it is conceivable that it could be isolated in observation from other changes and useful information derived.

APPENDIX.

Equations of Tidal Motion on a Yielding Nucleus.

When the motion of the solid earth is according to the equilibrium theory, the tides on it have been calculated by Sir G. H. Darwin* in two

* "On the Bodily Tides of Viscous and Semi-Elastic Spheroids, and on the Ocean Tides upon a Yielding Nucleus," *Phil. Trans. Roy. Soc.*, Vol. CLXX, Pt. 1, p. 1 (1879).

special cases. The first of these cases is that in which the equilibrium theory is used; the second is the case of a semi-diurnal tide in an equatorial canal. Both his results can be obtained as particular cases of the following, which is quite general both as regards the motion of the earth and ocean, provided only that both these motions satisfy the usual conditions of being infinitesimal and tidal.

Referred to axes which are rotating with uniform angular velocity, the general equations of hydrodynamics may be written quite accurately in the vectorial form

$$\frac{\partial \bar{u}}{\partial t}+2[\bar{\omega}.\bar{u}]=-\nabla P, \tag{74}^*$$

where $\bar{u}$, $\bar{\omega}$ are vectors, the first being the velocity of the fluid relative to the rotating axes, the second the total vorticity of the fluid. $[\bar{\omega}.\bar{u}]$ denotes the usual vector product of $\bar{\omega}$, $\bar{u}$, and ∇P the gradient of P, which itself is given by

$$P=\int\frac{dp}{\rho}+\tfrac{1}{2}|\bar{u}|^2+W-\tfrac{1}{2}\omega^2\varpi^2. \tag{75}$$

Here p is the pressure intensity, ρ the density of the fluid, $|\bar{u}|^2$ the square of the relative speed of the water, W the potential of the applied forces, ω the angular speed of the axes, and ϖ the distance from the axis of rotation.

Now let us use spherical polar coordinates r, θ, ϕ, and let the axes be rotating round the line $\theta=0$. Neglecting squares and products of relative velocities, we obtain, from (74),

$$\left.\begin{aligned}\frac{\partial u'}{\partial t}-2\omega v'\cos\theta &= -\frac{\partial P}{r\partial\theta}\\ \frac{\partial v'}{\partial t}+2\omega u'\cos\theta+2\omega w'\sin\theta &= -\frac{\partial P}{r\sin\theta\,\partial\phi}\\ \frac{\partial w'}{\partial t}-2\omega v'\sin\theta &= -\frac{\partial P}{\partial r}\end{aligned}\right\}, \tag{76}$$

where u', v', w' are the spherical polar components of $\bar{u}$, w' being radial.

Also to the same approximation, and for a constant density, we have, from (75),

$$P=\frac{p}{\rho}+W-\tfrac{1}{2}\omega^2r^2\sin^2\theta. \tag{77}$$

* This equation may be easily obtained from the equations in their Cartesian form, taking one of the axes as that of rotation.

At time t let the equation of the surface of the solid nucleus be

$$r = a + h_0 + \zeta_0,$$

and the equation of the surface of the water be

$$r = a + h_0 + \zeta_0 + h + \zeta,$$

where a is a constant, and h_0, ζ_0, h, ζ are functions of θ, ϕ, such that when the whole system is rotating in relative equilibrium, $\zeta_0 = \zeta = 0$. Thus h_0, h are independent of t, while ζ_0, ζ involve t. We take a so that

$$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} (h_0 + h) \sin\theta \, d\theta \, d\phi = 0.$$

Also, let u_0, v_0, w_0 be the components of the velocity of the surface of the nucleus, relative to the rotating axes, and u, v, w the components of the velocity of the water, relative to the surface of the nucleus. We suppose that the squares and products of these velocities may be neglected, and that the vertical components may always be neglected in comparison with the greatest values which the horizontal components can attain. We also suppose that ζ/h is small, that ζ_0/ζ is not large, and that for points inside the water $(r-a)/a$ is small.

We then obtain from the equations (76), on the assumption of a time factor $e^{i\sigma t}$,

$$\left.\begin{aligned} i\sigma(u+u_0) - 2\omega(v+v_0)\cos\theta &= -\frac{\partial P}{a\,\partial\theta} \\ i\sigma(v+v_0) + 2\omega(u+u_0)\cos\theta &= -\frac{\partial P}{a\sin\theta\,\partial\phi} \\ -2\omega(v+v_0)\sin\theta &= -\frac{\partial P}{\partial r} \end{aligned}\right\}. \tag{78}$$

Solving, algebraically, the first two of these equations, we obtain

$$\left.\begin{aligned} u+u_0 &= \frac{1}{\sigma^2 - 4\omega^2\cos^2\theta} \left\{ i\sigma \frac{\partial P}{a\,\partial\theta} + 2\omega\cos\theta \frac{\partial P}{a\sin\theta\,\partial\phi} \right\} \\ v+v_0 &= \frac{1}{\sigma^2 - 4\omega^2\cos^2\theta} \left\{ i\sigma \frac{\partial P}{a\sin\theta\,\partial\phi} - 2\omega\cos\theta \frac{\partial P}{a\,\partial\theta} \right\} \end{aligned}\right\}. \tag{79}$$

Now $$\bar{p} = p_0 + g\rho(\zeta + \zeta_0),$$

where p_0 is the (atmospheric) pressure on the free surface at time t, and $\bar{p}$ denotes the value of p at the surface $r = a + h_0 + h = \bar{r}$ (say). In this we may give to g (the acceleration due to gravity) the value which it has for free relative equilibrium.

Also,
$$W = W_0 + \Pi_0 + \Omega + \Pi,$$

where W_0 is the potential when the system is rotating in free relative equilibrium, Π_0 is the change in potential due to deformation of the nucleus, Ω is the external disturbing potential, and Π the change in potential due to deformation of the ocean.

Then we have, from (77) and the above,

$$\bar{P} = \frac{\bar{p}}{\rho} + \bar{W} - \tfrac{1}{2}\omega^2 \bar{r}^2 \sin^2\theta$$

$$= g(\zeta + \zeta_0) + \bar{W}_0 + \bar{\Pi}_0 + \bar{\Omega} + \bar{\Pi} - \tfrac{1}{2}\omega^2 \bar{r}^2 \sin^2\theta,$$

on taking p_0 to be constant and dropping it. But from the conditions of free relative equilibrium

$$\bar{W}_0 - \tfrac{1}{2}\omega^2 \bar{r}^2 \sin^2\theta = \text{constant}.$$

Therefore we may take

$$\bar{P} = g(\zeta + \zeta_0) + \bar{\Pi}_0 + \bar{\Omega} + \bar{\Pi}. \tag{80}$$

If now we substitute a for r in Π_0, Ω, Π, and use the resulting functions of θ, ϕ instead of $\bar{\Pi}_0$, $\bar{\Omega}$, $\bar{\Pi}$ in (80), we shall have a value for P which never differs by more than a small fraction of itself from $\bar{P}$, and this may be used in (79) as referring to *any* point in the water whose coordinates are θ, ϕ.

Now the equation of continuity may be written in the form

$$i\sigma\zeta = -\frac{1}{a\sin\theta}\left\{\frac{\partial}{\partial\theta}(hu\sin\theta) + \frac{\partial}{\partial\phi}(hv)\right\}.$$

Substituting in this from (79) and (80) we have

$$\frac{g}{a^2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{h\sin\theta}{\sigma^2 - 4\omega^2\cos^2\theta}\frac{\partial P}{\partial\theta}\right) + \frac{\partial}{\sin\theta\,\partial\phi}\left(\frac{h}{\sigma^2 - 4\omega^2\cos^2\theta}\frac{\partial P}{\partial\phi}\right)\right.$$
$$\left. + \frac{2\omega}{i\sigma}\left\{\frac{\partial}{\partial\theta}\left(\frac{h\cos\theta}{\sigma^2 - 4\omega^2\cos^2\theta}\right)\frac{\partial P}{\partial\phi} - \frac{\partial}{\partial\phi}\left(\frac{h\cos\theta}{\sigma^2 - 4\omega^2\cos^2\theta}\right)\frac{\partial P}{\partial\theta}\right\}\right] + P$$
$$= \frac{g}{i\sigma a\sin\theta}\left\{\frac{\partial}{\partial\theta}(hu_0\sin\theta) + \frac{\partial}{\partial\phi}(hv_0)\right\} + g\zeta_0 + \Pi_0 + \Omega + \Pi. \tag{81}$$

This then is the general equation of tidal motion on a rotating nucleus which is approximately spherical, when the motion of the ocean bed is regarded as known, and fulfils the same general conditions as regards

order of magnitude as the "tidal" motion of the surface of the ocean itself.

The boundary conditions are as follows:—

(i) For a vertical wall

$$\frac{1}{\sigma^2-4\omega^2\cos^2\theta}\left\{i\sigma\frac{\partial P}{\partial n}+2\omega\cos\theta\frac{\partial P}{\partial s}\right\}$$

must be equal to the outward normal motion of the boundary itself referred to the moving axes.

(ii) For a shelving beach P must be finite.

If now we neglect II and write

$$\bar{\zeta} = -\zeta_0 - \frac{1}{g}(\Pi_0+\Omega), \tag{82}$$

we shall have $$P = g(\zeta-\bar{\zeta}).$$

Application to Limiting Forms.

Taking steady motions in the solid nucleus to be impossible, the limiting form of its motion will be determined by the equilibrium theory. This means that we shall have $u_0 = v_0 = 0$, though in general ζ_0 and therefore Π_0 will not be zero. The theory of the preceding paper then holds providing that to Ω we add $g\zeta_0+\Pi_0$.

When II is neglected and $\bar{\zeta}$ is used, the equations of the paper hold as they stand, provided that $\bar{\zeta}$ is given by (82). In particular the equilibrium theory gives $\zeta = \bar{\zeta}+$constant, aud the equation (14) remains true.

When Ω is a surface harmonic of the second order, we have, for purely elastic yielding, the limiting form of ζ_0 given by*

$$\zeta_0 = -\frac{5\Omega}{2g+\dfrac{19\eta}{\rho_0 a}},$$

where η is the coefficient of rigidity, and ρ_0 the (uniform) density of the nucleus.

For elastico-viscous yielding we cannot use the limiting form as an approximation to the tides we wish to consider for the reason given in § 1. However, in calculating the deformation of the nucleus we may neglect

* See Thomson and Tait, *l.c.*

the inertia terms, and then finally the horizontal motion of the surface. The work is given in the paper of Darwin's already referred to, and, quoting the result, we have

$$\zeta_0 = -\frac{5\Omega}{2g}\,\frac{\cos\chi}{\cos\psi}\,e^{i(\sigma t+\psi-\chi)},$$

where $$\tan\psi = \sigma\tau, \quad \tan\chi = \sigma\tau\left(1+\frac{19\eta}{2g\rho_0 a}\right),$$

τ being the modulus of the time of relaxation of rigidity.

We have also $$\Pi_0 = \tfrac{3}{5}g\rho_0\zeta_0,$$

so that we may write $$\bar{\zeta} = -\frac{K}{g}\,e^{ik}\Omega, \tag{83}$$

where K, k depend on η and τ, being calculable from the above.

The distortion of the ocean bed, due to the tidal load itself, is here neglected.

ON THE LINEAR INTEGRAL EQUATION

By E. W. HOBSON.

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IN most of the literature in which the linear integral equation

$$f(s) = \phi(s) - \lambda \int_a^b K(s, t)\, \phi(t)\, dt$$

is the subject of consideration, somewhat narrow restrictions are placed upon the nature of the given function $f(s)$ and of the nucleus $K(s, t)$. The nucleus $K(s, t)$ is usually taken to be continuous except at points lying on a finite number of curves with continuously turning tangents, and such that any straight line parallel to either of the axes of s or t intersects these curves in a finite number of points only. The function $f(s)$ is usually taken to be continuous, and the continuous solution of the equation is then obtained by Fredholm's method. In the present communication a wider range is given to the functions $K(s, t)$, $f(s)$. The definite integrals employed are supposed throughout to be defined in accordance with the definition of Lebesgue, and the function $f(s)$ is restricted only to be a summable function. It is shown to be sufficient, in order that the solution of the equation may have only the same discontinuities as $f(s)$, that $K(s, t)$ be restricted in respect of its discontinuities in a much less stringent manner than that referred to above. It is shown that the only summable solution of the integral equation is that given by Fredholm's formula, the nucleus being limited in the square for which it is defined. A large part of the communication is concerned with the cases in which the nucleus is unlimited. Fredholm himself considered the case in which the nucleus and a finite number of the repeated nuclei are unlimited, the remaining repeated nuclei being limited. This case has been treated more fully by Poincaré and others, but is here treated with greater generality than hitherto, at least so far as I am aware. The theory of the canonical sub-groups of the resolvant of a limited nucleus, in the form given by Lalesco, has been here applied to the complete investigation of the case of an unlimited nucleus just referred to. As this

case is the one which actually arises in the application of the theory of integral equations to Dirichlet's problem and other problems in the theory of the potential function, it appears to be desirable that the extension of Fredholm's method to such cases should be fully investigated. Certain cases in which all the repeated nuclei are unlimited, but in which Fredholm's method is still applicable, have been here considered.

The Continuity of Integrals with respect to a Parameter.

1. With a view to a discussion of the nature of the solutions of linear integral equations the following theorems will prove useful.

(*a*) *If $K(s, t)$ is limited and summable in the square defined by $a \leqslant s \leqslant b$, $a \leqslant t \leqslant b$, and if for any fixed value s_0 of s the set of values of t for which $K(s, t)$ is not continuous with respect to s, for $s = s_0$, forms a set of linear measure zero, then $\int_a^b K(s, \xi)\, \phi(\xi)\, d\xi$ is a continuous function of s at $s = s_0$; where $\phi(t)$ is any summable function, whether limited or unlimited in the interval (a, b) of t.*

Those values of s for which $K(s, \xi)$ is not summable with respect to ξ, if such exist, form a set of linear measure zero. Such values may in the theorem and in the proof be simply disregarded. The theorem holds good whenever the integral exists.

Let M denote the upper limit of $|K(s, t)|$ in the square for which it is defined; and let E_N denote that set of points ξ for which

$$|\phi(\xi)| \geqslant N,$$

where N is some positive number. Let $s_1, s_2, \ldots, s_n, \ldots$ be a sequence of values of s which converges to the limit s_0.

The integral of $[K(s_n, \xi) - K(s_0, \xi)]\, \phi(\xi)$ with respect to ξ, taken over the set of points E_N, does not exceed in absolute value

$$2M \int_{(E_N)} |\phi(\xi)|\, d\xi;$$

and this is less than the arbitrarily chosen positive number $\frac{1}{2}\epsilon$, provided N is chosen so great, and therefore the measure of E_N so small, that

$$\int_{(E_N)} |\phi(\xi)|\, d\xi < \frac{\epsilon}{4M}.$$

To estimate the integral of $[K(s_n, \xi) - K(s_0, \xi)]\, \phi(\xi)$ taken over the set of points $C(E_N)$ complementary to E_N, let $u_n(\xi)$ denote $K(s_n, \xi)\, \phi(\xi)$, and let

$u(\xi)$ denote $K(s_0, \xi)\phi(\xi)$. Then $|u_n(\xi)|$ is less than some fixed positive number for all values of n and for all values of ξ that belong to the set $C(E_N)$; hence, in accordance with a known theorem, we have

$$\lim_{n\sim\infty}\int_{C(E_N)} u_n(\xi)\,d\xi = \int_{C(E_N)} u(\xi)\,d\xi,$$

since $u_n(\xi)$ converges to $u(\xi)$ for all values of ξ belonging to $C(E_N)$, with the possible exception of those belonging to a set of which the linear measure is zero. It follows that the absolute value of the integral of $[K(s_n, \xi)-K(s_0, \xi)]\,\phi(\xi)$ taken over the set $C(E_N)$ of values of ξ is less than $\frac{1}{2}\epsilon$, provided n is greater than some fixed integer m.

The numbers N, m being chosen as has been explained, we now see that

$$\left|\int_a^b [K(s_n, \xi)-K(s_0, \xi)]\,\phi(\xi)\,d\xi\right| < \epsilon,$$

provided $n > m$. Since ϵ is arbitrary, we see that

$$\lim_{n\sim\infty}\int_a^b K(s_n, \xi)\,\phi(\xi)\,d\xi = \int_a^b K(s_0, \xi)\,\phi(\xi)\,d\xi.$$

As this holds however the sequence of values of s which converges to s_0 is chosen, it has now been proved that

$$\int_a^b K(s, \xi)\,\phi(\xi)\,d\xi$$

is a function of s that is continuous at $s = s_0$.

2. In the theorem (a) we may replace the variable s by a pair of variables (s, t), and no essential change will be needed in the proof of the theorem. The sequence which converges to s_0 will be replaced by sequences $s_1, s_2, \ldots, s_n, \ldots$ converging to s_0, and $t_1, t_2, \ldots, t_n, \ldots$ converging to t_0; $u_n(\xi)$ will now denote $K(s_n, t_n, \xi)$, and $u(\xi)$ will denote $K(s, t, \xi)$. We have now the theorem :—

(b) *If $K(s, t, \xi)$ is limited and summable in the domain defined by $a \leqslant s \leqslant b$, $a \leqslant t \leqslant b$, $a \leqslant \xi \leqslant b$, and if for the values s_0, t_0 of s, t the values of ξ such that $K(s, t, \xi)$ is not continuous with respect to the plane domain (s, t), at (s_0, t_0), form at most a set of which the linear measure is zero, then $\int_a^b K(s, t, \xi)\,\phi(\xi)\,d\xi$ is continuous with respect to (s, t) at the point (s_0, t_0); where $\phi(\xi)$ is any summable function, limited or unlimited in the interval (a, b) of ξ.*

In case $K(s, t, \xi)$ is a function of the form $f(s, \xi)\,g(t, \xi)$, it is suffi-

cient, in order that $K(s, t, \xi)$ may be continuous with respect to (s, t), that $f(s, \xi)$ be continuous with respect to s, and $g(t, \xi)$ be continuous with respect to t. Accordingly we have the theorem :*—

(c) *If $f(s, \xi)$, $g(t, \xi)$ are limited and summable in the two-dimensional domains $a \leqslant s \leqslant b$, $a \leqslant \xi \leqslant b$ and $a \leqslant t \leqslant b$, $a \leqslant \xi \leqslant b$, and if $f(s, \xi)$ is continuous at $s = s_0$ with respect to s for all values of ξ with the possible exception of those belonging to a set of measure zero, and $g(t, \xi)$ is continuous at $t = t_0$ with respect to t for all values of ξ, with a similar exception, then $\int_a^b f(s, \xi)\, g(t, \xi)\, \phi(\xi)\, d\xi$ is continuous with respect to (s, t) at the point (s_0, t_0); where $\phi(\xi)$ is any summable function, whether it be limited or not in the interval (a, b).*

3. The particular case of the theorem (c) which arises when

$$f(s, \xi) = K(s, \xi), \qquad g(t, \xi) = K(\xi, t)$$

will be useful later. It follows from the theorem that

$$\int_a^b K(s, \xi)\, K(\xi, t)\, \phi(\xi)\, d\xi,$$

or, in particular,
$$\int_a^b K(s, \xi)\, K(\xi, t)\, d\xi,$$

is continuous with respect to (s, t) at a point (s_0, t_0), if $K(s, t)$ is continuous with respect to s at $s = s_0$ for all values of t which do not belong to some set of linear measure zero, and if it is also continuous with respect to t at $t = t_0$ for all values of s that do not belong to some set of linear measure zero.

If $K(s, t)$ is a function defined for $a \leqslant s \leqslant b$, $a \leqslant t \leqslant b$, such that, for each value of s, it is continuous with respect to s for every value of t with the possible exception of a set of linear measure zero (dependent in general on the particular value of s), and also such that a similar condition holds for each value of t as regards continuity with respect to t, the discontinuities of $K(s, t)$ will be said to be *regularly distributed*. We now have, by employing the theorems (a), (c) above :—

(d) *If $K(s, t)$ is limited and summable in the domain $a \leqslant s \leqslant b$, $a \leqslant t \leqslant b$, and if it have its discontinuities regularly distributed, the*

* The particular case of these theorems (b) and (c) which arises when the function $\phi(\xi)$ is taken to be a constant has been given by Prof. W. H. Young, in an article "On Parametric Integration," see the *Monatshefte für Mathematik und Physik* for 1910, p. 141.

integral $\int_a^b K(s, \xi)\, K(\xi, t)\, d\xi$, *or, more generally,* $\int_a^b K(s, \xi)\, K(\xi, t)\, \phi(\xi)\, d\xi$, *where* $\phi(\xi)$ *is a limited or unlimited summable function, is continuous with respect to the two-dimensional domain* (s, t) *throughout the whole plane domain. Moreover,* $\int_a^b K(s, \xi)\, \phi(\xi)\, d\xi$ *is a continuous function of* s *for all the values of* s, *and* $\int_a^b K(\xi, t)\, \phi(\xi)\, d\xi$ *is a continuous function of* t *for all the values of* t.

As in the case of the former theorems, those values of s for which $K(s, \xi)$ is not summable with respect to ξ, and those of t for which $K(\xi, t)$ is not summable with respect to ξ, if such exist, may, as they form sets of linear measure zero, be disregarded. The theorems hold whenever the integrals exist.

By some writers* the discontinuities of $K(s, t)$ are said to be regularly distributed in the square for which the function is defined if all the discontinuities with respect to (s, t) lie on a finite number of curves with continuously turning tangents, no one of which is met by a line parallel to the axis of s or of t in more than a finite number of points. The theorem (d) shews that the more general definition given above introduces a sufficient restriction on the function for the purposes of the theory of integral equations. It will be observed that the term "regularly distributed," as here used, refers only to the discontinuities of the function with respect to the variables s and t separately, whereas the narrower definition hitherto employed has reference to the discontinuities with respect to (s, t).

In case the discontinuities of $K(s, t)$ with respect to (s, t) form a set of linear measure zero on every straight line parallel to either axis, the above integrals exist without any exception.

The Method of Successive Substitution.

4. The well known method of successive substitution, when applied to the equation

$$f(s) = \phi(s) - \lambda \int_a^b K(s, t)\, \phi(t)\, dt,$$

* See, for example, Bôcher's Tract, *An Introduction to the Study of Integral Equations*, p. 3.

gives us

$$\phi(s) = f(s) + \lambda \int_a^b [K(s, t) + \lambda K_2(s, t) + \ldots + \lambda^{n-1} K_n(s, t)] f(t)\, dt$$
$$+ \lambda^{n+1} \int_a^b \int_a^b \ldots K(s, \xi_1) K(\xi_1, \xi_2) \ldots K(\xi_n, t) \phi(t)\, dt\, d\xi_1 d\xi_2 \ldots d\xi_n,$$

where $K_2(s, t)$, $K_3(s, t)$, ... denote the successive repeated nuclei corresponding to $K(s, t)$; provided this expression has a definite meaning.

We shall in the first instance suppose that $K(s, t)$ is limited and summable in the square for which it is defined, and we shall suppose that $f(s)$ is summable in the interval (a, b), whether it be limited in that interval or not.

In accordance with a known theorem,* since $K(s, t)$ is summable in the square $a \leqslant s \leqslant b$, $a \leqslant t \leqslant b$, it is summable with respect to s for each value of t, with the possible exception of a set of values of linear measure zero; and a similar statement holds good as regards summability with respect to t. The function $K(s, t)$ being limited, it follows that $\int_a^b K(s, t) f(t)\, dt$ exists for each value of s, with the possible exception of a linear set of measure zero. The function

$$K_2(s, t) \equiv \int_a^b K(s, \xi) K(\xi, t)\, d\xi,$$

exists for all values of s and t with the possible exception of those belonging to sets of linear measure zero. Moreover $K_2(s, t)$ is a continuous function with respect to (s, t), in case the discontinuities of $K(s, t)$ are regularly distributed. It is easily seen that similar statements apply to $K_3(s, t)$, $K_4(s, t)$, ...

The series $\qquad K(s, t) + \lambda K_2(s, t) + \lambda^2 K_3(s, t) + \ldots$

converges uniformly and absolutely for all values of s, t, λ, such that the terms of the series have a definite meaning, and such that

$$0 \leqslant |\lambda| \leqslant |\lambda_1| < \frac{1}{M(b-a)};$$

where M denotes the upper limit of $|K(s, t)|$ in the square for which it is defined. We denote the sum of the series by $-\bar{K}(s, t)$.

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* See my paper on "Some Fundamental Properties of Lebesgue Integrals," in the *Proceedings*, Ser. 2, Vol. 8 (1909), p. 25.

It follows that, whether $f(t)$ is limited or not, the series

$$\int_a^b K(s, t) f(t)\,dt + \lambda \int_a^b K_2(s, t) f(t)\,dt + \lambda^2 \int_a^b K_3(s, t) f(t)\,dt + \dots$$

converges to
$$-\int_a^b \bar{K}(s, t) f(t)\,dt,$$

uniformly for all values of s and λ for which the terms of the series have a definite meaning, such that

$$0 \leqslant |\lambda| \leqslant |\lambda_1|.$$

It will be shewn that the integral equation is satisfied by

$$\phi(s) = f(s) - \lambda \int_a^b \bar{K}(s, t) f(t)\,dt,$$

for
$$0 \leqslant |\lambda| \leqslant |\lambda_1|,$$

which determines $\phi(s)$ for every value of s, with the possible exception of a set of linear measure zero.

Since

$$\phi(t) = f(t) + \lambda \int_a^b K(t, t') f(t')\,dt' + \lambda^2 \int_a^b K_2(t, t') f(t')\,dt' + \dots,$$

we may, on account of the uniform convergence of the series with respect to (t, λ), apply term by term integration with respect to t, after multiplying both sides by the limited function $K(s, t)$. We have thus

$$\int_a^b K(s, t)\,\phi(t)\,dt = \int_a^b K(s, t) f(t)\,dt + \lambda \int_a^b K_2(s, t) f(t)\,dt$$
$$+ \lambda^2 \int_a^b K_3(s, t) f(t)\,dt + \dots$$
$$= \frac{\phi(s) - f(s)}{\lambda};$$

remembering that the order of repeated integration in a Lebesgue integral is immaterial.

It has thus been shewn that the value of $\phi(s)$ assumed above satisfies the integral equation. The following theorem has now been established:—

(e) *If the nucleus $K(s, t)$ of the integral equation*

$$f(s) = \phi(s) - \lambda \int_a^b K(s, t)\,\phi(t)\,dt$$

be limited and summable in the square for which it is defined, then if $f(s)$ *be any summable function, limited or unlimited, the equation is satisfied for every value of* λ, *such that*

$$0 \leqslant |\lambda| \leqslant \frac{1}{M(b-a)},$$

by
$$\phi(s) = f(s) - \lambda \int_a^b \bar{K}(s, t) f(t)\, dt,$$

where $-\bar{K}(s, t)$ *is the sum-function of the series*

$$K(s, t) + \lambda K_2(s, t) + \lambda^2 K_3(s, t) + \dots .$$

The value of $\phi(s)$ *is determinate for every value of* s, *with the possible exception of those of a set of linear measure zero. In case* $K(s, t)$ *has its discontinuities regularly distributed,* $\bar{K}(s, t)$ *has the same points of discontinuity with respect to* (s, t), *and* $\int_a^b \bar{K}(s, t) f(t)\, dt$ *is a continuous function; thus in this case* $\phi(s)$, $f(s)$ *have the same points of discontinuity, and if* $f(s)$ *is continuous, so also is* $\phi(s)$.

5. There is one case, in which the nucleus and all the repeated nuclei are unlimited, to which the method may be readily extended. Let us suppose that $K(s, t)$ is of the form $\mu(s)\,\nu(t)\,P(s, t)$, where $P(s, t)$ is a limited summable function, and $\mu(s)$, $\nu(t)$ are unlimited, but such that $\mu(s)\,\nu(s)$ is a summable function in the interval (a, b) of s.

Let
$$\int_a^b |\mu(s)\,\nu(s)|\, ds = A,$$

and let the upper limit of $|P(s, t)|$ be B. We have

$$K_2(s, t) = \mu(s)\,\nu(t) \int_a^b \nu(\xi)\, P(s, \xi)\, \mu(\xi)\, P(\xi, t)\, d\xi$$
$$= \mu(s)\,\nu(t)\, P_2(s, t),$$

where
$$|P_2(s, t)| \leqslant AB^2.$$

Similarly
$$K_3(s, t) = \int_a^b K_2(s, \xi)\, K(\xi, t)\, d\xi$$
$$= \mu(s)\,\nu(t) \int_a^b \nu(\xi)\, P_2(s, \xi)\, \mu(\xi)\, P(\xi, t)\, d\xi$$
$$= \mu(s)\,\nu(t)\, P_3(s, t),$$

where
$$|P_3(s, t)| \leqslant A^2B^3.$$
In general
$$K_n(s, t) = \mu(s)\,\nu(t)\,P_n(s, t),$$
where
$$|P_n(s, t)| \leqslant A^{n-1}B^n.$$
We have
$$-\bar{K}(s, t) = \mu(s)\,\nu(t)\,\{P(s, t)+\lambda P_2(s, t)+\lambda^2 P_3(s, t)+\ldots\},$$
where the series converges uniformly for all the values of s, t, λ, such that
$$0 \leqslant |\lambda| \leqslant |\lambda_1| < 1/AB.$$

Let it be assumed that $f(s)$, whether it be limited or not, is such that $f(s)\,\nu(s)$ is a summable function. We see then that
$$-\int_a^b \bar{K}(s, t) f(t)\,dt$$
$$= \mu(s)\left\{\int_a^b f(t)\,\nu(t)\,P(s, t)\,dt+\lambda\int_a^b f(t)\,\nu(t)\,P_2(s, t)\,dt\right.$$
$$\left.+\lambda^2\int_a^b f(t)\,\nu(t)\,P_3(s, t)\,dt+\ldots\right\},$$
where the series in the bracket on the right-hand side converges uniformly for all values of s and λ, such that
$$0 \leqslant |\lambda| \leqslant |\lambda_1| < 1/AB.$$
The integral equation is satisfied by
$$\phi(s) = f(s)-\lambda\int_a^b \bar{K}(s, t)\,f(t)\,dt$$
$$= f(s)+\lambda\int_a^b K(s, t)\,f(t)\,dt+\lambda^2\int_a^b K_2(s, t)\,f(t)\,dt+\ldots,$$
provided
$$0 \leqslant |\lambda| < 1/AB;$$
for we have
$$\int_a^b \phi(t)\,K(s, t)\,dt$$
$$= \mu(s)\left\{\int_a^b \nu(t)\,f(t)\,P(s, t)\,dt+\lambda\int_a^b \nu(t)\,f(t)\,P_2(s, t)\,dt\right.$$
$$\left.+\lambda^2\int_a^b \nu(t)\,f(t)\,P_3(s, t)\,dt+\ldots\right\},$$
where the series on the right-hand side converges uniformly for all values of s and of λ, such that
$$0 \leqslant |\lambda| \leqslant |\lambda_1| < 1/AB.$$

The expression is equivalent to $\frac{\phi(s)-f(s)}{\lambda}$, and therefore the integral equation is satisfied by the value of $\phi(s)$ assumed.

We have thus:—

(*f*) *If the nucleus $K(s, t)$ of the integral equation*

$$f(s) = \phi(s) - \lambda \int_a^b K(s, t)\, \phi(t)\, dt$$

is of the form $\mu(s)\,\nu(t)\,P(s, t)$, where $P(s, t)$ is a limited summable function, and one or both of the functions $\mu(s)$, $\nu(t)$ are unlimited but such that $\mu(s)\,\nu(s)$ is summable in the interval (a, b); then the equation is satisfied by

$$\phi(s) = f(s) - \lambda \int_a^b \bar{K}(s, t)\, f(t)\, dt,$$

for all sufficiently small values of $|\lambda|$, where $-\bar{K}(s, t)$ denotes the sum of the series $K(s, t) + \lambda K_2(s, t) + \ldots$; provided $f(s)$ is a summable function, and such that $f(s)\,\nu(s)$ is also summable in the interval (a, b) of **s**.

As an example of the application of this theorem we may take

$$K(s, t) = \frac{P(s, t)}{|s-\alpha|^p \; t-\beta\,|^q},$$

where p and q are both less than 1, and α, β are both in the interval (a, b), but are unequal. In this case $f(s)\;|s-\beta|^{-q}$ must be summable.

Again, we may take

$$K(s, t) = \frac{P(s, t)}{|s-\alpha|^p \, |t-\alpha|^q},$$

where $p+q<1$, and α is in the interval (a, b). As before $f(s)\;|s-\alpha|^{-q}$ must be summable.

6. Let us suppose that $K(s, t)$, $K_2(s, t)$, ..., $K_{n-1}(s, t)$ are all unlimited in the square for which $K(s, t)$ is defined, but that $K_n(s, t)$ is a limited function. It will further be assumed that $\int_a^b |K(s, t)|\, ds$ is limited for all values of t in (a, b), and that $\int_a^b |K(s, t)|\, dt$ is limited for all values of s in (a, b). Let α, β denote the upper limits of these integrals.

We have $$K_{n+1}(s, t) = \int_a^b K_n(s, \xi)\, K(\xi, t)\, d\xi,$$

and therefore $$|K_{n+1}(s, t)| \leqslant aM_n,$$

where M_n denotes the upper limit of $|K_n(s, t)|$. Again we find that

$$K_{n+2}(s, t) \leqslant a^2 M_n, \quad K_{n+3}(s, t) \leqslant a^3 M_n, \quad \dots .$$

It follows that the series

$$\lambda^{n-1} K_n(s, t) + \lambda^n K_{n+1}(s, t) + \dots$$

is uniformly convergent with respect to (s, t, λ), provided

$$0 \leqslant |\lambda| \leqslant \lambda_1 < 1/a.$$

If we had expressed the repeated integral $K_{n+1}(s, t)$ in the form

$$\int_a^b K(s, \xi)\, K_n(\xi, t)\, d\xi,$$

and proceeded as before, it would have been shown that the series is uniformly convergent if $0 \leqslant |\lambda| \leqslant \lambda_2 < 1/\beta$. The radius of convergence of the series is therefore not less than the larger of the two numbers $1/a$, $1/\beta$.

We shall assume that $f(s)$ is either limited and summable, or that it is unlimited and summable, and also such that

$$K(s, t) f(t), \; K_2(s, t) f(t), \; \dots, \; K_{m-1}(s, t) f(t)$$

are all summable in the interval (a, b) of t, for each value of s (with the possible exception of a set of values of linear measure zero).

Subject to these assumptions, it can be verified that, if $\phi(s)$ denote the sum of the series

$$f(s) + \lambda \int_a^b K(s, t) f(t)\, dt + \dots + \lambda^{n-2} \int_a^b K_{n-1}(s, t) f(t)\, dt$$
$$+ \lambda^{n-1} \int_a^b K_n(s, t) f(t)\, dt + \dots,$$

this value satisfies the integral equation, for every value of s for which the terms have a definite meaning. We have thus obtained the following theorem :—

(*g*) *If there exists a repeated nucleus* $K_n(s, t)$ *that is limited, and if* $\int_a^b |K(s, t)|\, ds$ *exists, and has* a *for its upper limit for all values of* t, *and* $\int_a^b |K(s, t)|\, dt$ *exists, and has* β *for its upper limit for all values of* s,

then the solution of the integral equation in the form

$$\phi(s) = f(s) - \lambda \int_a^b \bar{K}(s, t) f(t)\, dt,$$

for all values of $|\lambda|$ *which do not exceed the greater of the two numbers* $1/a$, $1/\beta$ *is obtained by taking* $-\bar{K}(s, t)$ *as the sum of the series*

$$K(s, t) + \lambda K_1(s, t) + \dots ;$$

provided $f(s)$, *if it be not limited, is such that*

$$K(s, t) f(t),\ K_2(s, t) f(t),\ \dots,\ K_{n-1}(s, t) f(t)$$

are all summable with respect to t, *for every value of* s (*with the possible exception of values belonging to a set of linear measure zero*).

As an example of the application of this theorem, the well known case

$$K(s, t) = \frac{P(s, t)}{|s-t|^a},$$

where $a < 1$, may be cited.

7. It has been shewn by E. Schmidt, by a method depending upon the use of Schwarz's inequality, that

$$[K_n(s, t)]^2 \leqslant \left[\int_a^b \int_a^b \{K(s, t)\}^2\, ds\, dt\right]^{n-2} \int_a^b \{K(s, t)\}^2\, ds \int_a^b \{K(s, t)\}^2\, dt,$$

where it must be assumed that the integrals on the right-hand side have a definite meaning. This may be applied to the method of successive substitution in certain cases when $K(s, t)$ is unlimited.

If we assume that $\{K(s, t)\}^2$ is summable in the square for which $K(s, t)$ is defined, and also that $\int_a^b \{K(s, t)\}^2\, ds$ has a finite upper limit for all values of t in the interval (a, b), and further that $\int_a^b \{K(s, t)\}^2\, ds$ has a finite upper limit for all the values of s, we see that the series

$$K(s, t) + \lambda K_2(s, t) + \lambda^2 K_3(s, t) + \dots$$

converges uniformly for all values of s, t, λ, such that

$$0 \leqslant |\lambda| \leqslant |\lambda_1| < \left\{\int_a^b \int_a^b \{K(s, t)\}^2\, ds\, dt\right\}^{-\frac{1}{2}},$$

and thus that the method is applicable.

This theorem is less general than the theorem (g) of § 6, because it is

applicable only in cases in which $K_2(s, t)$ is limited in the fundamental square.

Fredholm's Solution of the Integral Equation.

8. Fredholm's solution of the integral equation is given by

$$\phi(s) = f(s) - \lambda \int_a^b \bar{K}(s, t) f(t)\, dt,$$

where
$$-\bar{K}(s, t) = \frac{D\left(\lambda \begin{smallmatrix} s \\ t \end{smallmatrix}\right)}{D(\lambda)},$$

the quotient of two integral functions of λ, expressible in the forms

$$D\left(\lambda \begin{smallmatrix} s \\ t \end{smallmatrix}\right) = K(s, t) - \lambda \int_a^b K\begin{pmatrix} s, & \xi_1 \\ t, & \xi_1 \end{pmatrix} d\xi_1 + \dots$$
$$+ \frac{(-1)^m \lambda^m}{m!} \int_a^b \int_a^b \dots \int_a^b K\begin{pmatrix} s, & \xi_1, & \xi_2, & \dots, & \xi_m \\ t, & \xi_1, & \xi_2, & \dots, & \xi_m \end{pmatrix} d\xi_1 d\xi_2 \dots d\xi_m + \dots,$$

$$D(\lambda) = 1 - \lambda \int_a^b K(\xi_1, \xi_1)\, d\xi_1 + \dots$$
$$+ \frac{(-1)^m \lambda^m}{m!} \int_a^b \int_a^b \dots K\begin{pmatrix} \xi_1, & \xi_2, & \dots, & \xi_m \\ \xi_1, & \xi_2, & \dots, & \xi_m \end{pmatrix} d\xi_1 d\xi_2 \dots d\xi_m + \dots,$$

the notation $K\begin{pmatrix} \xi_1 \xi_2 \dots \xi_m \\ \xi_1' \xi_2' \dots \xi_m' \end{pmatrix}$ being used for the determinant

$$\begin{vmatrix} K(\xi_1, \xi_1'), & K(\xi_2, \xi_1'), & \dots, & K(\xi_m, \xi_1') \\ K(\xi_1, \xi_2'), & K(\xi_2, \xi_2'), & \dots, & K(\xi_m, \xi_2') \\ \dots & \dots & \dots & \dots \\ K(\xi_1, \xi_m'), & K(\xi_2, \xi_m'), & \dots, & K(\xi_m, \xi_m') \end{vmatrix}.$$

It is known to be sufficient to ensure that $D\left(\lambda \begin{smallmatrix} s \\ t \end{smallmatrix}\right)$, $D(\lambda)$ are integral functions, that $K(s, t)$ should be a summable function which is limited in the square for which it is defined. It simplifies the statements to assume that $K(s, t)$ is summable with respect to s for every value of t, and with respect to t for every value of s. This assumption will here be made, though it is not necessary for the essential validity of the results obtained. It is clear, by employing the usual method of verification, that in this case there is a solution of the integral equation, given by Fredholm's

formula, provided λ is not a zero of $D(\lambda)$, and provided $f(s)$ is a summable function, limited or unlimited. We have, by taking out the term $K(s, t)$ multiplied by its minor,

$$K\begin{pmatrix} s, & \xi_1, & \xi_2, & \ldots, & \xi_m \\ t, & \xi_1, & \xi_2, & \ldots, & \xi_m \end{pmatrix} = K(s, t)\, K\begin{pmatrix} \xi_1, & \xi_2, & \ldots, & \xi_m \\ \xi_1, & \xi_2, & \ldots, & \xi_m \end{pmatrix} + E_m(s, t),$$

where $E_m(s, t)$ denotes the determinant

$$\begin{vmatrix} 0, & K(\xi_1, t), & K(\xi_2, t), & \ldots, & K(\xi_m, t) \\ K(s, \xi_1), & K(\xi_1, \xi_1), & K(\xi_2, \xi_1), & \ldots, & K(\xi_m, \xi_1) \\ K(s, \xi_2), & K(\xi_1, \xi_2), & K(\xi_2, \xi_2), & \ldots, & K(\xi_m, \xi_2) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ K(s, \xi_m), & K(\xi_1, \xi_m), & K(\xi_2, \xi_m), & \ldots, & K(\xi_m, \xi_m) \end{vmatrix}.$$

On substitution in the expression for $D\left(\lambda \begin{smallmatrix} s \\ t \end{smallmatrix}\right)$, we have

$$D\left(\lambda \begin{matrix} s \\ t \end{matrix}\right) = D(\lambda)\, K(s, t) + \Sigma \frac{(-1)^m \lambda^m}{m!} \int_a^b \int_a^b \ldots E_m(s, t)\, d\xi_1 d\xi_2 \ldots d\xi_m.$$

Hence the solution of the integral equation takes the form

$$\phi(s) = f(s) + \lambda \int_a^b K(s, t) f(t)\, dt$$
$$+ \frac{1}{D(\lambda)} \Sigma \frac{(-1)^m \lambda^m}{m!} \int_a^b \int_a^b \ldots E_m(s, t) f(t)\, dt\, d\xi_1 d\xi_2 \ldots d\xi_m.$$

By applying Hadamard's theorem to the determinant $E_m(s, t)$ it is seen that, for any value of λ which is not a zero of $D(\lambda)$, the series in the last term of the right hand side converges uniformly with respect to (s, t). The determinant $E_m(s, t)$, when expounded, has the form

$$\Sigma \pm K(s, \xi_p)\, K(\xi_q, t)\, K(\xi_{a_1}, \xi_{\beta_1})\, K(\xi_{a_2}, \xi_{\beta_2}) \ldots K(\xi_{a_{m-1}}, \xi_{\beta_{m-1}}),$$

where the indices $q, a_1, a_2, \ldots, a_{m-1}$ are all different, and also the indices $p, \beta_1, \beta_2, \ldots, \beta_{m-1}$ are all different. On substitution in the expression for $\phi(s)$ we see that the coefficient of $\dfrac{1}{D(\lambda)} \dfrac{(-1)^m \lambda^m}{m!}$ consists of two kinds of terms. The terms of the first of these kinds, corresponding to $q = p$, are of the form

$$A \int_a^b \int_a^b K(s, \xi_p)\, K(\xi_p, t)\, f(t)\, d\xi_p\, dt,$$

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where A denotes a constant. In case the discontinuities of $K(s, t)$ are regularly distributed, we see by theorem (d), that

$$\int_a^b K(s, \xi_p) K(\xi_p, t) d\xi_p$$

is a continuous function of (s, t), and it then follows that the term is a continuous function of s. The terms of the second kind are of the form

$$A \int_a^b \int_a^b \dots \int_a^b K(s, \xi_p) K(\xi_p, \xi_{p'}) K(\xi_{p'}, \xi_{p''}) \dots K(\xi_{p^{(r)}}, \xi_q) K(\xi_q, t) f(t) \, dt \, d\xi_p \dots d\xi_{p^{(r)}}.$$

On the same assumption, that discontinuities of $K(s, t)$ are regularly distributed, we see that $\int_a^b K(s, \xi_p) K(\xi_p, \xi_{p'}) d\xi_p$ is a continuous function of $(s, \xi_{p'})$, say $C(s, \xi_{p'})$; then $\int_a^b C(s, \xi_{p'}) K(\xi_{p'}, \xi_{p''}) d\xi_{p'}$ is continuous relative to $(s, \xi_{p''})$, and so on. The term ultimately reduces to the form

$$\int_a^b \int_a^b F(s, \xi_q) K(\xi_q, t) f(t) \, dt \, d\xi,$$

where $F(s, \xi_q)$ is continuous with respect to (s, ξ_q); this reduces to

$$\int_a^b G(s, t) f(t) \, dt,$$

where $G(s, t)$ is continuous relative to (s, t), and therefore by the theorem (d) the term is continuous relative to s, for any summable function $f(t)$. Since the series in the expression for $\phi(s)$ converges uniformly with respect to s, for any fixed value of λ, and since its terms are continuous, it follows that its sum-function is continuous. Therefore, when $K(s, t)$ has its discontinuities regularly distributed, $\phi(s)$ and $f(s)$ have only the same points of discontinuity; and in particular $\phi(s)$ is continuous when $f(s)$ is so.

It can be shewn that Fredholm's solution is the only possible solution which is summable in the interval (a, b). For, let $w(s)$ be any summable solution of the integral equation, thus

$$f(t) = w(t) - \lambda \int_a^b K(t, \xi) w(\xi) d\xi.$$

Multiplying the equation by $\bar{K}(s, t)$, and integrating with respect to t

through the interval (a, b), we have

$$\int_a^b \bar{K}(s, t) f(t)\, dt = \int_a^b \bar{K}(s, t)\, w(t)\, dt - \lambda \int_a^b \int_a^b \bar{K}(s, t)\, K(t, \xi)\, w(\xi)\, d\xi dt$$

$$= \int_a^b w(t) \left\{ \bar{K}(s, t) - \lambda \int_a^b \bar{K}(s, \xi)\, K(\xi, t)\, d\xi \right\} dt$$

$$= -\int_a^b K(s, t)\, w(t)\, dt,$$

in virtue of the fundamental relation

$$K(s, t) + \bar{K}(s, t) = \lambda \int_a^b \bar{K}(s, \xi)\, K(\xi, t)\, d\xi.$$

Since now $\int_a^b K(t, \xi)\, w(\xi)\, d\xi$ has been shewn to be equal to

$$-\int_a^b \bar{K}(t, \xi)\, f(\xi)\, d\xi,$$

we see that
$$w(t) = f(t) - \lambda \int_a^b \bar{K}(t, \xi)\, f(\xi)\, d\xi,$$

or
$$w(s) = f(s) - \lambda \int_a^b \bar{K}(s, t)\, f(t)\, dt,$$

and thus $w(s)$ is identical with Fredholm's solution.

The following results have now been established:—

If the nucleus $K(s, t)$ is limited in the square for which it is defined, and $f(s)$ is any summable function, limited or unlimited in the interval (a, b), then for any value of λ that is not a characteristic value, the only summable solution of the integral equation

$$f(s) = \phi(s) - \lambda \int_a^b K(s, t)\, \phi(t)\, dt$$

is that of Fredholm. Moreover, in case the discontinuities of $K(s, t)$ are regularly distributed, the solution has only the same points of discontinuity as $f(s)$, and is continuous if $f(s)$ be so.

A non-summable solution of the integral equation may exist which will not be given by Fredholm's method. Assuming that, in the equation, the integral is restricted to be of the Lebesgue type, it may happen that, although $\phi(t)$ is not summable in the interval (a, b), $K(s, t)\, \phi(t)$ is so.

For example, the equation

$$1 = \phi(s) - \int_0^1 K(s, t)\, \phi(t)\, dt,$$

where

$$K(s, t) = t^s - \tfrac{1}{2} t^{\frac{1}{2}}$$

admits of the non-summable solution

$$\phi(s) = 1/s.$$

An example of such a solution has been given by Bôcher for the case of Volterra's equation ;* he states that such solutions must necessarily be non-integrable.

The Integral Equation with Unlimited Nucleus.

9. We proceed to consider cases in which the nucleus $K(s, t)$ of the integral equation is unlimited. Let it be assumed that one of the repeated nuclei $K_n(s, t)$ is limited, $K_{n-1}(s, t)$, $K_{n-2}(s, t)$, ... being all unlimited.

The method of successive substitutions discussed in § 4 shows that

$$\phi(s) = f(s) + \lambda \int_a^b K(s, t) f(t)\, dt + \lambda^2 \int_a^b K_2(s, t) f(t)\, dt + \ldots$$
$$+ \lambda^{n-1} \int_a^b K_{n-1}(s, t) f(t)\, dt + \lambda^n \int_a^b K_n(s, t)\, \phi(t)\, dt,$$

it being assumed that $K(s, t)$ and the limited or unlimited summable function $f(t)$ are such that

$$K(s, t) f(t), \; K_2(s, t) f(t), \; \ldots, \; K_{n-1}(s, t) f(t)$$

are all summable in the interval (a, b) of t, for all (or almost all) the values of s.

If the integral equation has a solution, that solution must satisfy the equation

$$\phi(s) - \lambda^n \int_a^b K_n(s, t)\, \phi(t)\, dt = f_n(s), \tag{A}$$

where $f_n(s)$ denotes

$$f(s) + \lambda \int_a^b K(s, t) f(t)\, dt + \ldots + \lambda^{n-1} \int_a^b K_{n-1}(s, t) f(t)\, dt.$$

---

* See Bôcher's tract, p. 17. It has been remarked by Prof. W. H. Young that the solutions contemplated by Bôcher are examples rather of unlimited than of discontinuous function; see his paper "On Integral Equations," *Quarterly Journal of Math.*, Vol. XLI, p. 184.

If it be now assumed that the integrals

$$\int_a^b K(s, t)\,dt, \quad \int_a^b K_2(s, t)\,dt, \quad \ldots, \quad \int_a^b K_{n-1}(s, t)\,dt,$$

$$\int_a^b K(s, t) f(t)\,dt, \quad \int_a^b K_2(s, t) f(t)\,dt, \quad \ldots, \quad \int_a^b K_{n-1}(s, t) f(t)\,dt$$

all exist, and are summable functions of s, it follows that $f_n(s)$ is a summable function of s, and that the equation (A) has a single solution given by Fredholm's expression, provided λ^n is not a characteristic value.

Conversely it will be shewn that this solution $\phi(s)$ of (A) also satisfies the given integral equation.

Let the functions $\phi_1(s), \phi_2(s), \ldots, \phi_n(s)$ be defined by

$$\phi_1(s) = \lambda \int_a^b K(s, t)\,\phi(t)\,dt + f(s),$$

$$\phi_2(s) = \lambda \int_a^b K(s, t)\,\phi_1(t)\,dt + f(s),$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\phi_n(s) = \lambda \int_a^b K(s, t)\,\phi_{n-1}(t)\,dt + f(s).$$

It will be shewn that, in virtue of the hypotheses made as to the nature of the functions $f(t)$, $K(s, t)$, these functions $\phi_1(s), \phi_2(s), \ldots, \phi_n(s)$ are all summable in the interval (a, b) of s.

The function $\phi(t)$, being given by Fredholm's formula, is equal to $f_n(t) + \chi(t)$, where $\chi(t)$ is a limited summable function of t, for a fixed value of λ. Since

$$\int_a^b K(s, t)\,\phi(t)\,dt = \int_a^b K(s, t) f_n(t)\,dt + \int_a^b K(s, t)\,\chi(t)\,dt,$$

it follows from the hypotheses made that $\int_a^b K(s, t)\,\phi(t)\,dt$ is a summable function of s; therefore $\phi_1(s)$ is a summable function of s.

Again, we have

$$\phi_2(s) = f(s) + \lambda \int_a^b K(s, t) f(t)\,dt + \lambda^2 \int_a^b K_2(s, t)\,\phi(t)\,dt,$$

and it then follows that $\phi_2(s)$ is a summable function of s. Similarly it

may be shewn that $\phi_3(s) \dots \phi_n(s)$ are all summable functions. We have

$$\phi_n(s) = f(s) + \lambda \int_a^b K(s, t) f(t) dt + \dots + \lambda^{n-1} \int_a^b K_{n-1}(s, t) f(t) dt$$
$$+ \lambda^n \int_a^b K_n(s, t) \phi(t) dt$$
$$= f_n(s) + \lambda^n \int_a^b K_n(s, t) \phi(t) dt;$$

and therefore $$\phi_n(s) = \phi(s).$$

By adding the equations which define $\phi_1(s)$, $\phi_2(s)$, ..., $\phi_n(s)$, we see that

$$\frac{\phi_1(s) + \phi_2(s) + \dots + \phi_n(s)}{n} = f(s) + \lambda \int_a^b \frac{\phi(t) + \phi_1(t) + \dots + \phi_{n-1}(t)}{n} K(s, t) dt,$$

and since $$\phi(t) = \phi_n(t),$$

it follows that $\dfrac{\phi_1(s) + \phi_2(s) + \dots + \phi_n(s)}{n}$ satisfies the given integral equation, and it therefore satisfies the integral equation (A). Since (A) has a unique summable solution, it follows that

$$\phi(s) = \frac{\phi_1(s) + \phi_2(s) + \dots + \phi_n(s)}{n},$$

and thus that $\phi(s)$ satisfies the given integral equation.

The following theorem has now been established:—

If $K(s, t)$, $K_2(s, t)$, ..., $K_{n-1}(s, t)$ are unlimited, and $K_n(s, t)$ is limited in the square $a \leqslant s \leqslant b$, $a \leqslant t \leqslant b$, and if $\int_a^b K_r(s, t) dt$ exists as a summable function of s, for $r = 1, 2, 3, \dots, n-1$, then if $f(s)$ be any summable function, limited or unlimited, such that $\int_a^b K_r(s, t) f(t) dt$ is a summable function of s, for $r = 1, 2, 3, \dots, n-1$, the integral equation

$$f(s) = \phi(s) - \lambda \int_a^b K(s, t) \phi(t) dt$$

has a unique summable solution given by

$$\phi(s) = f_n(s) + \lambda^n \int_a^b \frac{D_n\left(\lambda^n \begin{matrix} s \\ t \end{matrix}\right)}{D_n(\lambda^n)} f_n(t) dt,$$

where $D_n\left(\lambda^n \begin{matrix} s \\ t \end{matrix}\right)$ *denotes the integral function*

$$K_n(s,\, t)-\lambda^n\int_a^b K_n\begin{pmatrix} s, & \xi_1 \\ t, & \xi_1 \end{pmatrix} d\xi_1+\frac{\lambda^{2n}}{2!}\int_a^b\int_a^b K_n\begin{pmatrix} s, & \xi_1, & \xi_2 \\ t, & \xi_1, & \xi_2 \end{pmatrix} d\xi_1 d\xi_2-\ldots,$$

and $D_n(\lambda^n)$ *denotes the integral function*

$$1-\lambda^n\int_a^b K_n(\xi,\, \xi_1)\, d\xi_1+\frac{\lambda^{2n}}{2!}\int_a^b\int_a^b K_n\begin{pmatrix} \xi_1, & \xi_2 \\ \xi_1, & \xi_2 \end{pmatrix} d\xi_1 d\xi_2-\ldots,$$

and $K_n\begin{pmatrix} \xi_1, & \xi_2, & \ldots, & \xi_r \\ \xi_1', & \xi_2', & \ldots, & \xi_r' \end{pmatrix}$ *denotes the determinant*

$$\begin{vmatrix} K_n(\xi_1,\, \xi_1), & K_n(\xi_1,\, \xi_2), & \ldots, & K_n(\xi_1,\, \xi_r) \\ K_n(\xi_2,\, \xi_1), & K_n(\xi_2,\, \xi_2), & \ldots, & K_n(\xi_2,\, \xi_r) \\ \ldots & \ldots & \ldots & \ldots \\ K_n(\xi_r,\, \xi_1), & K_n(\xi_r,\, \xi_2), & \ldots, & K_n(\xi_r,\, \xi_r) \end{vmatrix}.$$

The function $f_n(s)$ *denotes*

$$f(s)+\lambda\int_a^b K(s,\, t)\, f(t)\, dt+\ldots+\lambda^{n-1}\int_a^b K_{n-1}(s,\, t)\, f(t)\, dt.$$

The value of λ *must not be a zero of* $D_n(\lambda^n)$.

10. Using the notation $\bar{K}_n(s,\, t,\, \lambda^n)$ for the reciprocal function of $K_n(s,\, t)$ when the parameter is λ^n, we have

$$\bar{K}_n(s,\, t,\, \lambda^n) = -\frac{D_n\left(\lambda^n \begin{matrix} s \\ t \end{matrix}\right)}{D_n(\lambda^n)}.$$

Thus the solution of the integral equation

$$F(s) = \phi(s)-\lambda^n\int_a^b K_n(s,\, t)\, \phi(t)\, dt,$$

is

$$\phi(s) = F(s)-\lambda^n\int_a^b \bar{K}_n(s,\, t,\, \lambda^n)\, F(t)\, dt.$$

Writing the solution of the equation

$$f(s) = \phi(s)-\lambda\int_a^b K(s,\, t)\, \phi(t)\, dt$$

in the form

$$\phi(s) = f(s)-\lambda\int_a^b \bar{K}(s,\, t)\, f(t)\, dt,$$

we obtain from the expression obtained in § 9 the relation between $\bar{K}(s, t)$ and $\bar{K}_n(s, t, \lambda^n)$.

It will be convenient to use the notation $U_s\chi(s)$ for $\int_a^b K(s, t)\chi(t)\,dt$, the symbol U_s denoting therefore an operation. We have then also

$$U_s^2\chi(s) = U_s\int_a^b K(s, t)\,\chi(t)\,dt = \int_a^b K(s, t')\,dt'\int_a^b K(t', t)\,\chi(t)\,dt$$

$$= \int_a^b K_2(s, t)\,\chi(t)\,dt,$$

and generally
$$U_s^r\chi(s) = \int_a^b K_r(s, t)\,\chi(t)\,dt.$$

In a similar manner we denote by $V_s\chi(s)$, $\int_a^b K(t, s)\,\chi(t)\,dt$, and therefore $V_s^r\chi(s)$ denotes $\int_a^b K_n(t, s)\,\chi(t)\,dt$. With this notation the integral equation can be written in the form

$$f(s) = \phi(s) - \lambda U_s\phi(s),$$

and the associated equation in the form

$$f(s) = \phi(s) - \lambda V_s\phi(s).$$

We have now, from § 9,

$$\bar{K}(s, t) = -\{K(s, t) + \lambda K_2(s, t) + \lambda^2 K_3(s, t) + \ldots + \lambda^{n-2}K_{n-1}(s, t)\}$$
$$+ \lambda^{n-1}\{1 + \lambda V_t + \lambda^2 V_t^2 + \ldots + \lambda^{n-1}V_t^{n-1}\}\,\bar{K}_n(s, t, \lambda^n).$$

If $G_n(s, t, \lambda^n)$ denotes that part of $\bar{K}(s, t, \lambda^n)$ which consists of the sum of terms that contain negative powers of $\lambda^n - \lambda_1^n$, where λ_1^n is a characteristic value, or zero of $D_n(\lambda^n)$, then that part of $\bar{K}(s, t)$ which becomes infinite when λ has any one of the values $\lambda_1, \lambda_1\omega, \lambda_1\omega^2, \ldots, \lambda_1\omega^{n-1}$, where ω is a primitive n-th root of unity, is that part of

$$\lambda^{n-1}\{1 + \lambda V_t + \lambda^2 V_t^2 + \ldots + \lambda^{n-1}V_t^{n-1}\}\,G(s, t, \lambda^n) \text{ or } H(s, t, \lambda),$$

which consists of negative powers of $\lambda - \lambda_1, \lambda - \omega\lambda_1, \ldots, \lambda - \omega^{n-1}\lambda_1$. The remaining part of $\bar{K}(s, t)$ remains finite for the values $\lambda_1, \omega\lambda_1, \omega^2\lambda_1, \ldots$ of λ.

We find easily that

$$\frac{1}{n\lambda^{n-1}}\{H(s,t,\lambda)+\omega H(s,t,\omega\lambda)+\omega^2 H(s,t,\omega^2\lambda)+\ldots+\omega^{n-1}H(s,t,\omega^{n-1}\lambda)\} = G(s,t,\lambda^n);$$

thus $G(s, t, \lambda^n)$ is expressed in terms of the function $H(s, t, \lambda)$.

In accordance with Lalesco's theory* of the canonical forms of the resolvant $G(s, t, \lambda^n)$, that resolvant is expressible as the sum of a number of canonical sub-groups, each one of which is of the form

$$\frac{C_1(s,t)}{\lambda^n-\lambda_1^n}+\frac{C_2(s,t)}{(\lambda^n-\lambda_1^n)^2}+\ldots+\frac{C_p(s,t)}{(\lambda^n-\lambda_1^n)^p},$$

where
$$C_1(s,t) = \phi_1(s)\psi_1(t)+\phi_2(s)\psi_2(t)+\ldots+\phi_p(s)\psi_p(t),$$
$$C_2(s,t) = a_1\phi_1(s)\psi_2(t)+a_2\phi_2(s)\psi_3(t)+\ldots+a_{p-1}\phi_{p-1}(s)\psi_p(t),$$
$$C_3(s,t) = a_1a_2\phi_1(s)\psi_3(t)+\ldots+a_{p-2}a_{p-1}\phi_{p-2}(s)\psi_p(t),$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$C_p(s,t) = a_1a_2\ldots a_p\phi_1(s)\psi_p(t).$$

The sets of principal functions

$$\phi_1(s),\ \phi_2(s),\ \ldots,\ \phi_p(s) \quad \text{and} \quad \psi_1(t),\ \psi_2(t),\ \ldots,\ \psi_p(t)$$

form a biorthogonal system. Of these only $\phi_1(s)$ is a fundamental function, *i.e.*, a solution of the equation

$$\phi(s)-\lambda_1^n\int_a^b K_n(s,t)\,\phi(t)\,dt = 0.$$

Only $\psi_p(s)$ is a solution of the reciprocal equation

$$\psi(s)-\lambda_1^n\int_a^b K(t,s)\,\psi(t)\,dt = 0.$$

If, in the expression for $H(s, t, \lambda)$ in terms of $G(s, t, \lambda^n)$, we substitute all the sub-groups of the above form of which $G(s, t, \lambda^n)$ is composed, we obtain an expression for $H(s, t, \lambda)$ which consists of terms each of which involves a negative power of $\lambda^n-\lambda_1^n$. These terms may be expressed by resolution into partial fractions, each as the sum of terms involving negative powers of $\lambda-\lambda_1$, $\lambda-\omega\lambda_1$, ..., $\lambda^n-\omega^{n-1}\lambda_1$.

It thus appears that the part of $\bar{K}(s, t)$ that becomes infinite when λ

* See his *Introduction à la théorie des équations intégrales*, Chapter II.

has one of the values $\lambda_1, \omega\lambda_1, \omega^2\lambda_1, \ldots, \omega^{n-1}\lambda_1$, is of the form

$$\left[\frac{C_1^{(1)}(s, t)}{\lambda-\lambda_1}+\frac{C_2^{(1)}(s, t)}{(\lambda-\lambda_1)^2}+\ldots+\frac{C_p^{(1)}(s, t)}{(\lambda-\lambda_1)^p}\right]+\left[\frac{C_1^{(2)}(s, t)}{\lambda-\omega\lambda_1}+\frac{C_2^{(2)}(s, t)}{(\lambda-\omega\lambda_1)^2}+\ldots\right]+\ldots$$
$$+\left[\frac{C_1^{(n)}(s, t)}{\lambda-\omega^{n-1}\lambda_1}+\frac{C_2^{(n)}(s, t)}{(\lambda-\omega^{n-1}\lambda_1)^2}+\ldots\right],$$

where all the functions C are expressed in terms of the principal functions $\phi(s)$, $\psi(t)$, of $\bar{K}_n(s, t, \lambda^n)$.

The function $\bar{K}(s, t)$ must satisfy the equation

$$K(s, t)+\bar{K}(s, t) = \lambda\int_a^b \bar{K}(s, t')\, K(t', t)\,dt',$$

which is a necessary condition that

$$\phi(s) = f(s)-\lambda\int_a^b K(s, t) f(t)\, dt,$$

may satisfy the integral equation

$$f(s) = \phi(s)-\lambda\int_a^b K(s, t)\, \phi(t)\, dt.$$

It is clear that the parts of $\bar{K}(s, t)$ which involve negative powers of $\lambda-\lambda_1, \lambda-\omega\lambda_1, \ldots, \lambda-\omega^{n-1}\lambda_1$ must each separately satisfy this equation. It follows that Lalesco's theory of the canonical sub-groups must be applicable to

$$\frac{C_1^{(1)}(s, t)}{\lambda-\lambda_1}+\frac{C_2^{(1)}(s, t)}{(\lambda-\lambda_1)^2}+\ldots+\frac{C_n^{(1)}(s, t)}{(\lambda-\lambda_1)^n},$$

as also to each of the other such portions of $\bar{K}(s, t)$.

This part of $\bar{K}(s, t)$ is therefore expressible as the sum of a number of canonical sub-groups, each one of which is of the form

$$\frac{B_1(s, t)}{\lambda-\lambda_1}+\frac{B_2(s, t)}{(\lambda-\lambda_1)^2}+\ldots+\frac{B_q(s, t)}{(\lambda-\lambda_1)^q},$$

where
$$B_1(s, t) = \Phi_1(s)\,\Psi_1(t)+\Phi_2(s)\,\Psi_2(t)+\ldots+\Phi_q(s)\,\Psi_q(t),$$
$$B_2(s, t) = \beta_1\Phi_1(s)\,\Psi_2(t)+\ldots+\beta_{q-1}\Phi_{q-1}(s)\,\Psi_q(t),$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$B_q(s, t) = \beta_1\beta_2\ldots\beta_{q-1}\Phi_1(s)\,\Psi_q(t).$$

The sets of functions

$$\Phi_1(s),\ \Phi_2(s),\ \ldots,\ \Phi_q(s) \quad \text{and} \quad \Psi_1(t),\ \Psi_2(t),\ \ldots,\ \Psi_q(t)$$

are principal functions forming a biorthogonal system. The function $\Phi_1(s)$ is the only one of these functions that satisfies the equation

$$\Phi(s) - \lambda_1 U_s \Phi(s) = 0;$$

and $\Psi_p(s)$ is the only one which satisfies the equation

$$\Psi(s) - \lambda_1 V_s \Psi(s) = 0.$$

We proceed to form that part of $\bar{K}_n(s, t, \lambda^n)$ that corresponds to one of the canonical sub-groups of $\bar{K}(s, t)$. Corresponding to

$$\frac{B_1(s, t)}{\lambda - \lambda_1} + \frac{B_2(s, t)}{(\lambda - \lambda_1)^2} + \ldots + \frac{B_q(s, t)}{(\lambda - \lambda_1)^q},$$

the part of $\bar{K}_n(s, t, \lambda^n)$ is

$$\frac{1}{n\lambda^{n-1}}\Bigg[\ B_1(s, t)\left\{\frac{1}{\lambda - \lambda_1} + \frac{\omega}{\omega\lambda - \lambda_1} + \ldots + \frac{\omega^{n-1}}{\omega^{n-1}\lambda - \lambda_1}\right\}$$

$$+ B_2(s, t)\left\{\frac{1}{(\lambda - \lambda_1)^2} + \frac{\omega}{(\omega\lambda - \lambda_1)^2} + \ldots + \frac{\omega^{n-1}}{(\omega^{n-1}\lambda - \lambda_1)^2}\right\}$$

$$+ \ldots$$

$$+ B_q(s, t)\left\{\frac{1}{(\lambda - \lambda_1)^q} + \frac{\omega}{(\omega\lambda - \lambda_1)^q} + \ldots + \frac{\omega^{n-1}}{(\omega^{n-1}\lambda - \lambda_1)^q}\right\}\Bigg].$$

Employing the identity

$$\frac{1}{(\lambda - \lambda_1)^s} + \frac{\omega}{(\omega\lambda - \lambda_1)^s} + \ldots + \frac{\omega^{n-1}}{(\omega^{n-1}\lambda - \lambda_1)^s} = \frac{n\lambda^{n-1}}{(s-1)!}\,\frac{d^{s-1}}{d\lambda_1^{s-1}}\,\frac{1}{\lambda^n - \lambda_1^n},$$

where $s = 1, 2, 3, \ldots, q$, this expression may be written in the form

$$\frac{B_1(s, t)}{\lambda^n - \lambda_1^n} + \frac{1}{1!} B_2(s, t)\frac{d}{d\lambda_1}\,\frac{1}{\lambda^n - \lambda_1^n} + \frac{1}{2!} B_3(s, t)\frac{d^2}{d\lambda_1^2}\,\frac{1}{\lambda^n - \lambda_1^n} + \ldots$$

$$+ \frac{1}{(q-1)!} B_q(s, t)\frac{d^{q-1}}{d\lambda_1^{q-1}}\,\frac{1}{\lambda^n - \lambda_1^n}.$$

It will be observed that the first term in this expression is the only one which involves the first power of $\dfrac{1}{\lambda^n - \lambda_1^n}$, and that the last term is the only one which involves the q-th power of $\dfrac{1}{\lambda^n - \lambda_1^n}$; the other terms

contain more than one power of $\dfrac{1}{\lambda^n-\lambda_1^n}$. Each canonical sub-group corresponding to any one of the values $\lambda_1, \omega\lambda_1, \ldots, \omega^{n-1}\lambda_1$ of λ in $\bar{K}(s, t)$ gives rise to a part of $\bar{K}_n(s, t, \lambda^n)$ of the above form. The sum of all the parts of $\bar{K}_n(s, t, \lambda^n)$ so obtained must be equivalent to the expression of the same resolvant as the sum of canonical sub-groups. It will, however, be shewn that

$$\frac{B_1(s, t)}{\lambda^n-\lambda_1^n}+\frac{1}{1!}B_2(s, t)\frac{d}{d\lambda_1}\frac{1}{\lambda^n-\lambda_1^n}+\ldots+\frac{1}{(q-1)!}B_q(s, t)\frac{d^{q-1}}{d\lambda_1^{q-1}}\frac{1}{\lambda^n-\lambda_1^n},$$

is itself equivalent to one of the canonical sub-groups of $\bar{K}_n(s, t, \lambda^n)$. To see this we observe that this expression satisfies the condition that it is the resolvant of the nucleus

$$\frac{B_1(s, t)}{\lambda_1^n}+\frac{1}{1!}B_2(s, t)\frac{d}{d\lambda_1}\frac{1}{\lambda_1^n}+\frac{1}{2!}B_3(s, t)\frac{d^2}{d\lambda_1^2}\frac{1}{\lambda_1^n}+\ldots$$
$$+\frac{1}{(q-1)!}B_q(s, t)\frac{d}{d\lambda_1^{q-1}}\frac{1}{\lambda_1^n},$$

with the parameter λ^n. Denoting this last expression by $k_n(s, t)$, and the former one by $\bar{k}_n(s, t, \lambda^n)$, it is sufficient to shew that

$$k_n(s, t)+\bar{k}_n(s, t, \lambda^n) = \lambda^n\int_a^b k_n(s, t')\,\bar{k}_n(t', t, \lambda^n)\,dt'.$$

In accordance with Lalesco's theory, the functions B satisfy the conditions

$$\int_a^b B_\alpha(s, t')\,B_\beta(t', t)\,dt' = B_{\alpha+\beta-1}(s, t),$$

for $1 \leqslant \alpha+\beta-1 \leqslant q$. Forming the expression for

$$\lambda^n\int_a^b k_n(s, t')\,\bar{k}_n(t', t, \lambda^n)\,dt',$$

the coefficient of $B_p(s, t)$ becomes, in virtue of the relation quoted,

$$\frac{1}{\lambda_1^n}\frac{1}{(p-1)!}\frac{d^{p-1}}{d\lambda_1^{p-1}}\frac{1}{\lambda^n-\lambda_1^n}+\frac{1}{1!}\frac{d}{d\lambda_1}\frac{1}{\lambda_1^n}\frac{d^{p-2}}{d\lambda_1^{p-2}}\frac{1}{\lambda^n-\lambda_1^n}$$
$$+\frac{1}{2!}\frac{d^2}{d\lambda_1^2}\frac{1}{\lambda_1^n}\frac{d^{p-3}}{d\lambda_1^{p-3}}\frac{1}{\lambda^n-\lambda_1^n}+\ldots$$
$$+\frac{1}{(p-1)!}\frac{d^{p-1}}{d\lambda_1^{p-1}}\frac{1}{\lambda_1^n}\frac{1}{\lambda^n-\lambda_1^n},$$

which is equal to $$\frac{1}{(p-1)!}\frac{d^{p-1}}{d\lambda_1^{p-1}}\frac{1}{\lambda_1^n(\lambda^n-\lambda_1^n)},$$

or to $$\frac{\lambda^n}{(p-1)!}\frac{d^{p-1}}{d\lambda_1^{p-1}}\left(\frac{1}{\lambda_1^n}+\frac{1}{\lambda^n-\lambda_1^n}\right),$$

and this is the coefficient of $B_p(s, t)$ in $k_n(s, t)+\bar{k}^n(s, t, \lambda^n)$. It has accordingly been verified that

$$\frac{B_1(s, t)}{\lambda^n-\lambda_1^n}+\frac{1}{1!}B_2(s, t)\frac{d}{d\lambda_1}\frac{1}{\lambda^n-\lambda_1^n}+\ldots+\frac{1}{(q-1)!}B_q(s, t)\frac{d^{q-1}}{d\lambda_1^{q-1}}\frac{1}{\lambda^n-\lambda_1^n},$$

is a resolvant, for the parameter λ^n. This resolvant can be expressed as a canonical group, or else as the sum of a number of canonical sub-groups. But the latter case cannot arise, because since the expression contains a term in $\frac{1}{(\lambda^n-\lambda_1^n)^q}$, the order of one of the sub-groups must be q, and if there were other sub-groups the total number of principal functions in s would exceed q, being the sum of the orders of all the sub-groups, and this cannot be the case, because the total number of linearly independent functions of s involved in the expression for the resolvant is equal to q. Therefore the above resolvant is reducible to a single canonical sub-group in $\bar{K}_n(s, t, \lambda^n)$.

It must therefore be reducible to the form

$$\frac{C_1(s, t)}{\lambda^n-\lambda_1^n}+\frac{C_2(s, t)}{(\lambda^n-\lambda_1^n)^2}+\ldots+\frac{C_q(s, t)}{(\lambda^n-\lambda_1^n)^q},$$

where the functions $C_1, C_2, \ldots, C_q$ are expressed in terms of principal functions $\phi_1(s), \ldots, \phi_q(s), \psi_1(s), \ldots, \psi_q(s)$.

On equating the terms in $\frac{1}{\lambda^n-\lambda_1^n}$, $\frac{1}{(\lambda^n-\lambda_1^n)^q}$, we have

$$\Phi_1(s)\Psi_1(t)+\Phi_2(s)\Psi_2(t)+\ldots+\Phi_q(s)\Psi_q(t)=\phi_1(s)\psi_1(t)+\ldots+\phi_q(s)\psi_q(t),$$

and $$\beta_1\beta_2\ldots\beta_{q-1}\Phi_1(s)\Psi_q(t)=a_1a_2\ldots a_{q-1}\phi_1(s)\psi_q(t);$$

from which it follows that the fundamental functions $\phi_1(s)$, $\Phi_1(s)$ must be identical, as also the fundamental solutions $\psi_q(s)$, $\Psi_q(s)$ of the reciprocal equations. The other principal functions $\phi_2(s), \ldots, \phi_q(s)$ can be expressed as linear functions of $\Phi_1(s), \ldots, \Phi_p(s)$; and a similar statement applies to $\psi_1(s), \psi_2(s), \ldots, \psi_{q-1}(s)$.

It has now been shewn that:—

To each canonical sub-group in $\bar{K}(s, t)$ *there corresponds a single*

canonical sub-group in $\bar{K}_n(s, t, \lambda^n)$; *the fundamental solutions of the integral equation, and its reciprocal in the one case being identical with the fundamental solutions of the integral equation and its reciprocal in the other case. The corresponding canonical sub-groups have the same order.*

11. Let it now be still assumed that all the repeated nuclei $K_1(s, t)$, $K_2(s, t)$, ..., $K_{n-1}(s, t)$ are unlimited, but that $K_n(s, t)$ is limited in the fundamental square. It will also be assumed, as in § 6, that $\int_a^b |K(s, t)|\,ds$ and $\int_a^b |K(s, t)|\,dt$ are limited functions. The trace $\int_a^b K_r(s, s)\,ds$ corresponding to $K_r(s, t)$ being denoted by k_r, we see that k_n, k_{n+1}, ... are all finite.

Let the function $Q_n(\lambda)$, defined for all values of λ, be such that, for sufficiently small values of $|\lambda|$, it is the sum-function of the series

$$-\frac{k_n}{n}\lambda^n-\frac{k_{n+1}}{n+1}\lambda^{n+1}-\frac{k_{n+2}}{n+2}\lambda^{n+2}-\dots,$$

which has a radius of convergence >0. The function $Q_n(\lambda)$ outside the circle of convergence is determined as the analytical continuation of the sum-function. It will be shewn that $e^{Q_n(\lambda)}$ and $e^{Q_n(\lambda)}\dfrac{D_n\left(\lambda^n \begin{matrix} s \\ t \end{matrix}\right)}{D_n(\lambda^n)}$ are both integral functions of λ. This theorem has been established by Poincaré* in the special case in which each pole of $\bar{K}_n(s, t, \lambda^n)$ is of the first order, and in which there is only one fundamental function corresponding to each such pole. The theorem will here be established, independently of any such restrictions, by means of the results developed in § 10.

We have, for sufficiently small values of $|\lambda|$,

$$-\frac{dQ_n(\lambda)}{d\lambda}=k_n\lambda^{n-1}+k_{n+1}\lambda^n+\dots$$

$$=\lambda^{n-1}\sum_{p=0}^{\infty}\lambda^{np}k_{n+np}+\lambda^n\sum_{p=0}^{\infty}\lambda^{np}k_{n+np+1}+\dots+\lambda^{2n-2}\sum_{p=0}^{\infty}\lambda^{np}k_{2n-1+np}.$$

* See "Remarques diverses sur l'équation de Fredholm," in the *Acta Mathematica*, Vol. 33, 1910.

It follows that the value of $-\dfrac{dQ_n(\lambda)}{d\lambda}$, for all values of λ, is given by

$$\lambda^{n-1}\int_a^b \frac{D_n\left(\lambda^n \begin{smallmatrix} s \\ s \end{smallmatrix}\right)}{D_n(\lambda^n)}\,ds + \lambda^n \int_a^b\int_a^b \frac{D_n\left(\lambda^n \begin{smallmatrix} s \\ t' \end{smallmatrix}\right)}{D_n(\lambda^n)} K_1(t', s)\,ds\,dt'$$

$$+\sum_{q=2}^{q=n-1} \lambda^{n-1+q} \int_a^b\int_a^b \frac{D_n\left(\lambda^n \begin{smallmatrix} s \\ t' \end{smallmatrix}\right)}{D_n(\lambda^n)} K_q(t', s)\,ds\,dt',$$

provided it be assumed that

$$\int_a^b |K_1(s, t)|\,ds,\quad \int_a^b |K_2(s, t)|\,ds,\quad \ldots,\quad \int_a^b |K_{n-1}(s, t)|\,ds$$

are all limited functions of t.

Hence we have

$$-\frac{dQ_n(\lambda)}{d\lambda} = \lambda^{n-1}\int_a^b \left[(1+\lambda V_t + \lambda^2 V_t^2 + \ldots + \lambda^{n-1} V_t^{n-1}) \frac{D_n\left(\lambda^n \begin{smallmatrix} s \\ t \end{smallmatrix}\right)}{D_n(\lambda^n)}\right]_{t=s} ds.$$

In accordance with the procedure in § 9 for finding that part of $\bar{K}(s, t)$ which becomes infinite when λ^n has the value λ_1^n, a zero of $D_n(\lambda^n)$, we now see that the part of $\dfrac{dQ_n(\lambda)}{d\lambda}$ that corresponds to a canonical sub-group of $\bar{K}(s, t)$ is

$$\int_a^b \left[\frac{\phi_1(s)\,\psi_1(s) + \ldots + \phi_q(s)\,\psi_q(s)}{\lambda-\lambda_1} + \frac{\beta_1\phi_1(s)\,\psi_2(s) + \ldots + \beta_{q-1}\phi_{q-1}(s)\,\psi_q(s)}{(\lambda-\lambda_1)^2} + \ldots\right] ds,$$

and this is equal to $\dfrac{q}{\lambda-\lambda_1}$. Here we have taken λ_1 to be the characteristic value for $\bar{K}(s, t)$ corresponding to the characteristic value λ_1^n for $\bar{K}_n(s, t, \lambda^n)$. A similar result would hold for a canonical sub-group for which $\omega\lambda_1$ or any of the numbers $\omega^2\lambda_1, \ldots, \omega^{n-1}\lambda_1$ is the characteristic value. The sum of all the integers q taken for all the sub-groups which belong to all the characteristic values $\lambda_1, \omega\lambda_1, \ldots, \omega^{n-1}\lambda_1$ is the sum of the orders of all the canonical sub-groups belonging to $\bar{K}_n(s, t, \lambda^n)$ for the characteristic value λ_1^n, and is therefore the degree of

multiplicity of the zero λ_1^n of the function $D_n(\lambda^n)$, in accordance with Lalesco's theory.

The complete value of $\dfrac{dQ_n(\lambda)}{d\lambda}$ consists of the sum, taken for all values of λ_1, of terms of the form

$$\frac{p_1}{\lambda-\lambda_1}+\frac{p_2}{\lambda-\omega\lambda_1}+\ldots+\frac{p_n}{\lambda-\omega^{n-1}\lambda_1},$$

where $p_1, p_2, \ldots, p_n$ are integers, any of which may be zero, and such that $p_1+p_2+\ldots+p_n$ is the degree of multiplicity of the zero λ_1^n of $D_n(\lambda^n)$, together with a function which has no singularities for any finite value of λ. It follows that $e^{Q_n(\lambda)}$ is of the form

$$e^{G(\lambda)}\Pi\left(1-\frac{\lambda}{\lambda_1}\right)^{p_1}e^{p_1 u(\lambda\,\lambda_1)}\left(1-\frac{\lambda}{\omega\lambda_1}\right)^{p_2}e^{p_2 u(\lambda\,\omega\lambda_1)}\ldots\left(1-\frac{\lambda}{\omega^{n-1}\lambda_1}\right)^{p_n}e^{p_n u(\lambda\,\omega^{n-1}\lambda_n)},$$

and it is therefore an integral function of λ.

To show that $e^{Q_n(\lambda)}\dfrac{D_n\left(\lambda^n\,{s \atop t}\right)}{D_n(\lambda^n)}$ is an integral function of λ, we observe that if a_1 is the order of infinity of $\lambda-\lambda_1$ in $\dfrac{D_n\left(\lambda^n\,{s \atop t}\right)}{D_n(\lambda^n)}$, there must be a canonical sub-group in the expression for $\bar{K}(s, t)$ of order a_1, and therefore $p_1 \geqslant a_1$; it follows then that λ_1 is not an infinity of $e^{Q_n(\lambda)}\dfrac{D_n\left(\lambda^n\,{s \atop t}\right)}{D_n(\lambda^n)}$, and similarly that $\omega\lambda_1, \omega^2\lambda_1, \ldots, \omega^{n-1}\lambda_1$ are not infinities of that function. Since $e^{Q_n(\lambda)}\dfrac{D_n\left(\lambda^n\,{s \atop t}\right)}{D_n(\lambda^n)}$ has no infinities for finite values of λ, and all its zeros are of integral order, it is an integral function of λ.

Since

$$\bar{K}(s, t) = -\left[K(s, t)+\lambda K_2(s, t)+\ldots+\lambda^{n-2}K_{n-1}(s, t)\right]$$

$$-\lambda^{n-1}\{1+\lambda V_t+\lambda^2 V_t^2+\ldots+\lambda^{n-1}V_t^{n-1}\}\frac{e^{Q_n(\lambda)}\dfrac{D_n\left(\lambda^n\,{s \atop t}\right)}{D_n(\lambda^n)}}{e^{Q_n(\lambda)}},$$

it now follows that $\bar{K}(s, t)$ can be expressed as the quotient of two integral functions of λ. Hence we have the following theorem :—

If $K_1(s, t), K_2(s, t), \ldots, K_{n-1}(s, t)$ be unlimited, but $K_n(s, t)$ is limited in the fundamental square, and if

$$\int_a^b |K(s, t)|\,dt, \quad \int_a^b |K(s, t)|\,ds, \quad \int_a^b |K_2(s, t)|\,ds, \quad \ldots, \quad \int_a^b |K_{n-1}(s, t)|\,ds$$

are all limited functions, the resolvant $\bar{K}(s, t)$ of the integral equation

$$f(s) = \phi(s) - \lambda \int_a^b K(s, t)\,\phi(t)\,dt$$

is the quotient of two integral functions of λ.

12. In the ordinary case in which $K_1(s, t), K_2(s, t), \ldots$ are all limited functions, and thus $k_1, k_2, \ldots$ are all finite, we have

$$e^{-k_1\lambda - \frac{1}{2}k_2\lambda^2 - \frac{1}{3}k_3\lambda^3 - \ldots} = 1 - \lambda \int_a^b K(s_1, s_1)\,ds_1 + \frac{\lambda^2}{2!}\int_a^b\int_a^b K\begin{pmatrix} s_1, & s_2 \\ s_1, & s_2 \end{pmatrix} ds_1\,ds_2 + \ldots$$
$$+ \frac{(-1)^m \lambda^m}{m!}\int_a^b \ldots \int_a^b K\begin{pmatrix} s_1, & s_2, & \ldots, & s_m \\ s_1, & s_2, & \ldots, & s_m \end{pmatrix} ds_1\,ds_2 \ldots ds_m$$
$$+ \ldots .$$

Let $K_0\begin{pmatrix} s_1, & s_2, & \ldots, & s_m \\ s_1, & s_2, & \ldots, & s_m \end{pmatrix}$ denote the determinant

$$\begin{vmatrix} 0, & K(s_2, s_1), & \ldots, & K(s_m, s_1) \\ K(s_1, s_2), & 0, & \ldots, & K(s_m, s_2) \\ \ldots & \ldots & \ldots & \ldots \\ K(s_1, s_m), & \ldots & \ldots, & 0 \end{vmatrix},$$

which is obtained by putting zero for the terms $K(s_1, s_1), K(s_2, s_2), \ldots, K(s_m, s_m)$ in the diagonal of $K\begin{pmatrix} s_1, & s_2, & \ldots, & s_m \\ s_1, & s_2, & \ldots, & s_m \end{pmatrix}$.

It is easily seen that

$$\int_a^b\int_a^b\dots\int_a^b K\begin{pmatrix}s_1, s_2, \dots, s_m\\ s_1, s_2, \dots, s_m\end{pmatrix} ds_1 ds_2\dots ds_m$$

$$= \int_a^b\int_a^b\dots\int_a^b K_0\begin{pmatrix}s_1, s_2, \dots, s_m\\ s_1, s_2, \dots, s_m\end{pmatrix} ds_1 ds_2\dots ds_m$$

$$+mk_1\int_a^b\int_a^b\dots\int_a^b K_0\begin{pmatrix}s_1, s_2, \dots, s_{m-1}\\ s_1, s_2, \dots, s_{m-1}\end{pmatrix} ds_1 ds_2\dots ds_{m-1}$$

$$+\frac{m(m-1)}{2!}k_1^2\int_a^b\int_a^b\dots\int_a^b K_0\begin{pmatrix}s_1, \dots, s_{m-2}\\ s_1, \dots, s_{m-2}\end{pmatrix} ds_1 ds_2\dots ds_{m-2}$$

$$+\dots.$$

It follows that

$$1-k_1\lambda+\frac{\lambda^2}{2!}\int_a^b\int_a^b K\begin{pmatrix}s_1, s_2\\ s_1, s_2\end{pmatrix} ds_1 ds_2-\dots$$

$$+\frac{(-1)^m\lambda^m}{m!}\int_a^b\dots\int_a^b K\begin{pmatrix}s_1, s_2, \dots, s_m\\ s_1, s_2, \dots, s_m\end{pmatrix} ds_1 ds_2\dots ds_m$$

$$=\left\{1-k_1\lambda+k_1^2\frac{\lambda_1^2}{2!}-\dots+(-1)^m k_1^m\frac{\lambda^m}{m!}+\dots\right\}$$

$$\times\left\{1+\frac{\lambda^2}{2!}\int_a^b\int_a^b K_0\begin{pmatrix}s_1, s_2\\ s_1, s_2\end{pmatrix} ds_1 ds_2+\dots\right.$$

$$\left.+\frac{(-1)^m\lambda^m}{m!}\int_a^b\int_a^b\dots K_0\begin{pmatrix}s_1, s_2, \dots, s_m\\ s_1, s_2, \dots, s_m\end{pmatrix} ds_1 ds_2\dots ds_m+\dots\right\}.$$

That the second series on the right-hand side is an integral function follows from the fact that it differs from the expression for $D(\lambda)$ only in the assumption that the value zero is assigned to $K(s, t)$ when $s=t$, the function remaining unaltered otherwise. We thus see that

$$e^{-\frac{1}{2}k_2\lambda^2-\frac{1}{3}k_3\lambda^3-\dots}=1+\frac{\lambda^2}{2!}\int_a^b\int_a^b K_0\begin{pmatrix}s_1, s_2\\ s_1, s_2\end{pmatrix} ds_1 ds_2+\dots$$

$$+\frac{(-1)^m\lambda^m}{m!}\int_a^b\int_a^b\dots\int_a^b K_0\begin{pmatrix}s_1, s_2, \dots, s_m\\ s_1, s_2, \dots, s_m\end{pmatrix} ds_1 ds_2\dots ds_m$$

$$+\dots,$$

and this holds good for all values of λ. The coefficients in the series on the right-hand side involve powers of $k_2, k_3, \dots$, but are independent of k_1.

Thus $\int_a^b\int_a^b \dots \int_a^b K_0\begin{pmatrix} s_1, s_2, \dots, s_m \\ s_1, s_2, \dots, s_m\end{pmatrix} ds_1\,ds_2 \dots ds_m$

$$= \Sigma \frac{1}{a!\,b!\,c!\dots}\,\frac{k_\alpha^a k_\beta^a \dots}{a^a\beta^b\dots}(-1)^{a+b+\dots-m},$$

the summation being taken for all integral values of a, b, c, and of $\alpha, \beta, \gamma, \dots, (> 1)$, such that $a\alpha+b\beta+\dots = m$.

Let us now consider the special case of the method of § 11 which arises when $n = 2$, *i.e.*, we suppose $K(s, t)$ to be unlimited, but $K_2(s, t)$, $K_3(s, t)$, ... to be limited. It has been shewn that $e^{-\frac{1}{2}k_2\lambda^2-\frac{1}{3}k_3\lambda^3-\dots}$ is then an integral function. In its expression in powers of λ the coefficients involve powers of $k_2, k_3, \dots$, and these coefficients are of the same form as has been obtained above. It follows that in this case the series

$$1+\frac{\lambda^2}{2!}\int_a^b\int_a^b K_0\begin{pmatrix} s_1, s_2 \\ s_1, s_2\end{pmatrix} ds_1\,ds_2+\dots$$

$$+\frac{(-1)^m\lambda^m}{m!}\int_a^b\dots\int_a^b K_0\begin{pmatrix} s_1, s_2, \dots, s_m \\ s_1, s_2, \dots, s_m\end{pmatrix} ds_1\,ds_2\dots ds_m+\dots$$

is an integral function, and it represents $e^{Q_n(\lambda)}$ the denominator in the expression for $\bar{K}(s, t)$. The numerator $-e^{Q_n(\lambda)}\dfrac{D_n\left(\lambda^n\,\begin{matrix} s\\ t\end{matrix}\right)}{D_n(\lambda)}$ has been shewn in § 11 to be an integral function.

It can be shewn in the same manner as above that the numerator in the expression for $\bar{K}(s, t)$ differs only from that in Fredholm's formula in having zero in all the diagonal terms of the coefficients, instead of $K(s_1, s_1)$, $K(s_2, s_2)$, ...

We have now established the following theorem :—

If $K(s, t)$ is unlimited, but such that $\int_a^b |K(s, t)|\,ds$, $\int_a^b |K(s, t)|\,dt$ are limited functions, and if $K_2(s, t)$ is limited, then the solution of Fredholm's equation

$$f(s) = \phi(s)-\lambda\int_a^b K(s, t)\,\phi(t)\,dt$$

is given by the modified form of Fredholm's expression that arises when zero is substituted for $K(s_1, s_1)$, $K(s_2, s_2)$, ... in the diagonal terms of $K\begin{pmatrix} s, s_1, \dots, s_m \\ t, s_1, \dots, s_m\end{pmatrix}$, $K\begin{pmatrix} s_1, \dots, s_m \\ s_1, \dots, s_m\end{pmatrix}$, which occur in the integrals that express the coefficients of the two integral functions. The function $f(s)$ may be

any summable function, limited or unlimited, such that $\int_a^b K(s, t) f(t)\, dt$ *is a summable function of s.*

This theorem is a generalization of the well known theorem of Hilbert applicable to the special case

$$K(s, t) = \frac{P(s, t)}{|s-t|^{a}},$$

where $a < \frac{1}{2}$, and $P(s, t)$ is a limited function.

In the more general case in which the order of the first repeated function that is limited is greater than 2, the forms of the integral functions that occur in the expression for $\bar{K}(s, t)$ are of a less simple character. These forms have been investigated by Poincaré (*loc. cit.*).

13. In the case in which $K(s, t)$ is of the form $\mu(s)\nu(t)P(s, t)$, where $\mu(s)\nu(s)$ is summable in the interval (a, b), already considered in § 5, it can be shewn that Fredholm's formula is applicable to obtain the solution of the integral equation. In this case all the successive repeated nuclei are unlimited, containing $\mu(s)\nu(t)$ as factor.

We have $K\begin{pmatrix} s, & t_1, & t_2, & \ldots, & t_m \\ t, & t_1, & t_2, & \ldots, & t_m \end{pmatrix}$ equal to

$$\mu(s)\,\nu(t)\,\mu(t_1)\,\nu(t_1)\,\mu(t_2)\,\nu(t_2)\ldots\mu(t_m)\,\nu(t_m),$$

multiplied by the determinant

$$\begin{vmatrix} P(s, t), & P(s, t_1), & \ldots, & P(s, t_m) \\ P(t_1, t), & P(t_1, t_1), & \ldots, & P(t_1, t_m) \\ \ldots & \ldots & \ldots & \ldots \\ P(t_m, t), & P(t_m, t_1), & \ldots, & P(t_m, t_m) \end{vmatrix}.$$

Hence, since the numerical value of the determinant is, by Hadamard's theorem, not greater than $M^m m^{\frac{1}{2}m}$, where M is the upper limit of $|P(s, t)|$, and $\int_a^b |\mu(t_1)\nu(t_1)|\, dt_1$ has a definite value γ, the series in Fredholm's expression is of the form

$$\mu(s)\,\nu(t)\left(a_0 - a_1\lambda + a_2\frac{\lambda^2}{2!} - \ldots + (-1)^m a_m \frac{\lambda^m}{m!} + \ldots\right),$$

where

$$|a_m| < M^m m^{\frac{1}{2}m} \gamma^m.$$

It follows that the series in the bracket is an integral function of λ. In a similar manner it can be shewn that the denominator in Fredholm's expression is also an integral function of λ; therefore Fredholm's expression

is equivalent to $\mu(s)\nu(t)$ multiplied by the quotient of two integral functions. The value of $-\bar{K}(s, t)$ so determined is the analytical continuation of the expression given in § 5, and satisfies the necessary condition for being the resolvant of $K(s, t)$, as may easily be verified.

We have therefore the theorem :—

If $K(s, t)$ is of the form $\mu(s)\nu(t)P(s, t)$, where $P(s, t)$ is limited and $\mu(s)$, $\nu(t)$ are one or both unlimited, but such that $\mu(s)\nu(s)$ is summable, the solution of the integral equation is given by Fredholm's formula, in case $f(s)$ is such that $f(s)\nu(s)$ is summable in the interval (a, b).

Finally, it may be remarked that this theorem could be extended to cases in which $P(s, t)$ is unlimited, for example, to a nucleus of the form $\mu(s)\nu(t)\dfrac{Q(s, t)}{|s-t|^{\alpha}}$, where $\alpha < \frac{1}{2}$, and $Q(s, t)$ is limited, the formula of Fredholm being modified in the manner explained in § 12.

SOME RESULTS ON THE FORM NEAR INFINITY OF REAL CONTINUOUS SOLUTIONS OF A CERTAIN TYPE OF SECOND ORDER DIFFERENTIAL EQUATION

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I.

1. This paper is the outcome of an attempt to apply to the differential equation

$$P(x,\ y,\ y',\ y'') = 0,$$

where P is a polynomial, the methods used by Mr. Hardy* to discuss the equation

$$P(x,\ y,\ y') = 0.$$

It is concerned with the form near infinity of a solution,

$$y = y(x), \tag{1}$$

of a simple form of second order differential equation, namely,

$$y'' = R(x,\ y) = P(x,\ y)/Q(x,\ y), \tag{2}$$

where R is a rational function of x and y, and P and Q are polynomials.† The solution (1) is always supposed to be real and continuous, and to have real continuous differential coefficients of the first two orders, at least when $x > x_0$. Moreover all the constants in $R(x,\ y)$ are supposed to be real. I propose to consider how the solution (1) behaves as $x \to \infty$.

The problem to be considered includes, for this simple form of second order equation, the problem proposed by Borel in his *Mémoire sur les*

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 10, pp. 457–468.

† P and Q will always be polynomials in y, but we shall find that it is unnecessary to assume that they are polynomials in x as well.

*Séries Divergentes.** He indicates the general lines of a proof that

$$P(x, y, y', y'') = 0,$$

where P is a polynomial, cannot have a solution y such that (in the notation of Mr. Hardy's tract on *Orders of Infinity*)†

$$y \neq O(e_3(x)).$$

Borel's proof is not complete even for the simple equation (2), and the most important lacuna is the omission of a proof that *it is impossible that solutions should exist such that*

$$y > e_3(x), \quad \{x_s\},$$

$$y < x^\kappa, \quad \{\xi_s\},$$

where κ *is some finite constant, and* $\{x_s\}$, $\{\xi_s\}$ *are different sequences tending to infinity.*

It is precisely here that the difficulty of the problem lies. Even for the equation (2) it seems to be the case that the more unlikely a solution appears to be, the harder it is to prove that it cannot exist.

2. The results which it is my object to prove are closely analogous to the results of Messrs. Lindelöf and Hardy‡ concerning solutions of the general polynomial equation of the first order

$$P(x, y, y') = 0; \tag{3}$$

the solutions of equations (2) and (3) behave in very much the same ways

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* *Annales de l'École Normale*, t. 16, pp. 26 *sqq.*

† The notation of this tract, *Cambridge Mathematical Tracts*, No. 12, will be used throughout this paper. I shall also frequently make use of relations that hold for a sequence of values of x tending to infinity. Such a sequence will be denoted by $\{x_s\}$ or $\{\xi_s\}$, and the fact that the relation in question holds for these values of x will be denoted by writing $\{x_s\}$ or $\{\xi_s\}$ after the relation. Thus, for example,

$$y \succ x^3, \quad \{x_s\},$$

means that $y/x^3 \to \infty$ as $x \to \infty$ through the sequence of values denoted by $\{x_s\}$, *i.e.*, as $s \to \infty$.

It should also be observed that O and o are always supposed to be uniform with respect to any parameters that may occur.

‡ (1) Lindelöf, *Bulletin de la Société Mathématique de France*, t. 27, p. 205; (2) Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 10, pp. 451–468. In the latter further references will be found, to which I have none to add.

as $x \to \infty$.* If the solutions which I have already specified in § 1 be called *proper solutions*, my results may be summed up thus :—

If $y = y(x)$ is a proper solution of equation (2), *then either there exists a number K such that*

$$y(x) = O(x^K),$$

or there exist numbers A and ρ, such that

$$y(x) = e^{Ax^\rho(1+\epsilon)},$$

where $\epsilon \to 0$ as $x \to \infty$, and ρ is rational.

The proof falls into four parts. In the first part I prove that, if $P(x, y)$ is a polynomial in *both x and y*, any proper solution $y = y(x)$ of

(4) $$y'' = P(x, y),$$

is such that $$y(x) = O(x^K),$$

for some value of K, if the degree of P in y is greater than unity. In the second part this result is extended to the equation

(2) $$y'' = P(x, y)/Q(x, y),$$

for which the degree of P in y does not exceed the degree of Q by unity, and Q has no real roots at least when $x > x_0$. P and Q are no longer restricted to be polynomials in x; a less stringent assumption about the nature of P and Q as functions of x is sufficient for my purpose. In the third part the condition that Q should have no real roots is shown to be irrelevant, and in the fourth the results are completed by the consideration of the case in which the degree of P exceeds the degree of Q by unity.

II.

3. Let us consider equation (4),

(4) $$y'' = P(x, y),$$

where $P(x, y)$ is *a polynomial in both x and y, and of degree in y greater than the first.* We proceed to prove that, *if $y(x)$ is any proper solution of equation* (4), *then*

$$y(x) = O(x^K),$$

for some finite value of K.

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* Cf., *loc. cit.*, (2), p. 464.

If there is no finite value of K, such that

$$y(x) = O(x^K),$$

then either

(1) $$\pm y \succ x^K,$$

for all values of K,

or else

(2) *for any sufficiently large value of κ there exists a sequence of isolated* points $\{P\}$, tending to infinity, at which the curve $y = y(x)$ crosses one of the curves $y = x^\kappa$, $y = -x^\kappa$.*

The nature of the proof that cases (1) and (2) are both impossible depends on the form of $P(x, y)$ when x is large and y is large compared to x. More precisely, we can find a number λ such that if

$$|y| \geqslant x^\lambda,$$

then as $x \to \infty$,

(5) $$P(x, y) = bx^m y^n (1+\epsilon), \quad (\epsilon \to 0),$$

where m and n are positive integers. The nature of the proof depends on the signs of b and $(-)^n$.

Case 1.—It is easy to show that Case 1 is impossible. Positive and negative values of $y(x)$ may be identically treated, so we shall confine ourselves to the former. Since, for any value of K,

(6) $$y(x) \succ x^K,$$

equation (5) holds. If therefore $b < 0$, then

$$y'' < 0, \quad (x > x_0),$$

and this is inconsistent with $y(x) \succ x^2$.

In dealing with the case $b > 0$, we shall make use of the fact that, when (6) holds,

(7) $$y'(x) > 0, \quad (x > x_0).$$

This is proved below.† From (5) and (7) we obtain

$$y''y' > y'y^{n-\frac{1}{2}}, \quad (x > x_0).$$

* The existence of a sequence of limiting points of ntersections tending to infinity is barred by the continuity conditions imposed on y, y' and y''. Consequently at least when $x > x_0$, all the intersections of $y = y(x)$ and $y = x^\kappa$ will be isolated.

† If $y'(x) \not> 0$, $(x > x_0)$, then either (1) $y'(x) < 0$, $(x > x_0)$; or (2) a sequence of values of x, $\{x_\nu\}$, exists tending to infinity such that $y'(x) = 0$, $\{x_\nu\}$.

Both (1) and (2) are impossible. Case (1) obviously contradicts $y \succ x$. Case (2) implies that there exists a sequence of stationary values of $y(x)$ for all of which $y''(x)$ has the same sign [by (5) and (6)], which is absurd. Consequently (7) is true.

Integrating this from x_0 to x, and using the fact that $y(x) \to \infty$, we obtain

$$y'/y^{n/2+1/4} > M, \quad (x > x_0).$$

Since $n \geqslant 2$, $\qquad \frac{1}{2}(n-2)+1/4 = \tau > 0.$

Therefore, when we integrate again from x_0 to x, we obtain

$$[(y(x_0))^{-\tau}-(y(x))^{-\tau}] > M(x-x_0), \quad (x > x_0),$$

which obviously contradicts $y(x) \to \infty$. Case 1 is therefore impossible.

4. *Case* 2.—It is convenient to divide this case up into three parts. We shall consider separately

(i) $b > 0$;

(ii) $b < 0$, n even;

(iii) $b < 0$, n odd.

In this case the fundamental supposition is that $y = y(x)$ cuts one at least of $y = x^\kappa$ and $y = -x^\kappa$, for any value of κ ($> \kappa_0$), in a sequence of points tending to infinity. We can assure ourselves that it cuts $y = x^\kappa$ by changing (if need be) the sign of y'' and y throughout equation (4). The sign of b may thus be changed, but this does not matter. It is clearly sufficient to prove that $y = y(x)$ cannot cut $y = x^\kappa$ infinitely often in any of the three sub-cases (i), (ii), and (iii), in order to show that the whole of Case 2 is impossible.

5. *Case* 2 (i).

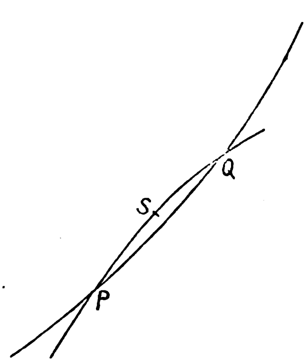

FIG. 1.

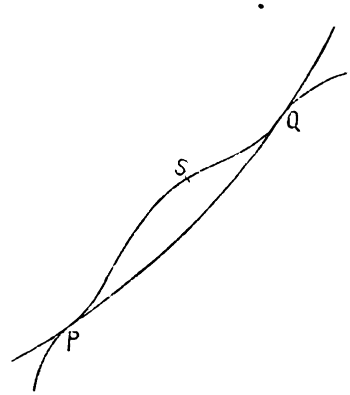

FIG. 2.

Since the sequence of isolated points $\{P\}$ exists, there must exist a

sequence of intervals $\{PQ\}$ whose least abscissæ tend to infinity, such that

$$y(x) \geqslant x^{\kappa}, \quad \{PQ\}.$$

Now at Q, $$y'(x) \leqslant \kappa x^{\kappa-1},$$

and at P, $$y'(x) \geqslant \kappa x^{\kappa-1}.$$

Therefore in each interval of $\{PQ\}$ there must exist at least one point S of abscissa x_s such that

$$y''(x_s) \leqslant \kappa(\kappa-1)(x_s)^{\kappa-2}.$$

In other words, there exists a sequence $\{x_s\}$ tending to infinity such that

$$\left.\begin{aligned} y''(x) &\leqslant \kappa(\kappa-1)(x)^{\kappa-2} \\ y(x) &\geqslant x^{\kappa} \end{aligned}\right\}, \quad \{x_s\}.$$

But it follows from (4), (5), and the conditions of Case 1 (i), (namely, $b > 0$, $n \geqslant 2$), that

$$y''(x) > x^{\frac{3}{2}\kappa}, \quad \{PQ\},$$

provided $\kappa > \kappa_0$. We have therefore arrived at a contradiction.

6. *Case* 2 (ii).—When $b < 0$, the intervals in which $y(x) > x^{\kappa}$ give us no information. By considering the intervals in which $y(x) < x^{\kappa}$, we can obtain a contradiction when n is even, but not when n is odd. The arguments already used show that there must exist a sequence of points of abscissæ $\{x_s\}$, such that

$$\left.\begin{aligned} y''(x) &\geqslant \kappa(\kappa-1)x^{\kappa-2} \\ y(x) &\leqslant x^{\kappa} \end{aligned}\right\}, \quad \{x_s\}. \tag{8}$$

It follows that

$$P(x, y) \geqslant \kappa(\kappa-1)x^{\kappa-2}, \quad \{x_s\}. \tag{9}$$

Now since the sequence $\{x_s\}$ exists for any value of κ, in order to satisfy (9), it must be the case that $|y(x)|$ is so large at points of $\{x_s\}$ that

$$P(x, y) = bx^m y^n(1+\epsilon), \quad (\epsilon \to 0, \ \{x_s\}).$$

If n is even the sign of $y(x)$ at points of $\{x_s\}$ is irrelevant, and we can deduce that

$$P(x, y) < 0, \quad \{x_s\},$$

which contradicts (9). Case 2 (ii) is accordingly impossible.

If n is odd we cannot deduce an impossibility. It is however true that

$$y(x) < -x^{\kappa}, \quad \{x_s\},$$

for any value of κ, a fact which it is convenient to remember in Case 2 (iii).

7. *Case* 2 (iii).—It follows from § 6 that $y = y(x)$ cuts infinitely often both $y = x^\lambda$ and $y = -x^\lambda$ for any fixed value of λ. We shall suppose that λ is chosen large enough to satisfy (5); then when $x > x_0$ all the roots of $P(x, y) = 0$ will lie inside the region C defined by

$$|y| \leqslant x^\lambda, \quad x > x_0.$$

A finite number of other conditions may have to be imposed on λ and x_0, all of which can be satisfied by choosing λ and x_0 sufficiently large. Let the sequence of abscissæ of points at which $y = y(x)$ emerges from the region C be denoted by $\{\xi'_s\}$, and the sequence at which it enters C by $\{\xi_s\}$, so that the sequence of intervals in which $y = y(x)$ lies outside C will be denoted by $\{\xi'_s \xi_{s+1}\}$. (See Fig. 3, p. 349.) It should be observed that ξ'_s and $\xi_s \to \infty$ as $s \to \infty$.

Now it follows from (5) and the conditions of Case 2 (iii) that

$$y''(x) < 0, \quad [y(x) \geqslant x^\lambda,\ x > x_0],$$

$$y''(x) > 0, \quad [y(x) \leqslant -x^\lambda,\ x > x_0].$$

Hence in the interval $\xi'_s \xi_{s+1}$ for all values of $s \geqslant s_0$,

$$|y(x)| < |y'(\xi'_s)|(x - \xi'_s) + (\xi'_s)^\lambda.$$

If therefore we can prove that, for all values of $s \geqslant s_0$,

(10) $$|y'(\xi'_s)| < H(\xi'_s)^q,$$

where H and q are positive numbers independent of s, we shall obtain the inequality

$$|y(x)| < H(\xi'_s)^q (x - \xi'_s) + (\xi'_s)^\lambda.$$

If (10) is true for q_0 it is true for any greater value of q. We may therefore suppose that $q > \lambda$, and therefore

$$|y(x)| < H(x)^{q+1}, \quad (\xi'_s \leqslant x \leqslant \xi_{s+1},\ s \geqslant s_0).$$

But at all points not in the sequence of intervals $\{\xi'_s \xi_{s+1}\}$,

$$|y(x)| < x^\lambda;$$

therefore $$y(x) = O(x^K),$$

for some finite value of K, and Case 2 (iii) is impossible.

Now it is easy to show how (10) may be deduced from the inequality

(11) $$|\,y'(\xi_{s+1})/y'(\xi'_s)\,| < (\xi_{s+1}/\xi'_s)^{q_0}, \quad (s \geqslant s_0),$$

where q_0 is independent of s. For suppose that (11) is true. It gives us an upper limit of the change in $y'(x)$ in the intervals of the set $\{\xi'_s \xi_{s+1}\}$. In any interval of the type $\{\xi_{s+1}\xi'_{s+1}\}$, $|\,y(x)\,| \leqslant x^{\lambda}$, and therefore

$$|\,y''(x)\,| < Gx^p, \quad (s \geqslant s_0),$$

where G and p are constants independent of s. Let us start to trace the changes in $y'(x)$ at the point where $x = \xi_{s_0+1}$. We can choose H and q to satisfy the following conditions:—

$$(a) \quad |\,y'(\xi_{s_0+1})\,| < H(\xi_{s_0+1})^q,$$

$$(b) \quad H > G,$$

$$(c) \quad q-1 \geqslant p, \quad q \geqslant q_0.$$

Then in the interval $\xi_{s+1}\xi'_{s+1}$ $(s \geqslant s_0)$,

(12) $$|\,y''(x)\,| < Hx^{q-1},$$

where H and q depend on s_0 *but not on* s. Applying (12) to the interval $\xi_{s_0+1}\xi_{s_0+1}$, and integrating over the interval, we find

$$|\,y'(\xi'_{s_0+1}) - y'(\xi_{s_0+1})\,| < H(\xi'_{s_0+1})^q - H(\xi_{s_0+1})^q.*$$

With the help of (a) it follows that

$$|\,y'(\xi'_{s_0+1})\,| < H(\xi'_{s_0+1})^q;$$

and thence with the help of (11),

$$|\,y'(\xi_{s_0+2})\,| < H(\xi_{s_0+2})^q.$$

Proceeding thus step by step we find that

(10) $$|\,y'(\xi'_s)\,| < H(\xi'_s)^q, \quad (s \geqslant s_0),$$

where H and q are independent of s, which is the relation we require. In order, therefore, to prove (10) and so show that Case 2 (iii) is impossible, it only remains to prove that

(11) $$|\,y'(\xi_{s+1})/y'(\xi'_s)\,| < (\xi_{s+1}/\xi'_s)^{q_0}, \quad (s \geqslant s_0),$$

where q_0 is independent of s.

* It is essential to observe that the H here may be taken to be the *same* H as the constant we have already chosen to fulfil conditions (a), (b), and (c).

8. Since a sequence of values of x can be found tending to infinity for which $y(x)$ is alternately positive and negative, there must exist a sequence of values of x tending to infinity, such that

$$y'(x) = 0, \quad \{x_r\}.$$

We cannot say that all members of $\{x_r\}$ must ultimately lie in the intervals $\{\xi_s' \xi_{s+1}\}$, nor that every interval of $\{\xi_s' \xi_{s+1}\}$ contains at least one member of $\{x_r\}$. Since, however, $y''(x)$ is one-signed in any interval of $\{\xi_s' \xi_{s+1}\}$, there cannot be more than one member of $\{x_r\}$ in any such interval. All the possibilities that we must take account of are shown in Fig. 3.

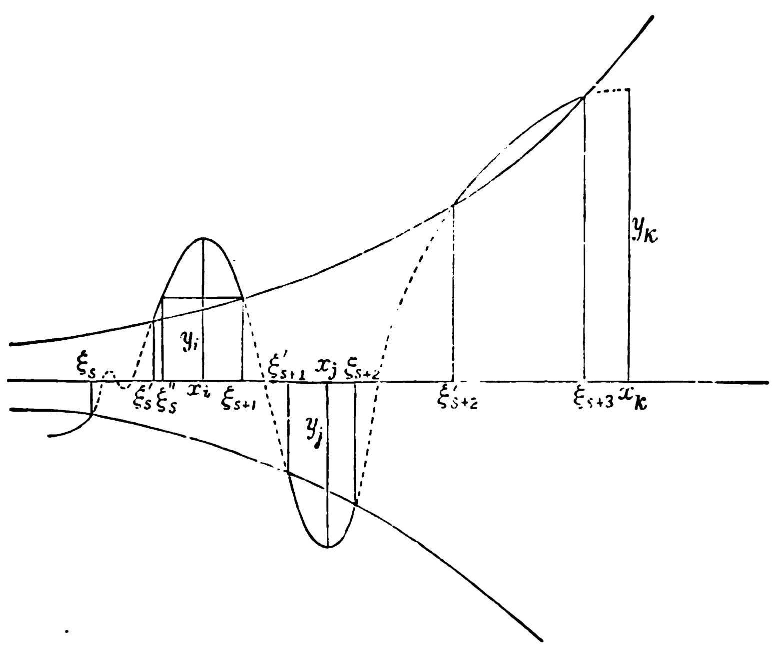

FIG. 3.

It may or may not be the case that $k-2 = j-1 = i$.

Throughout any interval of $\{\xi_s' \xi_{s+1}\}$ in which no member of $\{x_r\}$ occurs (such as $\xi_{s+2}' \xi_{s+3}$, Fig. 3), $y'(x)$ and $y''(x)$ have opposite signs, and therefore (11) is true for $q_0 = 0$, and therefore for any greater value of q. An interval such as $\xi_s' \xi_{s+1}$ (Fig. 3) in which there is one member of $\{x_r\}$, namely x_i, needs more careful discussion. It is convenient to compare $y'(\xi_{s+1})$ with $y'(\xi_s'')$ instead of with $y'(\xi_s')$, where ξ_s'' is that value of x in the interval $\xi_s' x_i$ which makes

$$y(\xi_s'') = y(\xi_{s+1}). \tag{13}$$

It is easily seen that

$$|y'(\xi_s'')/y'(\xi_s')| < 1 < (\xi_s''/\xi_s')^q,$$

for any value of $q \geqslant 0$; it remains to be proved that

(14) $$|y'(\xi_{s+1})/y'(\xi_s'')| < (\xi_{s+1}/\xi_s'')^q, \quad (s \geqslant s_0),$$

for some value of q independent of s.

9. In the range $\xi_s'' \leqslant x \leqslant x_i$,

$$-y''y' = -P(x, y)\,y',$$

both sides being $\geqslant 0$. Integrating this we have

$$[y'(\xi_s'')]^2 = \int_{y(\xi_s'')}^{y_i} -P(x, y)\,dy.*$$

To obtain (14) we require a lower limit for $-P(x, y)$ over the range of integration. This we shall get by proving that

$$F(x) \equiv -(x_i)^q P(x, y) + (x)^q P(x_i, y) \geqslant 0, \quad (s \geqslant s_0),$$

for all values of x such that $\xi_s'' \leqslant x \leqslant x_i$, provided only that $q \geqslant q_0$, where q_0 depends solely on the construction of $P(x, y)$.

$P(x, y)$ can be written in the form

$$-P(x, y) = \sum_1^N b_r x^{m_r} y^{n_r},$$

the leading term being the N-th, so that

$$b_N = -b > 0, \quad m_N = m, \quad n_N = n.$$

Therefore $$F(x) = \sum_1^N b_r [x^{m_r}(x_i)^q - (x_i)^{m_r} x^q]\, y^{n_r}.$$

$F(x)$ vanishes when $x = x_i$; when $\xi_s'' \leqslant x < x_i$ the ratio of any term of this expression to the leading term is

$$-\frac{b_r}{b}\frac{x^{m_r-m}}{y^{n-n_r}}\left[\frac{1-(x/x_i)^{q-m_r}}{1-(x/x_i)^{q-m}}\right].$$

---

* The work is arranged for an interval in which $y > x^\lambda$. For an interval in which $y < -x^\lambda$, the sign of y may be changed throughout the differential equation, and the arguments of §§ 9 and 10 applied unchanged.

If we take $q > m_r$, $(r = 1, 2, \ldots, N)$, it is easily verified that

$$\left[\frac{1-(x/x_i)^{q-m_r}}{1-(x/x_i)^{q-m}}\right] < h, \quad (\xi_s'' \leqslant x < x_i),$$

where h is independent of r and i. Moreover

$$\frac{x^{m_r-m}}{y^{n-n_r}} \to 0, \quad (x \to \infty,\ |y| \geqslant x^\lambda),$$

for a sufficiently large λ. Therefore

$$F(x) = -b[x^m (x_i)^q - (x_i)^m (x)^q]\, y^n (1+\epsilon), \quad (x \neq x_i),$$

where $|\epsilon| < \frac{1}{2}$, say, if $s \geqslant s_0$. Therefore $F(x)$ never vanishes in the range $\xi_s'' \leqslant x \leqslant x_i$ except for $x = x_i$; and, since $F(\xi_s'') > 0$, we have

$$F(x) \geqslant 0, \quad (\xi_s'' \leqslant x \leqslant x_i,\ s \geqslant s_0),$$

provided q be greater than the greatest index of x in $P(x, y)$. It follows at once that

(15a) $$-P(x, y) \geqslant -P(x_i, y)(x/x_i)^q$$
$$> -P(x_i, y)(\xi_s''/x_i)^q,$$

and therefore that

(15) $$[y'(\xi_s'')]^2 > (\xi_s''/x_i)^q \int_{y(\xi_s'')}^{y_i} -P(x_i, y)\,dy, \quad (s \geqslant s_0),$$

where q is independent of s.

10. Again, from the range $x_i \leqslant x \leqslant \xi_{s+1}$, we obtain

$$[y'(\xi_{s+1})]^2 = \int_{y(\xi_s'')}^{y_i} -P(x, y)\,dy.$$

In order to prove that

(16a) $$-P(x, y) \leqslant -P(x_i, y)(x/x_i)^q, \quad (x_i \leqslant x \leqslant \xi_{s+1},\ s \geqslant s_0),$$

we repeat the arguments of the last section, writing the ratio of the r-th to the N-th term of $P(x, y)$ in the form

$$-\frac{b_r}{b}\frac{(x_i)^{m_r-m}}{y^{n-n_r}}\left[\frac{1-(x_i/x)^{q-m_r}}{1-(x_i/x)^{q-m}}\right];$$

the result follows as before. We therefore find

(16) $$[y'(\xi_{s+1})]^2 < (\xi_{s+1}/x_i)^q \int_{y(\xi_s'')}^{y_i} -P(x_i, y)\,dy, \quad (s \geqslant s_0).$$

Combining (15) and (16) we obtain the required inequality, namely,

$$|y'(\xi_{s+1})/y'(\xi_s'')| < (\xi_{s+1}/\xi_s'')^q, \quad (s > s_0), \tag{14}$$

where q is independent of s. We have already seen how a contradiction of the assumptions of Case 2 (iii) may be deduced from (14). We have therefore proved the theorem that

If $y(x)$ is any proper solution of

$$y'' = P(x, y),$$

then

$$y(x) = O(x^K),$$

for some finite value of K, if the degree of P in y is > 1.

III.

11. The next step is the extension of this result to the equation

$$y'' = P(x, y)/Q(x, y), \tag{17}$$

where $Q(x, y)$ has no real roots when $x > x_0$, and the degree of $P(x, y)$ in y does not exceed the degree of $Q(x, y)$ by unity. $P(x, y)$ and $Q(x, y)$ are polynomials in y; it will be found sufficient to suppose that, *if $f(x)$ is the coefficient of any power of y in P or Q, then*

$$f(x) = x^{\mu} P_0\{(x)^{-\nu}\}, \tag{18}$$

where μ and ν are rational numbers and $\nu > 0$, and $P_0(\lambda)$ is a power series in λ convergent when $\lambda < \lambda_0$ and satisfying the inequality

$$P_0(0) \neq 0.$$

The fundamental inequalities on which all the reasoning in Part II depends are given by (5), (12), (15a), and (16a). We shall require corresponding inequalities here. Corresponding to (5) we obviously have

$$P(x, y)/Q(x, y) = bx^m y^n (1+\epsilon), \tag{19}$$

where $\epsilon \to 0$ when $x \to \infty$ and $|y| > x^{\lambda}$ for a sufficient value of λ. In this case n is any integer ($\neq 1$), and m is any rational number. To obtain the analogue of (12), namely,

$$|P(x, y)/Q(x, y)| < Hx^p, \quad (|y| \leqslant x^{\lambda},\ x > x_0), \tag{20}$$

where H and p positive constants, we need some such Lemma as the following, which we proceed to prove.

LEMMA.—*If* $Q(x, y)$ *has no real roots when* $x > x_0$, *and the coefficients of* y *in* Q *satisfy* (18), *then* $[Q(x,y)]^{-1}$ *satisfies* (20).

Let τ be the degree of $Q(x, y)$ in y. The implicit function y defined by

$$Q(x, y) = 0,$$

will have τ branches (distinct or not) near infinity, all of which can [by virtue of (18)] be expressed in the form*

$$x^{\mu_1}\Pi_1\{(x)^{-\nu_1}\}+ix^{\mu_2}\Pi_2\{(x)^{-\nu_2}\},$$

where $\mu_1, \nu_1, \Pi_1, \mu_2, \nu_2, \Pi_2$ have the properties of μ, ν, P_0 in (18). It should be observed that Π_1 and Π_2 are both real for real x. Now since

$$Q(x, y) = 0$$

has no real roots when $x > x_0$, all its roots must occur in conjugate pairs, of which the imaginary part is never null. We can therefore write

$$|Q(x, y)| = \prod_{r=1}^{\tau/2} \left\{[y - x^{\mu_{r,1}}\Pi_{r,1}\{(x)^{-\nu_{r,1}}\}]^2 + [x^{\mu_{r,2}}\Pi_{r,2}\{(x)^{-\nu_{r,2}}\}]^2\right\}$$

$$\geqslant \prod_{r=1}^{\tau/2} [x^{\mu_{r,2}}\Pi_{r,2}\{(x)^{-\nu_{r,2}}\}]^2$$

$$> hx^{-p}, \quad (x > x_0),$$

where h and p are positive constants. The Lemma is therefore proved, and the truth of (20) is established.

12. Before proceeding to obtain the analogues of (15*a*) and (16*a*), we shall consider the case $n \leqslant 0$, for the treatment of which only (19) and (20) are needed. For the reasons given in §§ 3 and 4 it is sufficient to consider only positive values of y.

(1) *It is impossible that, for all values of* K,

$$y \succ x^K.$$

For when $|y| \geqslant x^\lambda$ and $x > x_0$, we have, by (19),

$$y'' = bx^m y^n(1+\epsilon), \quad (\epsilon \to 0).$$

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* See, for instance, Goursat, *Cours d'Analyse*, Vol. II, p. 273 *sqq.*

Therefore either (i) $y'' < 0$, or (ii) $y'' \prec x^{m+1}$,

and in either case it cannot be true that $y \succ x^K$ for all values of K.

(2) *It is impossible that, for all values of* κ, $y = y(x)$ *intersect* $y = x^\kappa$ *in a sequence of points whose abscissæ* $\to \infty$.

Consider the sequence of intervals in which

$$y(x) < x^\kappa.$$

We can deduce that a sequence of points exists, of abscissæ $\{x_s\}$ tending to infinity, such that

$$y''(x) > \kappa(\kappa-1)\, x^{\kappa-2}, \ \{x_s\},$$

so that

$$P(x, y)/Q(x, y) > \kappa(\kappa-1)\, x^{\kappa-2}, \ \{x_s\}.$$

But when $n \leqslant 0$ and (20) is true, this is obviously impossible whatever the sign of b.

It follows that it must be true that

$$y(x) = O(x^K),$$

for some finite value of K, under the assumed conditions, which is the required result.

13. It remains to consider equation (17) when $n \geqslant 2$. The required result obviously follows at once (as in Part II) from (19), (20), and the analogues of (15a) and (16a) which we now proceed to obtain. They are only needed when the leading term in P/Q is negative, so we shall suppose that the leading term of P is negative, and the leading term of Q positive.

The analogues are

(21) $$-P(x, y)/Q(x, y) \geqslant (\xi_s''/x_i)^q\,[-P(x_i, y)/Q(x_i, y)], \quad (\xi_s'' \leqslant x \leqslant x_i, \ s \geqslant s_0),$$

(22) $$-P(x, y)/Q(x, y) \leqslant (\xi_{s+1}/x_i)^q\,[-P(x_i, y)/Q(x_i, y)], \quad (x_i \leqslant x \leqslant \xi_{s+1}, \ s \geqslant s_0),$$

for some value of q independent of s, and they follow at once from the inequalities

(23)
$$F(x) \equiv -(x_i)^q\, Q(x_i, y)\, P(x, y) + (x)^q\, P(x_i, y)\, Q(x, y) \geqslant 0, \ (\xi_s'' \leqslant x \leqslant x_i, \ s \geqslant s_0),$$
$$\leqslant 0, \ (x_i \leqslant x \leqslant \xi_{s+1}, s \geqslant s_0).$$

Let us write $P(x, y) = \sum_{\mu=0}^{\alpha} a_\mu f_\mu(x) y^\mu, \quad f_\mu(x) \sim x^{m_\mu};$

$$Q(x, y) = \sum_{\nu=0}^{\beta} b_\nu g_\nu(x) y^\nu, \quad g_\nu(x) \sim x^{m_\nu}.$$

Then the leading term of $F(x)$ is

$$a_\alpha b_\beta y^{\alpha+\beta}[-f_\alpha(x) g_\beta(x_i)(x_i)^q + f_\alpha(x_i) g_\beta(x)(x)^q],$$

and the ratio of any other term to the leading term when $x \neq x_i$ is

$$\frac{A}{y^{\alpha+\beta-\mu-\nu}}\left[\frac{-f_\mu(x) g_\nu(x_i)(x_i)^q + f_\mu(x_i) g_\nu(x)(x)^q}{-f_\alpha(x) g_\beta(x_i)(x_i)^q + f_\alpha(x_i) g_\beta(x)(x)^q}\right],$$

where A is an absolute constant.

For the range $\xi_s'' \leqslant x < x_i$, this may be put in the form

$$(24) \qquad \frac{A(x_i)^{m_\nu - m_\beta}(x)^{m_\mu - m_\alpha}}{y^{\alpha+\beta-\mu-\nu}}\left[\frac{b_{\mu,\nu}(x, x_i) - (x/x_i)^{q+m_\nu-m_\mu}}{b_{\alpha,\beta}(x, x_i) - (x/x_i)^{q+m_\beta-m_\alpha}}\right][1+o(1)],$$

where $o(1) \to 0$, as $s \to \infty$, uniformly for all values of x such that

$$\xi_s'' \leqslant x < x_i,$$

and $$b_{\mu,\nu}(x, x_i) = [f_\mu(x) g_\nu(x_i)/f_\mu(x_i) g_\nu(x)]\,(x_i/x)^{m_\mu - m_\nu}.$$

Now $$(x)^{-m_\mu} f_\mu(x) = 1+\theta_\mu(x),$$

$$(x)^{-m_\nu} g_\nu(x) = 1+\kappa_\nu(x),$$

where $\theta_\mu(x)$, $\kappa_\nu(x)$ are power series convergent when $x > x_0$, whose value $\to 0$ as $x \to \infty$. Therefore

(25) $b_{\mu,\nu}(x, x_i) - 1$

$$= \{[1+\theta_\mu(x)][1+\kappa_\nu(x_i)]/[1+\theta_\mu(x_i)][1+\kappa_\nu(x)]\} - 1$$

$$= \left[\theta_\mu(x)[1+\kappa_\nu(x)]\left(1 - \frac{\theta_\mu(x_i)}{\theta_\mu(x)}\right) - \kappa_\nu(x)[1+\theta_\mu(x)]\left(1 - \frac{\kappa_\nu(x_i)}{\kappa_\nu(x)}\right)\right][1+o(1)].$$

Now for all values of μ, ν, and i, and for all values of x, such that $x_0 \leqslant x \leqslant x_i$, we can find an a independent of μ, ν, i, and x, such that

$$a(1-x/x_i) > 1-[\theta_\mu(x_i)/\theta_\mu(x)] > 0,$$

$$a(1-x/x_i) > 1-[\kappa_\nu(x_i)/\kappa_\nu(x)] > 0.$$

This is an immediate deduction from the fact that $\theta_\mu(x)$, $\kappa_\nu(x)$ are con-

2 A 2

vergent power series whose values $\to 0$ as $x \to \infty$. It follows at once from (25) that

$$(25') \qquad b_{\mu,\nu}(x, x_i) = 1 + (1 - x/x_i)\, o(1),$$

where $o(1) \to 0$ uniformly for all values of x, such that $\xi_s'' \leqslant x \leqslant x_i$ as $s \to \infty$. It follows that the factor in square brackets in (24) takes the form

$$\left[\frac{1-(x/x_i)^{q+m_\nu-m_\mu}+(1-x/x_i)\, o(1)}{1-(x/x_i)^{q+m_\beta-m_a}+(1-x/x_i)\, o(1)}\right],$$

for all values of x in the interval $\xi_s'' \leqslant x < x_i$; we may let $x \to x_i$ if we like; only the point $x = x_i$ is excluded from consideration, and we already know that $F(x_i) = 0$. It is easily seen that if

$$q > m_\nu - m_\mu, \quad (\mu = 0, 1, \ldots, a;\ \nu = 0, 1, \ldots, \beta),$$

and $s \geqslant s_0$, then this expression is less than a constant K independent of s. Consequently the ratio of any term of $F(x)$ to the leading term is of the form

$$(26) \qquad O\left\{\frac{(x_i)^{m_\nu - m_\beta}\,(x)^{m_\mu - m_a}}{y^{a+\beta-\mu-\nu}}\right\}.$$

Now, in the range $\xi_s'' \leqslant x \leqslant x_i$,

$$|\, y(x)\,| > (x_i)^\lambda,$$

not merely $> (x)^\lambda$. Therefore

$$\frac{(x_i)^{m_\nu - m_\beta}\,(x)^{m_\mu - m_a}}{y^{a+\beta-\mu-\nu}} = o(1),$$

if λ be chosen sufficiently large. Using this result we have

$$F(x) = ky^{a+\beta}\left[f_a(x)\, g_\beta(x_i)(x_i)^q - f_a(x_i)\, g_\beta(x)(x)^q\right]\left[1+o(1)\right], \quad (\xi_s'' \leqslant x < x_i),$$

where k is a positive constant. With the help of (25′) we may put this in the form

$$F(x) = k(x)^{m_a}(x_i)^{q+m_\beta}\, y^{a+\beta}\left[1-(x/x_i)^{q+m_\beta-m_a}+(1-x/x_i)\, o(1)\right]\left[1+o(1)\right].$$

Hence $F(x)$ cannot vanish in the interval $\xi_s'' \leqslant x < x_i$ when $s \geqslant s_0$, unless

$$1-(x/x_i)^{q+m_\beta-m_a}+(1-x/x_i)\, o(1) = 0.$$

Since $o(1)$ is uniform with respect to x this cannot occur when $s \geqslant s_0$. It follows that

$$F(x) \geqslant 0, \quad (\xi_s'' \leqslant x \leqslant x_i,\ s \geqslant s_0),$$

which is the inequality we require. The proof that

$$F(x) \leqslant 0, \quad (x_t \leqslant x \leqslant \xi_{s+1},\ s > s_0),$$

is exactly similar. We have therefore established the relations (21) and (22). From relations (19), (20), (21), and (22) it follows exactly as in Part II that there exists a value of K, such that

$$y(x) = O(x^K),$$

provided the conditions of § 11 are satisfied by $P(x, y)$ and $Q(x, y)$.

IV.

14. The final step is to remove the restriction that $Q(x, y)$ has no real roots when $x > x_0$, leaving the other conditions of § 11 unaltered.

Suppose then that $Q(x, y) = 0$ has one or more real roots when $x > x_0$. They can be represented by power series convergent when $x > x_0'$, and for all values of x such that $x > x_0''$ they can be arranged in an order of magnitude which is independent of x. Let

$$y = A_1(x)$$

be the greatest root, and $\qquad y = A_2(x)$

be the least root when $x > x_0$. We may suppose that any common factors of $P(x, y)$ and $Q(x, y)$ have been eliminated; there must therefore exist a number k such that within the regions

$$y = A_1(x)(1 \pm 1/x^k), \quad (x > x_0),$$

$$y = A_2(x)(1 \pm 1/x^k), \quad (x > x_0),$$

there lies no root of $P(x, y) = 0$, and no root of $Q(x, y) = 0$ other than $A_1(x)$, $A_2(x)$ respectively.

Now, if $y = y(x)$ is a proper solution, such that for all values of K,

$$y(x) \neq O(x^K),$$

it must be possible to find a number x_0 such that when $x > x_0$, $y = y(x)$ lies entirely above $y = A_1(x)$, or entirely below $y = A_2(x)$; for otherwise $y''(x)$ would not be finite and continuous as $x \to \infty$. It is clearly sufficient to consider only one of these cases; we shall confine ourselves to

$$y(x) > A_1(x), \quad (x > x_0).$$

Now there will be no loss of generality if we suppose that $A_1(x) \equiv 0$. This is equivalent to making the substitution

$$y = A_1(x) + \eta,$$

so that η satisfies an equation

$$\eta'' = R_1(x, \eta) = \frac{P_1(x, \eta)}{\eta^p Q_1(x, \eta)}, \quad (p \geqslant 1),$$

of exactly the same form as y, for all the coefficients of η on the right-hand side are still power series convergent when $x > x_0$. We shall therefore suppose that $y(x)$ satisfies

$$y'' = R(x, y) = \frac{P(x, y)}{y^p Q_1(x, y)}, \quad (p \geqslant 1),$$

where $Q_1(x, y)$ has no roots $\geqslant 0$ when $x > x_0$.

[It follows that $Q(x, y)$ has no roots $> -x^{-k}$ for some positive value of k.]

Now, if we treat $Q_1(x, y)$ as we treated $Q(x, y)$ in the Lemma of § 11, we shall see that we can find a number λ, as large as we please, such that

$$(27) \qquad R(x, y) = bx^m y^n (1+\epsilon), \quad (\epsilon \to 0 \text{ as } x \to \infty,\ y \geqslant x^\lambda),$$

$$(28) \qquad |R(x, y)| < Hx^N, \qquad (x^\lambda \geqslant y \geqslant x^{-\lambda},\ x > x_0),$$

$$(29) \qquad R(x, y) = cx^l y^{-p}(1+\epsilon), \quad (\epsilon \to 0 \text{ as } x \to \infty,\ x^{-\lambda} \geqslant y > 0).$$

H and N are independent of x though not of λ, $n \neq 1$, and $p \geqslant 1$.

It follows from these inequalities that the proofs of §§ 3 and 12, that it is not possible that for all values of K $y \succ x^K$, hold unchanged. If therefore $y(x) \neq O(x^K)$ for some value of K, $y = y(x)$ must intersect infinitely often $y = x^\kappa$. By repeating the argument of § 6 we deduce from (28) and (29) that this is impossible *unless* $c > 0$, *and the curve* $y = y(x)$ *intersects the curve*

$$y = x^{-\kappa}$$

for any value of κ, *in a sequence of points tending to infinity.* Further, by the argument of § 5 it follows that, *if* $n > 1$, $y = y(x)$ *cannot intersect* $y = x^\kappa$ unless $b < 0$. [If $n \leqslant 0$, the sign of b is irrelevant.] The form of the curve in the remaining cases when

$$(1) \quad c > 0, \quad n \leqslant 0,$$

or

$$(2) \quad c > 0, \quad n \geqslant 2, \quad b < 0,^*$$

---

* The curve is drawn for the case $c > 0$, $b < 0$. If $b > 0$, its form above $y = x^{-\lambda}$ would be different, but its form below $y = x^{-\lambda}$ will always be of the type shown.

is shown in Fig. 4.

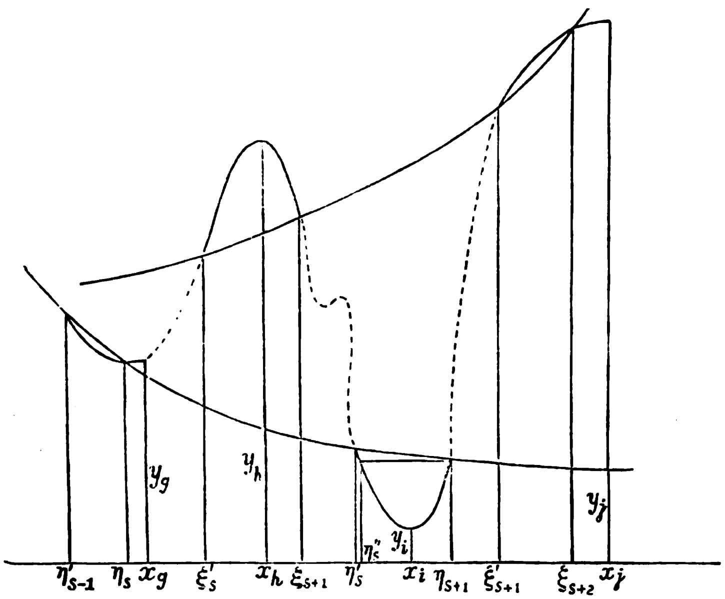

FIG. 4.

15. *Case* 1.—In this case it is easy to deduce that it is impossible that for all values of K, $y(x) \neq O(x^K)$, when the formula

$$|y'(\eta_{s+1})/y'(\eta_s')| < (\eta_{s+1}/\eta_s')^q, \quad (s > s_0), \tag{30}$$

where q is independent of s, has been established.

For suppose (30) is true. Since $n \leqslant 0$, the formula (28) holds for all values of x and y, such that $x > x_0$, $y > x^{-\lambda}$, without any restriction of the form $y \leqslant x^\lambda$. We can therefore deduce from (30) by the method of § 7 that

$$|y'(\eta_{s+1})| < H(\eta_{s+1})^q, \quad (s > s_0),$$

and hence that

$$|y'(\xi_s')| < H(\xi_s')^q, \quad (s > s_0),$$

for values of H and q independent of s. But in the interval $\xi_s' \xi_{s+1}$,

$$y'' \leqslant bx^m(1+\epsilon), \quad (\epsilon \to 0).$$

Therefore

$$y'(x) < H(x)^q, \qquad (\xi_s' \leqslant x \leqslant \xi_{s+1}),$$

for values of H and q independent of s. Therefore

$$y(x) < H(x)^{q+1} + (\xi_s')^\lambda, \quad (\xi_s' \leqslant x \leqslant \xi_{s+1}),$$

whence it follows that there exists a K such that

$$y(x) = O(x^K).$$

There remains Case 2 ($n \geqslant 2$) to be considered.

Case 2.—To establish the impossibility of the assumption that, for all values of K, $y(x) \neq O(x^K)$ in this last remaining case, it is clearly only necessary to prove that

$$(30) \qquad \begin{cases} \text{(i)} \quad |y'(\xi_{s+1})/y'(\xi_s')| < (\xi_{s+1}/\xi_s')^q, \quad (s \geqslant s_0), \ (n > 1), \\ \text{(ii)} \quad |y'(\eta_{s+1})/y'(\eta_s')| < (\eta_{s+1}/\eta_s')^q, \quad (s \geqslant s_0), \end{cases}$$

for some q independent of s. The result will then follow from (28), and [(30), (i), (ii)] just as it follows from the analogous relations in Part II.

The first of these inequalities we have already proved in Part III, for the proof of (23), from which [(30), (i)] follows immediately, does not make use of the assumption that $Q(x, y)$ has no real roots when $x > x_0$. It only remains to prove [(30), (ii)].

The proof of [(30), (ii)] is exactly analogous to the proof of [(30), (i)]. Since $c > 0$, [(30), (ii)] is obviously true for any interval in which $y'(x)$ does not vanish once (it cannot vanish more often than once). It is clearly sufficient, for an interval in which $y'(x)$ vanishes once, say at $x = x_i$, to establish [(30), (ii)] in the form

$$(31) \qquad |y'(\eta_{s+1})/y'(\eta_s'')| < (\eta_{s+1}/\eta_s'')^q, \quad (s \leqslant s_0),$$

where η_s'' is that value of x in the range $\eta_s' \leqslant x \leqslant x_i$, at which

$$y(x) = y(\eta_{s+1}).$$

Proceeding, as in § 9, we have

$$[y'(\eta_s'')]^2 = \int_{\eta_s''}^{x_i} [P(x, y)/Q_1(x, y)](-y'/y^p)\,dx,$$

$$[y'(\eta_{s+1})]^2 = \int_{x_i}^{\eta_{s+1}} [P(x, y)/Q_1(x, y)](y'/y^p)\,dx.$$

The proof that, for a value of q independent of s,

$$P(x, y)/Q_1(x, y) > (\eta_s''/x_i)^q P(x_i, y)/Q_1(x_i, y), \quad (\eta_s'' \leqslant x \leqslant x_i, \ s \geqslant s_0),$$

$$< (\eta_{s+1}/x_i)^q P(x_i, y)/Q_1(x_i, y), \quad (x_i \leqslant x \leqslant \eta_{s+1}, \ s \geqslant s_0),$$

is exactly similar to the proof of the analogous inequalities (21) and (22)

in § 13. We therefore obtain

(32) $$[y'(\eta_s'')]^2 > (\eta_s''/x_i)^q \int_{y_i}^{y(\eta_s'')} [P(x_i, y)/y^p Q_1(x_i, y)]\, dy,$$

(33) $$[y'(\eta_{s+1})]^2 < (\eta_{s+1}/x_i)^q \int_{y_i}^{y(\eta_s'')} [P(x_i, y)/y^p Q_1(x_i, y)]\, dy,$$

from which (31) follows at once. We have therefore proved the following theorem:—

THEOREM I.—*If $y = y(x)$ is a proper solution of the differential equation*

(34) $$y'' = P(x, y)/Q(x, y),$$

where P and Q are polynomials in x and y, then, if the degree of P in y does not exceed the degree of Q by unity,

$$y(x) = O(x^K),$$

for some finite value of K.

More generally, we may suppose that P and Q are polynomials in y whose coefficients are functions of x expressible when $x > x_0$ in convergent power series of the form

$$f(x) = x^\mu \left(\sum_{n=0}^{\infty} a_n x^{-\nu n} \right),$$

where μ and ν are rational, $\nu > 0$, and $a_0 \neq 0$.

V.

16. We have now to consider the case in which the degree of P in y exceeds the degree of Q by unity. As before, if $y(x) \neq O(x^K)$ for any value of K, then it must be the case that either

(1) for all values of K,

$$y(x) \succ x^K;*$$

or (2) for all values of κ there exists a sequence of points whose abscissæ tend to infinity at which $y = y(x)$ crosses $y = x^\kappa$.*

* All reference to the cases $-y \succ x^K$, and $y = y(x)$ crosses $y = -x^\kappa$ will be omitted. They differ in no way from the cases treated.

Now when for all values of K $y(x) \succ x^K$, that is to say when

$$y(x) \succ x^{\Delta},$$

$y = y(x)$ satisfies an equation of the form

$$y''/y = s(x, y) \equiv f(x)\left[1+O(x^{-\Delta})\right], \tag{35}$$

where $f(x)$ is of the form

$$bx^m \left(1+\sum_{n=1}^{\infty} a_n x^{-\nu n}\right),$$

where m and ν are rational, and $\nu > 0$, and the series converges when $x > x_0$. If now we write

$$y = e^V, \quad u = V',$$

$u = u(x)$ will be a proper solution of the equation

$$u' = -u^2+s(x, y) \equiv -u^2+f(x)\left[1+O(x^{-\Delta})\right]. \tag{36}$$

Before we can make any further progress with this equation we must prove some such proposition as the following :—

If $y = y(x)$ *is a proper solution of* (35) *such that*

$$y(x) \succ x^{\Delta},$$

then $y'(x)/y(x)$ $[= u(x)]$ *is such that the ratio of any two of the three terms in* (36) *is ultimately monotonic.*

Without some such proposition as this we cannot deduce from (36) any information concerning the asymptotic form of y. With this end in view we prove the following Lemmas.

17. LEMMA I.—*If* $y(x) \succ x^{\Delta}$, *then*

$$y'(x) < [y(x)]^{1+\mu}, \quad (x > x_0), \tag{37}$$

where μ *is any positive constant.*

If (37) is not true then it must be the case that either

(i) $y'/y^{1+\mu} > 1, \quad (x > x_0)$,

or (ii) there must exist a sequence $\{x_s\}$ tending to infinity such that

$$(y'/y^{1+\mu})-1 = 0, \quad \{x_s\}.$$

Both of these cases are impossible. In case (i) we can integrate from x_0 to x, obtaining

$$[y(x_0)]^{-\mu}-[y(x)]^{-\mu} > \mu(x-x_0), \quad (x > x_0),$$

which contradicts $y(x) \to \infty$. In case (ii) since $d/dx(y'/y^{1+\mu})$ exists and is continuous when $x > x_0$, there exists a sequence $\{x'_s\}$ tending to infinity such that

$$y'/y^{1+\mu} \geqslant 1, \quad d/dx\,(y'/y^{1+\mu}) = 0, \quad \{x_s\},$$

or, with the help of (35),

$$y' \geqslant y^{1+\mu}, \quad (1+\mu)(y')^2-bx^m y^2(1+\epsilon) = 0, \quad \{x_s\},$$

where $\epsilon \to 0$, $\{x_s\}$. Since $y(x) \succ x^{\Delta}$, this cannot be satisfied and our Lemma is proved.

COROLLARY.—*The function defined in* (35), *namely,*

$$S(x, y) = f(x)\,[1+O(x^{-\Delta})],$$

is such that

$$(38) \qquad (d/dx)^p\,[S(x, y)] = [(d/dx)^p f(x)]\,[1+O(x^{-\Delta})].$$

This is an immediate deduction from Lemma I, which was proved to establish (38).

LEMMA II.—*If* $y = y(x)$ *is a proper solution of* (34), $y(x) \succ x^{\Delta}$, $u(x) = y'(x)/y(x)$, *and* $H(x, u, S)$ *any rational function of* x, u, *and* S, *and if*

$$dH/dx = U(x, u, S, dS/dx)/V(x, u, S),$$

then either H *is ultimately monotonic, or else*

$$u = u(x)\,[1+O(x^{-\Delta})]$$

satisfies $U = 0$ *or* $V = 0$.

With the help of (38) it is easily seen that the proof of the similar theorem given by Mr. Hardy* can be adapted, with little alteration, to prove Lemma II. The proof of Lemma II will therefore be omitted. We are now in a position to proceed with our investigation.

18. When $x \to \infty$ and $|y| \geqslant x^{\lambda}$ for a sufficient value of λ, then

$$(39) \qquad y''/y = bx^m(1+\epsilon), \quad (\epsilon \to 0).$$

* *Proc. London Math. Soc.*, Ser. 2, Vol. 10, p. 455.

The signs of b and $(m+2)$ determine the possible forms of solutions. The facts to be established may be arranged as follows:—

PART 1.—If $y = y(x)$ intersects $y = x^\lambda$ (for a sufficient value of λ) in a sequence of points tending to infinity, then if $b < 0$ or $b > 0$ and $m+2 \leqslant 0$, there exists a K such that

$$y(x) = O(x^K).$$

PART 2.—If $b > 0$ and $m+2 > 0$ and $y(x) \neq O(x^K)$, then

$$y(x) \succ x^\Delta.$$

PART 3.—If $b < 0$ or $b > 0$ and $m+2 \leqslant 0$, then it is impossible that $y(x) \succ x^\Delta$. [Consequently, in these cases, there must exist a K such that any solution $y = y(x)$ satisfies

$$y(x) = O(x^K).]$$

PART 4.—If $b > 0$ and $m+2 > 0$, it is possible that $y(x) \succ x^\Delta$, and in that case

$$y = e^{b^{\frac{1}{2}} x^{\frac{1}{2}(m+2)}(1+\epsilon)}, \quad (\epsilon \to 0).$$

19. PART 1.—This is certainly true if $b < 0$, for the investigations of Parts II–IV cover this case as it stands. If $b > 0$ and $m+2 \leqslant 0$, in any interval of $\{\xi_s' \, \xi_{s+1}\}$ (see Figs. 3, 4), $y'' > 0$; and therefore $y' > 0$.* At the end of the interval $(x = \xi_{s+1})$, we have

$$y'(\xi_{s+1}) < \lambda (\xi_{s+1})^{\lambda - 1}.$$

It follows that

$$|y'(x)| < G(x)^{q_0}, \quad \{\xi_s\}, \tag{40}$$

where G and q_0 are independent of s. From (40) we can deduce by the arguments used in Parts II–IV, that

$$|y'(x)| < Hx^q, \quad \{\xi_s'\}, \tag{41}$$

where H and q are independent of s.

Now, with the help of (41) and the condition $m+2 \leqslant 0$, we shall prove that, in the interval $\xi_s' \, \xi_{s+1}$, if $s \geqslant s_0$, $y = y(x)$ lies below the curve

$$y = y_1(x) \equiv (\xi_s')^\lambda + H[x^p - (\xi_s')^p],$$

* Of course this only refers to $y > x^\lambda$. Similar arguments apply to the case $y < -x^\lambda$. Figs. 3 and 4 are drawn for $b < 0$, and are only here referred to for the notation.

for a sufficient value of p $(> q)$ independent of s, whence it follows at once that there exists a K such that

$$y(x) = O(x^K).$$

It follows from (41) that, at P (Fig. 5),

$$y_1'(x) > y'(x).$$

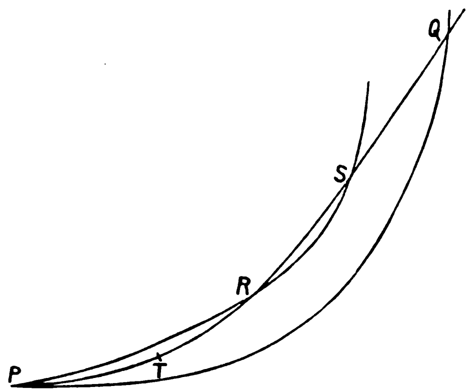

FIG. 5.

Therefore, if there is an interval RS in PQ in which $y = y(x)$ lies above $y = y_1(x)$, there must also exist an interval PR in which $y = y(x)$ lies below $y = y_1(x)$, and above $y = x^\lambda$. Therefore there exists a point T, such that

(42) $$y''(x) > y_1''(x) = p(p-1)Hx^{p-2}.$$

Now we may suppose p chosen so that

$$y_1(x) < H(x)^p,$$

at least if $s \geqslant s_0$. Therefore at T,

$$y(x) < H(x)^p;$$

and therefore, since $m+2 \leqslant 0$,

(43) $$y''(x) < bHx^{p-2}(1+\epsilon),$$

where $\epsilon \to 0$ as $s \to \infty$. (42) and (43) contradict each other, if $p(p-1) > b$. It follows that $y = y(x)$ lies below $y = y_1(x)$ in PQ if $s \geqslant s_0$, and therefore that

$$y(x) = O(x^K),$$

for some value of K. Part 1 is therefore established.

PART 2.—If, for any value of κ, $y = y(x)$ intersects infinitely often $y = x^\kappa$, then a sequence $\{x_s\}$ can be found such that

$$y''(x) \leqslant \kappa(\kappa-1)x^{\kappa-2}, \quad y(x) \geqslant x^\kappa, \quad \{x_s\}.$$

If $b > 0$ and $m+2 > 0$, this contradicts

$$y'' = bx^m y(1+\epsilon), \quad (\epsilon \to 0, \ \{x_s\}),$$

and is accordingly impossible. Part 2 is therefore proved.

PART 3.—It is obviously impossible that $y(x) \succ x^\Delta$ if $b < 0$, for this implies that $y'' < 0$, $(x > x_0)$, which is absurd. We have therefore to consider the case $b > 0$ when it is assumed that $y(x) \succ x^\Delta$.

If $u(x) = y'(x)/y(x)$, $u = u(x)$ satisfies

$$(36) \qquad u' = -u^2 + bx^m(1+\epsilon), \quad (\epsilon \to 0)$$
$$= -u^2 + f(x)[1+O(x^{-\Delta})].$$

Now, by Lemma II, u'/u^2 will be ultimately monotonic unless

$$u = u(x)[1+O(x^{-\Delta})]$$

satisfies
$$2f(x)[u^2 - f(x)] + uf'(x) = 0,$$

or
$$u = 0.$$

The latter is impossible. If $u = u(x)[1+O(x^{-\Delta})]$ satisfies the former, then

$$(44) \qquad u(x) = \left[-\frac{f'(x)}{4f(x)} \pm \tfrac{1}{2}\left(\frac{[f'(x)]^2}{4[f(x)]^2} + 4f(x)\right)^{\frac{1}{2}}\right][1+O(x^{-\Delta})].$$

Now when $m+2 \leqslant 0$, $u(x)$, if it is determined by (44), satisfies the relation

$$u(x) \preccurlyeq 1/x;$$

therefore
$$y'/y \preccurlyeq 1/x,$$

which may be integrated,* giving

$$y(x) = O(x^K).$$

It follows that $u(x)$ is not determined by (44), and therefore that u'/u^2 is ultimately monotonic. It must therefore be true that either

$$(a) \quad u'/u^2 \succcurlyeq 1,$$

or
$$(b) \quad u'/u^2 \prec 1.$$

---

* *Orders of Infinity*, p. 37.

In case (a) integration is legitimate and we obtain (without regard to sign)

$$1/u \succcurlyeq x,$$

which contradicts $y(x) \succ x^{\Delta}$. Case (a) is therefore impossible.

In case (b) we have, from (36),

$$u^2 \sim bx^m,$$

and since $m+2 \leqslant 0$, this again contradicts $y(x) \succ x^{\Delta}$. Since every case is impossible, it cannot be true that $y(x) \succ x^{\Delta}$, when $b > 0$ and $m+2 \leqslant 0$. Part 3 is therefore proved.

PART 4.—It is certainly possible in this case that $y(x) \succ x^{\Delta}$, for the solution $y = e^x$ of the equation $y'' = y$ is a case in point. We have only to determine the form of such a solution, when it exists for the general equation.

If $u(x)$ is determined by (44), then

$$u(x) \sim \pm (bx^m)^{\frac{1}{2}}.$$

If $u(x)$ is not determined by (44), then u'/u^2 is ultimately monotonic, and $u'/u^2 \succcurlyeq 1$ impossible as before. In either case therefore we have

$$u(x) \sim \pm (bx^m)^{\frac{1}{2}}.$$

Since $y'(x)$ must be positive when $y(x)$ is positive, we have

$$y'(x)/y(x) \sim +b^{\frac{1}{2}} x^{\frac{1}{2}m}.$$

This may be integrated, and we obtain finally

$$y(x) = e^{b^{\frac{1}{2}} x^{\frac{1}{2}(m+2)}(1+\epsilon)}, \quad (\epsilon \to 0).$$

This completes Part 4.

20. If we denote by the *leading term* of P/Q the term which represents the asymptotic value of P/Q, when $x \to \infty$ and $|y| \succcurlyeq x^{\lambda}$ for a sufficiently large λ, *i.e.*, $bx^m y$ in the notation we have been using, these results can be summed up as follows:—

THEOREM II.—*If* $y = y(x)$ *is a proper solution of*

$$y'' = P(x, y)/Q(x, y),$$

where P *and* Q *satisfy the conditions of Theorem I, and if the leading*

term of P/Q is $bx^m y$, then if $b < 0$ or $m+2 \leqslant 0$, there exists a K such that

$$y(x) = O(x^K);$$

but if $b > 0$ and $m+2 > 0$, then either there exists a K such that

$$y(x) = O(x^K),$$

or else

$$y(x) = e^{b^{\frac{1}{2}} x^{\frac{1}{2}(m+2)}(1+\epsilon)}, \quad (\epsilon \to 0).$$

This last result could evidently be made much more precise by greater attention to detail. It is fairly evident that $y(x)$ can be put in the form

$$Bx^A e^{\Pi(x^\nu)}(1+\epsilon), \quad (\epsilon \to 0),$$

where Π is a polynomial and ν is rational and > 0, but it does not seem worth while to proceed to this degree of accuracy.

VI.

21. There are one or two directions in which we might try to generalize these results without the introduction of any new kind of argument. We might try to establish results corresponding to Theorems I and II for equations in which the y'' of (34) is replaced by

$$(a) \quad (y'')^p, \qquad (b) \quad d/dx\,(y')^p, \qquad (c) \quad d/dx\,[f(x)\,y'],$$

where p is a positive integer,

$$f(x) = x^\mu \left(\sum_{n=0}^{\infty} a_n x^{-\nu n}\right),$$

μ and ν are rational, a_0, $\nu > 0$, and the series converges when $x > x_0$. We shall not consider the changes necessary to adapt the foregoing arguments to (a) and (b), but shall content ourselves with pointing out that, with mere verbal alterations, the arguments may be reapplied *in toto* to the form (c). We can state this result in the following theorem :—

THEOREM III.—*If $y = y(x)$ is a proper solution of*

$$d/dx\,[f(x)\,y'] = P(x, y)/Q(x, y), \tag{45}$$

where $f(x)$ and the coefficients of y in P and Q satisfy the conditions of Theorem I, then, if the leading term of P/Q is $bx^m y^n$ and $f(x) \sim a_0 x^\mu$,

$$y(x) = O(x^K),$$

unless $n = 1$, $b/a_0 > 0$, and $m-\mu+2 > 0$.

If these three conditions are all satisfied, then either

$$y(x) = O(x^K),$$

or else

$$y(x) = e^{(b/a_0)^{\frac{1}{2}} x^{\frac{1}{2}(m-n+2)}(1+\epsilon)}, \quad (\epsilon \to 0).$$

22. It is a question of some interest to see if more definite information can be obtained about the solutions of (45) say, that are $O(x^K)$. To take an example which is as free as possible from irrelevant detail, let us consider the equation

$$(46) \qquad d/dx\,(x^\rho y') = ax^m(y-a_1)\ldots(y-a_n), \quad (n>1),$$

where $m-\rho+2>0$, the a's are power series in x convergent when $x>x_0$, and no relation of the form $a_r \sim a_s$ holds for any values of r and s. Then it can easily be shown that the supposition, that $|y(x)-a_s|$ has, for all values of s, a lower limit other than zero as $x\to\infty$, leads to contradictions. It follows that *either*

$$y(x)-a_s \to 0,$$

for some value of s, *or else* $y=y(x)$ *cuts one at least of*

$$y = a_s, \quad (s=1, 2, \ldots, n)$$

in a sequence of points tending to infinity. In cases where $y=y(x)$ does not cut *more* than one of these curves infinitely often—for instance, it can be proved that a proper solution of

$$y'' = x^m(y^2-x^2), \quad (m+2>0)$$

cannot cut both $y=x$ and $y=-x$ infinitely often—we have obtained some definite information about $y=y(x)$. We can express it by saying that any solution is definitely associated with one particular real branch of $P(x,y)=0$.

This, however, is not true at any rate when, for some values of r and s, $a_r-a_s\to 0$. Consider the function

$$y(x) = x^\alpha\,\mathrm{sn}\,(\lambda x^\beta, i),$$

which is real when x is real, and oscillates between the limits x^α and $-x^\alpha$ as $x\to\infty$, if $\beta>0$. It satisfies the differential equation

$$(47) \qquad \frac{d}{dx}(y'/x^{2\alpha+\beta-1}) = -2\beta^2\lambda^2 x^{-4\alpha+\beta-1}\,y\left(y^2+\frac{\alpha(\alpha+\beta)}{2\beta^2\lambda^2}\,x^{2\alpha-2\beta}\right).$$

The three roots of the right-hand side are all real if

$$\alpha(\alpha+\beta) < 0,$$

and the conditions imposed on (46) are all satisfied if

$$\beta-\alpha > 0.$$

Since $\beta > 0$, $\alpha-\beta < \alpha$, and therefore the function $y=y(x)$ cuts infinitely often all three roots of the right-hand side of (47). It is, however, necessary that $\alpha < 0$, and the example throws no light on the question whether *if* $\alpha_r-\alpha_s$ *does not* $\rightarrow 0$ *for any values of* r *and* s, it may not be true that *any* solution is definitely associated with one particular real branch of $P(x, y) = 0$.

23. I shall conclude this paper with some remarks about the more general second order differential equations

$$y'' = P(x, y, y'), \quad y'' = P(x, y, y')/Q(x, y, y'), \tag{48}$$

where P and Q are polynomials in x, y, y'. Obviously all the foregoing arguments cannot be applied to them. It is possible to obtain fairly accurate information for solutions which are assumed to increase faster than x^K for a sufficient value of K, by arguments which are natural developments of the foregoing, but some new form of attack seems to be needed for solutions not thus restricted.

The much more stringent assumption that the solution under consideration is such that any rational function of x, y, and y' is ultimately monotonic (*i.e.*, the assumption of *strictly regular increase*), leads to results exactly analogous to the results of Part II of Mr. Hardy's paper already quoted. Mr. Hardy there proves that *any proper solution of*

$$y' = P(x, y)/Q(x, y), \tag{49}$$

where P *and* Q *are polynomials in* x *and* y, *is of strictly regular increase, and that if it* $\rightarrow \infty$ *its rate of increase is measured by one of*

(1) $Ax^B e^{\Pi(x)}$,

(2) $A(x^p \log x)^{1/q}$,

(3) $Ax^{p/q}$,

where $\Pi(x)$ *is a polynomial,* p *and* q *are integers,* $q \neq 0$, *and* A *and* B *are any real numbers.*

On the *assumption* that $y = y(x)$ *is a proper solution of* (48) *of strictly*

regular increase, which $\rightarrow \infty$, it can be proved that *its rate of increase is measured by one of*

(1)	$e^{(A+\epsilon)x^B e^{\Pi(x)}}$,	(9)	$A\,(\log\log x)^{1/p}$,
(2)	$e^{(A+\epsilon)(x^q \log x)^{1/p}}$,	(8)	$A\,[(\log x)^p \log\log x]^{1/q}$,
(3)	$e^{(A+\epsilon)x^B}$,	(7)	$A\,(\log x)^B$,
(4)	$e^{(A+\epsilon)(\log x)^{(p+1)/p}}$, $(p > 0)$,	(6)	$e^{(A+\epsilon)(\log x)^{(p-1)/p}}$, $(p > 0$,

$$(5) \qquad x^{A+\epsilon}.$$

In these formulæ p and q are integers other than zero, A, B are any real numbers, $\Pi(x)$ is a polynomial in x, and $\epsilon \rightarrow 0$. If these formulæ are compared with the formulæ for L-functions* of order 2, it will be seen that they differ therefrom, by the admission of certain irrational indices, in exactly the same way as the rates of increase of solutions of (49) differ from the rates of increase of L-functions of order 1.

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* G. H. Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 10, p. 87.

SURFACES WITH SPHERICAL LINES OF CURVATURE AND SURFACES WHERE SYSTEMS OF INFLEXIONAL TANGENTS BELONG TO SYSTEMS OF LINEAR COMPLEXES

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A CONSIDERABLE amount of attention has been devoted to the study of the system of surfaces on which the lines of curvature are plane or spherical, and the paper which follows is based on the exposition contained in Darboux's *Théorie Générale des Surfaces*, Livre IV, Ch. IX and XI, taken in conjunction with Lie's beautiful contact transformation, by which spheres are transformed into straight lines. Lie's transformation was first presented in his paper, "Über Complexe, inbesondere Linien- und Kugelcomplexe mit Anwendung auf die Theorie partieller Differentialgleichungen," *Math. Annalen*, 5 Bd. (1872), p. 145. A most interesting account of this line geometry and of the famous transformation will be found in the *Geometrie der Berührungstransformationen*, Lie-Scheffers, Vol. I; and I would draw attention to chapters 7, 9, 10, and 14, and in particular to Theorem 12 on p. 373, Theorem 25 on p. 639, and the relations between the line and sphere geometry illustrated on pp. 654 and 655.

In what follows I give a brief account of Lie's transformation; and show how by means of it he connects the geometry of lines of curvature with the geometry of asymptotic lines, and in particular the surfaces, whose lines of curvature are spherical, with surfaces whose asymptotic lines are such that the tangents to them belong to a system of linear complexes. Readers who desire fuller information will refer to the *Berührungstransformationen*. I then consider at much greater length the latter class of surfaces, a knowledge of whose properties has been shown to involve a knowledge of the properties of the first class, and in this investigation I make considerable use of the vectorial notation.

1. Lie's contact transformation has the generating equations

$$x'+\iota y'+xz'+z=0, \quad x(x'-\iota y')-y-z'=0,$$

leading to

$$p'(x-q)+1+qx=0, \quad q'(x-q)+\iota(1-qx)=0, \quad p+z'+q(x'-\iota y')=0.$$

Each element of space $x'y'z'$ can be expressed uniquely in terms of the corresponding element of space xyz. If, however, we wish to express an element of the latter space in terms of the first, we have two alternatives, viz.,

$$x=\frac{p'+\iota q'}{1-\sqrt{1+p'^2+q'^2}} \quad \text{or} \quad \frac{p'+\iota q'}{1+\sqrt{1+p'^2+q'^2}};$$

but we keep to the first of these alternatives.

Eliminating x, y, z between the generating equations and the equations of a straight line

$$a=mz-ny, \quad \beta=nx-lz, \quad \gamma=ly-mx,$$

we have

$$l(x'^2+y'^2+z'^2)-\beta(x'-\iota y')-m(x'+\iota y')+(n+\gamma)z'-a=0.$$

Writing the equation of this sphere in the form

$$x'^2+y'^2+z'^2+2gx'+2fy'+2hz'+c=0,$$

we see that the line whose six coordinates are

$$\frac{l}{1}=\frac{m}{-g+\iota f}=\frac{n}{h-r}=\frac{a}{-c}=\frac{\beta}{-g-\iota f}=\frac{\gamma}{h+r}$$

corresponds to that part of the sphere for which

$$r=(z'+h)\sqrt{1+p'^2+q'^2},$$

where r is the sphere's radius. We call this line the positive correspondent of the sphere. The line deduced from the positive correspondent by changing the sign of r is called the negative correspondent and corresponds to the other hemisphere of the sphere. For a plane, regarded as a sphere of infinite radius, we see that the corresponding lines will be perpendicular to the axis of x.

2. If l', m', n', a', β', γ' are the coordinates of a linear complex it is said to be special if

$$l'a'+m'\beta'+n'\gamma'=0.$$

If $l_1, m_1, n_1, a_1, \beta_1, \gamma_1$ and $l_2, m_2, n_2, a_2, \beta_2, \gamma_2$ are the coordinates of two

linear complexes, and p and q are two parameters, the complexes whose coordinates are

$$p_1 l_1+p_2 l_2,\ p_1 m_1+p_2 m_2,\ p_1 n_1+p_2 n_2,\ p_1 a_1+p_2 a_2,\ p_1 \beta_1+p_2 \beta_2,\ p_1 \gamma_1+p_2 \gamma_2$$

are said to form a pencil. In any pencil of complexes clearly there are two special ones.

Consider now the pencil whose two special complexes are the positive and negative correspondents of a sphere, viz.,

$$l',\ m',\ n',\ a',\ \beta',\ \gamma' \quad \text{and} \quad l',\ m',\ \gamma',\ a',\ \beta',\ n',$$

then the general complex of the pencil has the coordinates

$$l',\ m',\ n' \cos^2\frac{\theta}{2}+\gamma' \sin^2\frac{\theta}{2},\quad a',\ \beta',\ \gamma' \cos^2\frac{\theta}{2}+n' \sin^2\frac{\theta}{2}.$$

We now see that if l, m, n, a, β, γ are the six coordinates of any line which is the positive correspondent of a sphere, and if this line belongs to the above general complex, the sphere will cut at an angle θ, the sphere of which l', m', n', a', β', γ' is the positive correspondent.

3. We know that the two spheres, which touch any surface and whose radii are the principal radii of curvature of the surface at the point, touch the surface at two consecutive points (Salmon, *Solid Geometry*, 4th edition, p. 267). If we apply our contact transformation to the surface and the spheres, we obtain in the other space a surface and two inflectional tangents, and thus the lines of curvature are transformed into asymptotic lines.

If a sphere can be described through a line of curvature it will intersect the surface everywhere along that line at the same angle, and, therefore, all of the spheres of one system at the same angle. This is but a particular case of the well known theorem that two surfaces intersect at a constant angle if their line of intersection is a line of curvature on both surfaces. Conversely it must be shown that if a sphere intersects all of the spheres which have stationary contact with a surface along a line of curvature at the same angle, then the line of curvature is spherical, that is, lies on a sphere.

Let the surface, referred to its lines of curvature, be given vectorially by the equations

$$z_1 = a\lambda_1, \quad z_2 = b\lambda_2,$$

where the surface is traced out by the extremity of the vector z which depends on the two parameters u and v, and where λ is a unit vector

parallel to the normal. The suffix 1 denotes that the vector or scalar to which it is attached is the derivative of that vector or scalar with respect to u, and the suffix 2 has a similar meaning; a and b are scalars, which are the principal radii of curvature. The equation of the sphere which has stationary contact with the surface along the $v =$ constant, direction may be written

$$z' = z - a\lambda + a\mu,$$

l' being the vector to any point of it and μ a unit vector.

Let

$$z' = \gamma + c\mu'$$

be a fixed sphere of radius c, γ be the vector to its centre, and μ' a unit vector, and suppose that this sphere cuts all the spheres, obtained by varying u only in the equation

$$z' = z - a\lambda + a\mu,$$

at the same angle θ.

From first principles we have

$$(z - a\lambda - \gamma)^2 + a^2 + c^2 - 2ac\cos\theta = 0,$$

and therefore, differentiating with respect to u and remembering that

$$z_1 = a\lambda_1,$$

we get

$$a_1 S(z - a\lambda - \gamma)\lambda - aa_1 + a_1 c\cos\theta = 0,$$

or

$$S(z-\gamma)\lambda + c\cos\theta = 0.$$

Differentiating again with respect to u and remembering that $Sz_1\lambda$ is zero, we see that

$$S(z-\gamma)\lambda_1 = 0.$$

It follows that a sphere whose centre is at the extremity of γ can be described through the line of curvature along which only u varies.

4. From the theorem just proved it follows that if all the inflectional tangents along an asymptotic line belong to a linear complex whose coordinates are

$$l',\ m',\ n'\cos^2\frac{\theta}{2} + \gamma'\sin^2\frac{\theta}{2}, \qquad \alpha',\ \beta',\ \gamma'\cos^2\frac{\theta}{2} + n'\sin^2\frac{\theta}{2},$$

then, in the transformed surface, the corresponding line of curvature will be spherical. Conversely, from the original theorem, a spherical line of curvature will transform into an asymptotic line, all the tangents to which will belong to a linear complex of the kind we have discussed. If

the line of curvature is plane, the corresponding linear complex will have the special mark that l' is zero.

If the line of curvature is circular two spheres can be described through it, and therefore the inflectional tangents in the corresponding asymptotic line will belong to two linear complexes.

Suppose that all the lines of curvature of one system are spherical, then, in the corresponding surface, all the asymptotic lines along which v is constant will have the property that the corresponding inflectional tangents belong to a linear complex whose coordinates are functions of v only, and conversely.

5. Instead, therefore, of investigating the surfaces with plane or spherical lines of curvature, we investigate the surfaces whose systems of inflectional tangents belong to systems of linear complexes, and, on transforming back again, we have the surfaces with plane or spherical lines of curvature.

The equations

$$\alpha = mz - ny, \quad \beta = nx - lz, \quad \gamma = ly - mx,$$

representing a straight line whose six coordinates are $l, m, n, \alpha, \beta, \gamma$, the line belongs to the linear complex whose coordinates are $l', m', n', \alpha', \beta', \gamma'$, when

$$\alpha' l + \beta' m + \gamma' n + l'\alpha + m'\beta + n'\gamma = 0.$$

We can express the theory of the linear complex simply by aid of the vector notation. Let i, j, k be three unit vectors mutually at right angles, and let

$$a' = il' + jm' + kn', \quad b' = i\alpha' + j\beta' + k\gamma',$$

$$a = il + jm + kn, \quad b = i\alpha + j\beta + k\gamma;$$

then we say that the vectors a and b are the coordinates of the line, and a', b' are the coordinates of the linear complex.

If x and y are any two vectors, I find it more convenient to denote the vectorial part of xy by the symbol $\overset{\frown}{xy}$ than by the more usual symbol Vxy, and the scalar part by $\underset{\smile}{xy}$ rather than by Sxy.

The two coordinates of a straight line a and b are connected by the equation

$$\underset{\smile}{ab} = 0,$$

and, if the straight line belongs to the linear complex whose coordinates are a' and b', we have

$$\underset{\smile}{ab'} + \underset{\smile}{ba'} = 0.$$

The equation of the straight line whose coordinates are a, b is

$$\widehat{az'} = b,$$

z' being a vector drawn, as all the other vectors, from the origin, its extremity traces out the line. If the extremity of a vector z traces out any curve in space, we may take, as the two coordinates of any tangent line to this curve,

$$a = z_1, \quad b = \widehat{z_1 z},$$

where u is the parametric coordinate of any point on the curve and z_1 denotes the first derivative of z with respect to u. The equation of the tangent line to the curve will now be

$$\widehat{z_1 z'} = \widehat{z_1 z}.$$

If we now denote by a and b the coordinates of a linear complex, all the tangent lines to the curve traced out by the extremity of the vector z will belong to this linear complex if

$$Sz_1(az+b) = 0.$$

If a and b are fixed, this, then, is the equation of the curve whose tangent lines belong to the given complex.

6. Suppose next that we have a surface and that we choose the parametric coordinates so that the curves

$$v = \text{constant}, \quad u = \text{constant}$$

are the asymptotic lines, then, if the surface is traced out by the extremity of the vector z, we have

$$z_1 = \widehat{ll_1}, \quad z_2 = -\widehat{ll_2},$$

where the suffix 2 attached to any vector denotes the derivative of that vector with respect to v. This is true for any surface, and is for many purposes the most convenient way of studying the properties of the surface. The vector l depends on the two parametric coordinates u and v of the surface and is clearly parallel to the normal to the surface at the point u, v.

Since $$z_{12} = z_{21},$$

we must have $$\widehat{ll_{12}} = 0;$$

and therefore the vector l must satisfy the Laplacian equation

$$l_{12} = pl,$$

where p is a scalar.

This equation is said to be of the first rank if p is zero. If it is not of the first rank let

$$p' = p - \frac{\partial^2 \log p}{\partial u\, \partial v},$$

and let

$$l' = l_1,$$

then we see that

$$l'_{12} = \frac{p_1}{p} l'_2 + pl'.$$

The Laplacian invariants of this new equation are p and p'. If p' is zero we can solve the original equation by quadratures, and it is said to be of the second rank. If it is neither of the first or second rank, let

$$l'' = l'_1 - \frac{p_1}{p} l' = l_{11} - \frac{p_1}{p} l_1,$$

then we see that

$$l''_2 = p'l', \quad l''_{12} = p'_1 l' + p'l'_1,$$

so that

$$l''_{12} - \frac{\partial}{\partial u} \log (pp') l''_2 - p'l'' = 0.$$

The invariants of this Laplacian equation in l'' are

$$p' \quad \text{and} \quad p' - \frac{\partial^2 \log (pp')}{\partial u\, \partial v}.$$

If the second invariant is zero, the original equation

$$l_{12} = pl$$

is said to be of the third rank.

The above is a brief sketch of Laplace's transformations in so far as they will be required for the purpose of this paper. The method is fully explained in Darboux's *Theory of Surfaces*, II, p. 23 and onwards, as also in Forsyth's *Theory of Differential Equations*, Part IV, § 191, &c.

The fact that we are dealing with a vectorial equation

$$l_{12} = pl$$

rather than with a scalar one makes no real difference, for if

$$l = ix + jy + kz,$$

the vectorial equation is equivalent with the three scalar ones

$$x_{12} = px, \quad y_{12} = py, \quad z_{12} = pz.$$

7. Suppose now that the surface has the property that the inflectional tangents which touch the asymptotic line $v =$ constant belong to a linear complex whose coordinates a and b are functions of v, and that this is true for all values of v, we shall prove that the equation

$$l_{12} = pl$$

is of the first, second, or third rank. We have for any surface which satisfies our condition

$$Sz_1(az+b) = 0;$$

and therefore, differentiating with respect to u,

$$Sz_{11}(az+b).$$

Now the parametric lines being asymptotic, we have

$$z_{11} = mz_1+nz_2,$$

where m and n are scalars.

If n is zero we see at once that

$$z = ra+\beta,$$

where r is a scalar and a and β are vectors depending only on v; that is, the surface is a ruled one. A ruled surface obviously satisfies the condition of the question, and the coordinates a and b of the linear complex only need to satisfy one condition. It is also easily seen that for a ruled surface

$$l_{12} = pl$$

must be of the first or second rank.

Again, if

$$l_{11} = ml_1+sl,$$

where m and s are scalars, we have, since

$$z_1 = \widehat{ll_1},$$

$$z_{11} = mz_1;$$

and therefore the surface is ruled.

We shall not further consider the case of ruled surfaces, and therefore we may assume in future that

$$l_{11} \neq ml_1+sl \quad \text{and} \quad l_{11} \neq mz_1.$$

8. Leaving aside the particular case of ruled surfaces, we shall prove that not only must

$$l_{12} = pl$$

be of one of the first three ranks, but that, when the surface is given, the coordinates of the complex a and b are determined.

We shall also see that, if the equation is of the necessary rank, we cannot take any value of l which satisfies the equation, but only those which satisfy a particular further condition.

Since
$$z_{11} = mz_1 + nz_2,$$

and n is not zero, we have also

$$Sz_2(az+b) = 0.$$

Since, if
$$\widehat{az}+b = 0,$$

the surface is ruled, we need only consider the other alternative that it is parallel to the normal, and therefore

$$\widehat{az}+b = kl,$$

where k is some scalar which cannot be zero. Differentiate this equation with respect to u, and we obtain

$$Va\widehat{ll_1} = kl_1 + k_1 l,$$

that is,
$$l_1 \underset{\smile}{al} - l a l_1 = k_1 l + k_1 l,$$

so that
$$\underset{\smile}{al} = k, \quad k_1 = -\underset{\smile}{al_1}.$$

Differentiating the first of these equations with respect to u, we see that

$$k_1 = \underset{\smile}{al_1} = -\underset{\smile}{al_1};$$

and therefore k_1 must be zero; that is, k is a function of v only. As k is not zero we may by a transformation of the form

$$v' = \phi(v)$$

take it to be minus unity, and we thus have

$$\underset{\smile}{al} + 1 = 0, \quad l_{12} = pl.$$

By differentiation we now obtain

$$\underset{\smile}{l_1 a} = 0, \quad \underset{\smile}{l_1 a_2} = p, \quad p\,\underset{\smile}{la_2} + \underset{\smile}{l_1 a_{22}} = p_2.$$

9. We first consider the possibility that

$$a_2 = Aa,$$

where A is a scalar function of u. The vector a would then be fixed in direction and p would be zero, so that the equation would be of the first rank. We then have

$$l = \alpha + \beta,$$

where α is a vector depending on u only and v a vector depending on v only.

Since a must be a vector fixed in direction we may say that

$$a = mk,$$

where i, j, k are unit vectors, mutually at right angles, and m a scalar function of v. If $f(u)$ and $\phi(v)$ are respectively the coefficients of k in α and β, then, in order that we may have

$$\underset{\smile}{al} + 1 = 0,$$

it is necessary and sufficient that

$$m\,[f(u) + \phi(v)] = 1.$$

This involves $f(u)$ being a mere constant and defines m in terms of v, and thus the vector a.

We now leave aside the particular hypothesis as regards a. If, however, the equation is still of rank 1 we have

$$l_1 = m\widehat{aa_2};$$

and therefore
$$l_{11} = \frac{m_1}{m} l_1,$$

which leads to a ruled surface.

10. Differentiating with respect to u, the equation

$$p\,\underset{\smile}{la_2} + \underset{\smile}{l_1 a_{22}} = p\frac{\partial}{\partial v}\log p,$$

we get
$$pp' = \frac{p_1}{p}\,\underset{\smile}{l_1 a_{22}} - \underset{\smile}{l_{11} a_{22}},$$

where
$$p' = p - \frac{\partial^2 \log p}{\partial u\,\partial v}.$$

If the equation is of the second rank so that p' is zero, then, from what

we have said about Laplace's transformation, we know that

$$l_{11} - \frac{p_1}{p} l_1 = \zeta,$$

when ζ is a vector depending on u only. We therefore have, from what we have just proved, the further fact about this vector ζ, viz., that

$$\underset{\smile}{\zeta a_{22}} = 0.$$

But $\underset{\smile}{l_1 a} = 0, \quad \underset{\smile}{l_{11} a} = 0, \quad \underset{\smile}{l_{11} a_2} = p_1, \quad \underset{\smile}{l_1 a_2} = p;$

and therefore also $\underset{\smile}{\zeta a} = 0, \quad \underset{\smile}{\zeta a_2} = 0.$

We leave aside the case where ζ is zero as we have already considered it, since then

$$l_{11} = \frac{p_1}{p} l_1.$$

The vectors a, a_2, and a_{22} being each perpendicular to ζ must be coplanar, and therefore

$$a_{22} = A a_2 + B a,$$

where A and B are scalars. It follows that the vector $\widehat{aa_2}$ is fixed *in direction* in space; for this direction does not depend on u, and we have now seen that in the particular case before us it does not depend on v either.

It follows that γ denoting a unit vector in this fixed direction

$$\zeta = m\gamma,$$

where m is a scalar depending on u only. By a transformation of the form

$$du' = \sqrt{m}\, du$$

we can therefore bring the equations

$$l_{11} - \frac{p_1}{p} l_1 = m\gamma \quad \text{and} \quad l_{12} = pl,$$

to the respective forms

$$l_{11} - \frac{p_1'}{p'} l_1 = \gamma \quad \text{and} \quad l_{12} = p'l,$$

where $p' = pm^{-\frac{1}{2}}.$

We now omit accent on p and have to find the most general common integral of

$$p \frac{\partial}{\partial u} \frac{l_1}{p} = \gamma \quad \text{and} \quad l_{12} = pl,$$

where
$$p = \frac{\partial^2 \log p}{\partial u \, \partial v}.$$

Let
$$\theta_1 = \frac{1}{p},$$
then the first equation gives us

$$l_1 \theta_1 = \theta\gamma + \eta,$$

where η is a vector depending on v only.

Differentiating with respect to v and remembering that

$$l_{12} = pl,$$

we see that
$$l = \left(\frac{p_2}{p}\theta + \theta_2\right)\gamma + \frac{p_2}{p}\eta + \eta_2,$$

and it is easily verified that this value of l satisfies the condition of being the common integral of the two equations

$$l_{11} - \frac{p_1}{p} l_1 = \gamma \quad \text{and} \quad l_{12} = pl.$$

In order that we may also have

$$\underset{\smile}{al} + 1 = 0,$$

where a depends on v only, we see that it is necessary and sufficient that the vector a should satisfy the three conditions

$$\underset{\smile}{a\gamma} = 0, \quad \underset{\smile}{a\eta} = 0, \quad \underset{\smile}{a\eta_2} + 1 = 0;$$

for we saw from the definition of γ that the first of these equations must be true, and the second follows from the fact that $\underset{\smile}{al_1} = 0$. We thus have the solution required of the important particular case of

$$l_{12} = pl$$
being of the second rank.

It may be noticed in passing that it is only when the equation is of the first or second rank that the vectors a, a_2, a_{22}, and therefore all the other derived vectors a_{222}, ..., are coplanar.

11. We now assume that p' is not zero, and by differentiating with respect to v the equation

$$pp' = \frac{p_1}{p} \underset{\smile}{l_1 a_{22}} - \underset{\smile}{l_{11} a_{22}}$$

(obtained at the beginning of § 10), and remembering that

$$l_{112} = p_1 l + p l_1,$$

we have
$$pp' \frac{\partial}{\partial v} \log pp' = \frac{p_1}{p} \underset{\smile}{l_1 a_{222}} - \underset{\smile}{l_{11} a_{222}} - p' \underset{\smile}{l_1 a_{22}}.$$

Let
$$a_{222} = A a_{22} + B a_2 + C a,$$

where A, B, C are scalars depending on v only, then since

$$\frac{p_1}{p} \underset{\smile}{l_1 a_{22}} - \underset{\smile}{l_{11} a_{22}} = pp',$$

$$\frac{p_1}{p} \underset{\smile}{l_1 a_2} - \underset{\smile}{l_{11} a_2} = 0,$$

$$\frac{p_1}{p} \underset{\smile}{l_1 a} - \underset{\smile}{l_{11} a} = 0,$$

we see that
$$\frac{\partial}{\partial v} \log (pp') = A - \frac{\underset{\smile}{l_1 a_{22}}}{p}.$$

Differentiating once more with respect to u, we immediately deduce from the equation

$$pp' = \frac{p_1}{p} \underset{\smile}{l_1 a_{22}} - \underset{\smile}{l_{11} a_{22}},$$

that
$$p' = \frac{\partial^2 \log (pp')}{\partial u \partial v},$$

so that the equation must be of the third rank.

12. Now we have seen that if

$$l'' = l_{11} - \frac{p_1}{p} l_1,$$

$$l''_{12} - \frac{\partial}{\partial u} \log (pp') l''_2 - p' l'' = 0;$$

and therefore if the original equation is of the third rank a first integral of this equation is

$$l''_1 - \frac{\partial \log pp'}{\partial u} l'' = \zeta,$$

where ζ is a vector depending on u only. But we have just seen that

$$\underset{\smile}{l'' a} = 0, \quad \underset{\smile}{l'' a_2} = 0, \quad \underset{\smile}{l'' a_{22}} + pp' = 0;$$

and therefore $\quad \underset{\smile}{\zeta a} = 0, \quad \underset{\smile}{\zeta a_2} = 0, \quad \underset{\smile}{\zeta a_{22}} = 0.$

It follows, since $\quad Saa_2a_{22} \neq 0,$

that ζ must be zero.

We can now find l, for

$$l = \frac{p_2}{p^2}\, l_1 + \frac{\partial}{\partial v}\, \frac{l_1}{p} = \frac{p_2}{p^2}\, l' + \frac{\partial}{\partial v}\, \frac{l'}{p} = \frac{p_2}{p}\, \frac{l_2''}{pp'} + \frac{\partial}{\partial v}\, \frac{l_2''}{pp'}\,;$$

and, since $\quad pp'l_1'' = \dfrac{\partial}{\partial u}(pp')\, l'',$

$$l'' = pp'\zeta,$$

where ζ is a vector depending on v only; we therefore have

$$l = \frac{p_2}{p}\left(\zeta_2 + \zeta\, \frac{\partial}{\partial v} \log pp'\right) + \frac{\partial}{\partial v}\left(\zeta_2 + \zeta\, \frac{\partial}{\partial v} \log pp'\right),$$

or say $\quad l = \zeta_{22} + A\zeta_2 + B\zeta,$

where A and B are scalars defined by

$$A = \frac{\partial}{\partial v} \log p^2p', \qquad B = \frac{\partial}{\partial v} \log p\, \frac{\partial}{\partial v} \log pp' + \frac{\partial^2}{\partial v^2} \log pp'.$$

We have seen that

$$\underset{\smile}{l''a} = 0, \qquad \underset{\smile}{l''a_2} = 0, \qquad \underset{\smile}{l''a_{22}} + pp' = 0\,;$$

and therefore $\quad \underset{\smile}{\zeta a} = 0, \quad \underset{\smile}{\zeta a_2} = 0, \quad \underset{\smile}{\zeta a_{22}} + 1 = 0.$

It follows that the vector a is determined by the equations

$$\underset{\smile}{a\zeta} = 0, \qquad \underset{\smile}{a\zeta_2} = 0, \qquad \underset{\smile}{a\zeta_{22}} + 1 = 0.$$

We have now seen how to determine the vectors a and l which satisfy the three equations $\quad a_1 = 0, \quad \underset{\smile}{al} + 1 = 0, \quad l_{12} = pl,$

and seen that the latter equation must be of rank 1, 2 or 3 that this may be possible, and that this necessary condition is also sufficient.

Having obtained a and l we can find z by quadratures from

$$z_1 = \widehat{ll_1}, \qquad z_2 = -\widehat{ll_2}.$$

We now see that $\quad \widehat{az} + l$

is a vector which depends on v only, since it vanishes on differentiation with respect to u; it is, therefore equal to $-b$, and we have thus found

the vectors a and b uniquely which fit the particular surface. We notice that for a ruled surface, and only for a ruled surface, are these indeterminate.

13. We now pass on to consider the properties of surfaces, both of of whose systems of inflectional tangents belong to linear complexes, and we shall slightly change our notation for the sake of symmetry.

$$\text{Let}\quad A = \frac{\partial}{\partial u}\log p^2p', \qquad A' = \frac{\partial}{\partial u}\log p\,\frac{\partial}{\partial u}\log pp' + \frac{\partial^2}{\partial u^2}\log pp',$$

$$B = \frac{\partial}{\partial v}\log p^2p', \qquad B' = \frac{\partial}{\partial v}\log p\,\frac{\partial}{\partial v}\log pp' + \frac{\partial^2}{\partial v^2}\log pp'.$$

And let α denote a vector depending on u only, and β a vector depending on v only. We are assuming, for the sake of brevity from this onwards, that the surface is not ruled and that the equation

$$l_{12} = pl$$

is of the third rank.

If
$$l = \alpha_{11} + A\alpha_1 + A'\alpha,$$

then the surface, given by

$$z_1 = \widehat{ll_1}, \qquad z_2 = -\widehat{ll_2},$$

will have the property that its inflectional tangents of the u system will belong to a linear complex system which we can determine. If l can also be written in the form

$$l = \beta_{22} + B\beta_2 + B'\beta;$$

then the inflectional tangents of the v system will also belong to a linear complex system which we can determine.

Let
$$\xi \equiv \alpha_{11} + A\alpha_1 + A'\alpha,$$

we must try if a vector β can be found such that

$$\beta_{22} + B\beta_2 + B'\beta = \xi,$$

$$\beta_1 = 0.$$

Noticing that
$$B_1 = p, \qquad B'_1 = p\frac{\partial}{\partial v}\log pp',$$

we see that
$$\xi_1 = p\beta_2 + p\frac{\partial}{\partial v}\log pp'\beta;$$

and therefore
$$p'\beta = \frac{\partial}{\partial u}\frac{\xi_1}{p},$$

since
$$p' = \frac{\partial^2 \log pp'}{\partial u \partial v}.$$

We must therefore have
$$\frac{\partial}{\partial u}\frac{1}{p'}\frac{\partial}{\partial u}\frac{\xi_1}{p} = 0;$$

that is, we see that
$$\xi_{111} - A\xi_{11} + \xi_1(A' - A_1) = 0.$$

If we can find a ξ to satisfy this equation, then taking
$$p'\beta = \frac{\partial}{\partial u}\frac{\xi_1}{p},$$

we see that β is a vector depending on v only; and, remembering that
$$\xi_{12} = p\xi,$$

we see that
$$\xi_1 = p\beta_2 + p\frac{\partial}{\partial v}\log pp'\beta,$$

and differentiating this with respect to v, we see that
$$\xi = \beta_{22} + B\beta_2 + B'\beta.$$

We have therefore only to see if we can choose a, so that
$$\xi_{111} - A\xi_{11} + (A' - A_1)\xi_1 = 0,$$

where
$$\xi \equiv a_{11} + Aa_1 + A'a.$$

Expanding the first equation we see that it is necessary and sufficient that a vector a depending on u only should be found such that
$$\begin{aligned} a_{1111} &+ (2A_1 + 2A' - A^2)a_{111} + (3A_{11} + 3A'_1 - 3AA_1)a_{11} \\ &+ (A_{111} + 3A'_{11} - AA_{11} - 2AA'_1 + A'^2 - A_1^2)a_1 \\ &+ (A'_{111} - AA'_{11} + A'A'_1 - A_1A'_1)a = 0. \end{aligned}$$

This equation may be written
$$a_{1111} + 2aa_{111} + 3a_1a_{11} + (a_{11} + 2a')a_1 + a'_1 a = 0,$$

where $\quad 2a = 2A_1 + 2A' - A^2, \quad 2a' = 2A'_{11} - 2AA'_1 + A'^2.$

It may easily be verified that, from the definitions of A and A' and the

fact that the equation is of the third rank, a and a' are functions of u only. The only additional condition necessary in order that the surface generated from

$$l = a_{11} + A a_1 + A' a,$$

by

$$z_1 = \widehat{l l_1}, \quad z_2 = -\widehat{l l_2},$$

may have both systems of inflectional tangents belonging to systems of linear complexes, is that a, instead of being an arbitrary vectorial function of u, must be one which satisfies the equation

$$a_{11111} + 2 a a_{111} + 3 a_1 a_{11} + (a_{11} + 2a') a_1 + a_1' a = 0.$$

When a has been obtained, β is given uniquely, as we have seen.

SOME RULER CONSTRUCTIONS FOR THE COVARIANTS OF A BINARY QUANTIC

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I. *Introduction.*

IN Section III of this paper we give a discussion of the systems of conics which are associated with the unique $C_{2,4}$ of a binary sextic. Section IV is a lemma by which we construct certain joint covariants of a quadratic and a binary p-ic which are used in V to obtain ruler constructions for the lines which connect the marking points of the quadratic covariants of a binary sextic. Section VI deals with an interesting conic associated with the sextic. In Section VII a method is described for constructing by ruler the $C_{2,2}$ of a quantic of odd order, the work being illustrated by reference to the septimic and octavic.

II. *Notation.*

We shall represent a p-ic by means of p-points taken on a conic, which will be referred to as S. Following Prof. Elliott,* it will be convenient to take for S the parabola $y^2 = x$. The linear form $x-ty$ will then be represented by a point on the parabola whose coordinates are $x' = t^2$, $y' = t$. Throughout the following work x, y or X, Y will be used to denote *either* the variables in a binary form *or* the coordinates of a point on the conic $y^2 = x$. This need cause no confusion, however, since, if the *first* of the two meanings be intended, the equation or expression containing x, y will be homogeneous in these quantities. The advantage of the above representation lies in the fact that if

$$(x-t_1 y)(x-t_2 y) \equiv x^2-(t_1+t_2)xy+t_1 t_2 y^2$$

denotes a quadratic form, then the *equation* of the *line* which connects

* "On the Projective Geometry of some Covariants of a Binary Quintic," *Proc. London Math. Soc.*, Ser. 2, Vol. 6.

the points on the conic which mark the quadratic will be found to be $x-(t_1+t_2)y+t_1t_2=0$ referred to the axis and vertex tangent to the parabola $y^2=x$ as axes of coordinates. We thus pass immediately from a binary quadratic to the equation of the line that represents it.

In what follows we shall say that we have "linearly constructed a conic" when by a finite number of joins of points and intersections of lines we are able to construct with respect to the conic the polar of any point in the plane.

The notation employed for invariants and covariants is the one given in Prof. Elliott's *Algebra of Quantics*, Chap. XIII, pp. 322, 323.

III. *The $C_{2,4}$ of a Sextic.*

Writing the unique $C_{2,4}$ of the sextic in the form

$$A_0x^4+A_1x^3y+A_2x^2y^2+A_3xy^3+A_4y^4,$$

then the equation of any conic through its marking points is given by

$$[\Sigma] \quad 3A_0X^2+2A_2Y^2+3A_1XY+A_2X+3A_3Y+3A_4+\lambda I_2[Y^2-X]=0,$$

where I_2 denotes the invariant $a_0a_6-6a_1a_5+15a_2a_4-10a_3^2$, of the sextic which is taken to be $(a_0, a_1, a_2, \ldots, a_6)(x, y)^6$.

The point on S which corresponds to the linear form $\xi x+\eta y$ will have the coordinates

$$X=\frac{\eta^2}{\xi^2}, \qquad Y=-\frac{\eta}{\xi};$$

and the polar of this point with respect to any member of the above sytem of conics $[\Sigma]$ is

$$\begin{aligned} X[6A_0\eta^2-3A_1\xi\eta+A_2\xi^2]+Y[3A_1\eta^2-4A_2\xi\eta+3A_3\xi^2] \\ +[A_2\eta^2-3A_3\xi\eta+6A_4\xi^2] \\ -\lambda I_2[\xi^2X+2\xi\eta Y+\eta^2]=0. \end{aligned}$$

Now suppose $\xi x+\eta y$ to be a factor of $(a_0, a_1, a_2, \ldots, a_6)(x, y)^6$, and, further, for simplicity, take $\xi=0$. Then we may write the sextic in the form

$$y\,(b_0, b_1, b_2, \ldots, b_4, b_5)(x, y)^5.$$

Now call the point y, P_1 and denote the other five marking points of the sextic by $(P_2, P_3, P_4, P_5, P_6)$, these being given by $(b_0, b_1, b_2, \ldots, b_5)(x, y)^5$.

Then putting $\xi=0$ in the above equation of the polar of $\xi x+\eta y$, we shall obtain the equation of the polar of $P_1=(y)$ with respect to any

conic of the system [Σ] in the form [after expressing all the A's in terms of the b's].

$$[\mathrm{H}] \qquad [b_0 b_2 - b_1^2]\, X + [b_0 b_3 - b_1 b_2]\, Y + [b_1 b_3 - b_2^2] - \frac{5\lambda}{12}[b_0 b_4 - 4 b_1 b_3 + 3 b_2^2] = 0.$$

This system of polars, corresponding to different values of λ, all intersect the tangent at P_1 in the same point O. If we can construct the line for *two* values of λ, say $\lambda = 0$ and $\lambda = a$, we are able to construct it for every rational value of λ. For if the line corresponding to $\lambda = 0$ be OX_0 and that for $\lambda = \lambda_1$ be OX_1, then to determine OX corresponding to $\lambda = \lambda$ we have only to construct OX so that the anharmonic ratio

$$O[P_1 X_0 X_1 X] = \frac{\lambda}{\lambda_1}.$$

Let us consider first the case $\lambda = 0$. The equation [H] now becomes

$$(b_0 b_2 - b_1^2)\, X + (b_0 b_3 - b_1 b_2)\, Y + (b_1 b_3 - b_2^2) = 0.$$

This line represents the connector of the Hessian points of the cubic which consists of the polar 3-points of the point $P_1(y)$ with respect to the quintic $(P_2 P_3 P_4 P_5 P_6)$. It is therefore the line joining the poles with respect to S of the lines

$$b_0 X + 2 b_1 Y + b_2 = 0 \tag{i}$$

and

$$b_1 X + 2 b_2 Y + b_3 = 0. \tag{ii}$$

Now (i) is the connector of the polar 2-points of $P_1(y)$ with respect to the quintic $(P_2 P_3 P_4 P_5 P_6)$ [*i.e.* the quintic $(b_0, b_1, b_2, \ldots, b_5)(x, y)^5$]; while (ii) is the connector of the polar 2-points of the point denoted by x [the origin] with respect to the polar 3-points of P_1 with regard to the above quintic.

It is easy to give a construction for (i) by a process of induction such as Mr. C. F. Russell * has given.

The following direct method of finding the polar 2-points of a point P_1 with respect to a quintic $[P_2 P_3 P_4 P_5 P_6]$ may be noticed :—

Separate the quintic into $[P_2 P_3]$ and $[P_4 P_5 P_6]$. Construct the polar line of P_1 with respect to the triangle $[P_4 P_5 P_6]$, meeting the tangent at P_1 in M and $[P_2 P_3]$ in N. Join P_1 to the pole (with respect to S)

* "On the Geometrical Interpretation of Apolar Binary Forms," *Proc. London Math. Soc.*, Ser. 2, Vol. 4, 1906.

of P_2P_3, meeting S again in T. Let P_1N meet S again in R. Join M to the pole of RT, meeting P_2P_3 in X. Then X is a point on the line required which will be given by ten collinear points, corresponding to the ten separations of $(P_2P_3P_4P_5P_6)$ into sets of 2 and 3.

To verify this construction take P_1 to be y [point at infinity on axis of parabola] and take the quintic to be $x(x-y)(x^3-s_1x^2y+s_2xy^2-s_3y^3)$.

Then the polar 2-points of y with respect to the quintic are given by

$$10x^2-4(1+s_1)xy+(s_1+s_2)y^2,$$

which may be written

$$6x^2-4s_1xy+(s_1+s_2)y^2+4(x^2-xy). \tag{A}$$

Now the polar 2-points of $P_1(y)$ with respect to the cubic

$$x^3-s_1x^2y+s_2xy^2-s_3y^3$$

[the polar line of P_1 with regard to the triangle $P_4P_5P_6$] are given by

$$6x^2-4s_1xy+2s_2y^2, \tag{A$'$}$$

and the mate of (y), $[P_1]$ in the involution

$$6x^2-4s_1xy+2s_2y^2+\lambda(x^2-xy),$$

is

$$(3-2s_1)x+s_2y. \tag{B}$$

Now the harmonic conjugate of (y) with respect to x^2-xy is

$$2x-y. \tag{C}$$

Hence
$$\begin{aligned}(B)\times(C) &\equiv (2x-y)[(3-2s_1)x+s_2y] \\ &\equiv (6-4s_1)x^2+[2s_2+2s_1-3]xy-s_2y^2,\end{aligned}$$

and this is seen to be harmonically conjugate to the quadratic which occurs in the first part of the expression (A), *i.e.*, to

$$6x^2-4s_1xy+(s_1+s_2)y^2.$$

Further, this last expression will be seen to differ from the expression (A$'$) only in the coefficient of y^2. This justifies the construction given.

We have now to construct the line (ii), *i.e.*

$$b_1X+2b_2Y+b_3=0,$$

which we remarked was the polar line of x, the origin, with respect to the triangle consisting of the polar 3-points of y. It is clear we may take x to be one of the marking points P_2 of the quintic $[P_2P_3P_4P_5P_6]$. Then

it is easy to verify the following construction :—Construct the line joining the polar 2-points of P_2 with respect to $[P_1P_3P_4P_5P_6]$. Join P_1 to the pole of this line with respect to S, meeting S again in X_1. Join P_1 to the pole * of P_2X_1, meeting S again in X_2. Again construct the connector of the polar 2-points of P_1 with respect to $[P_2P_3P_4P_5P_6]$. Join P_2 to the pole of this line, meeting the conic S again in Y. Then P_1Y and P_2X_2 intersect in the pole of the line required.

We may observe that the line of the system (H) which corresponds to $\lambda = 0$ is the connector of the polar 2-points of P_1 with respect to the covariant $C_{2,4}$.

If the five corresponding lines be constructed one for each of the remaining points P_2, P_3, P_4, P_5, P_6, then the six lines so obtained touch a conic which passes through the Hessian points of the quartic $C_{2,4}$, *i.e.* through the marking points of the $C_{4,4}$ of Elliott's list.

We consider next the line obtained by putting $\lambda = -\frac{2}{5}$. This line represents the connector of the polar 2-points of P_1 with respect to the Hessian points of the quartic which consists of the polar 4-points of P_1 with respect to the quintic $(P_2P_3P_4P_5P_6)$.

Now we know that the connector of the polar 2-points of any point K with respect to the quartic of polar 4-points will envelope a conic which passes through the Hessian points of the quartic as k moves on S.

Now it is easy to show how to construct this connector when K is at any of the points $(P_2, P_3, P_4, P_5, P_6)$.

Taking K to be at P_2 we shall proceed as follows :—Construct O, the point conjugate to P_2 with respect to the 4-point system of conics, through $(P_3P_4P_5P_6)$. Let OP_1 meet the conic S again in X.

Again construct the connector of the polar 2-points of P_2 with respect to the quintic $(P_1P_3P_4P_5P_6)$. Let the connector of P_1 to the pole of this line meet the conic S again in Z. Let P_2Z meet the tangent at P_1 in Y. Then the required line is the line joining Y to the pole of P_2X.

We have thus constructed linearly a conic which passes through the Hessian points of the polar 4-points of P_1 with respect to $(P_2P_3P_4P_5P_6)$.

It will be useful for our present purpose and for future reference to give a ruler construction for the connector of the polar 2-points of any point on S with respect to a quartic which is given by the intersection of a given conic with the conic S.

Writing the quartic as

$$x^4 - s_1x^3y + s_2x^2y^2 - s_3xy^3 + s_4y^4,$$

* Unless otherwise stated, pole and polar refer to the conic S.

we consider the conic

$$3X^2+2s_2Y^2-3s_1XY+s_2X-3s_3Y+3s_4=0,$$

which passes through the marking points of the quartic. The polar of any point P on S with respect to this conic is the connector of the polar 2-points of P with respect to the quartic.

Further, this conic has the property that if K_1, K_2 are its intersections with the tangent at any point P of S, and if H_1, H_2 are the points of contact of the two remaining tangents which may be drawn from K_1, K_2 respectively to touch S, then the line H_1H_2 is the polar of P with respect to the conic.

We may now proceed as follows:—Take any point Q_1 on the tangent at P and let Q_2 be the mate of Q_1 in the involution determined by the system of conics on this tangent. Draw from Q_1 and Q_2 the remaining tangents to touch S at R_1, R_2 respectively. Then the line R_1R_2 will pass through O, the point conjugate to P with respect to the 4-point system. Construct (as is easily done) the polar of P with respect to the conic of the system which passes through Q_1, Q_2. Denote this polar by a; it is clear that a passes through O. Then for different pairs of mates of the involution like (Q_1Q_2) there will be a one-one correspondence between the corresponding lines (R_1R_2) and the lines a. One of the self-corresponding elements will be the tangent at P to S, while the other will be the connector of the polar 2-points of P with respect to the quartic, *i.e.* the line required.

We may note that if μ be the anharmonic ratio of the marking points of the quartic taken in any order on S, then the anharmonic ratio of these points taken in the same order on the above conic will be $\dfrac{2\mu-\mu^2}{2\mu-1}$.

By making use of this construction we are at once able to complete our determination of the conic which is got by putting $\lambda=-\frac{2}{5}$ in equation (Σ).

And by the preceding work we are able to construct the member of the system which corresponds to any rational value of λ.

Two cases are of interest.

Taking $\lambda=-\frac{1}{2}\frac{2}{5}$ we obtain a ruler construction for the connector of the polar 2-points of a given point (P_1) with respect to the sextic which represents the Hessian of a quintic $(P_2P_3P_4P_5P_6)$.

Corresponding to $\lambda=1$ we obtain the conic which has for a pole and polar pair any point P of S and the connector of the marking points of the covariant $C_{2,2}$ of the quintic which consists of the polar 5-points of P with respect to the sextic $(P_1P_2P_3P_4P_5P_6)$.

The line which represents the $C_{2,2}$ of a quintic was first constructed

by F. Morley in the case when the marking points of the quintic are assumed. We shall refer to it as Morley's line, and to the above conic, which is associated with a sextic, as Morley's conic.

We may notice that the polar reciprocals with respect to S of the system of conics (Σ) intersect S in the marking points of the system of covariants given by $C_{4,4}+\lambda I_2 C_{2,4}$.

In particular the reciprocal of Morley's conic passes through the marking points of $C_{4,4}+I_2 C_{2,4}$, a fact which will be used later on in connexion with the construction of the unique $C_{3,2}$ of the sextic.

IV. *Lemma on Joint Covariants.*

It will be useful to indicate here how to construct certain joint covariants of a quadratic and a quantic which are linear in the coefficients of both. A quadratic xy and a $(2p-1)$-ic given by

$$x^{2p-1}-s_1x^{2p-2}y+s_2x^{2p-3}y^2-\ldots-s_{2p-1}y^{2p-1},$$

have the pair of covariants $\qquad s_{p-1}x-s_py, \qquad (a)$

and $\qquad s_{p-1}x+s_py. \qquad (b)$

A quadratic xy and a $2p$-ic,

$$x^{2p}-s_1x^{2p-1}y+s_2x^{2p-2}y^2-\ldots+s_{2p}y^{2p},$$

have an involution of quadratic covariants given by

$$s_{p-1}x^2-\lambda s_pxy+s_{p+1}y^2. \tag{c}$$

If AB is the quadratic, then (a) denotes the polar 1-point of B with respect to the polar p-points of A with regard to the $(2p-1)$-ic.

Also (a) and (b) are harmonic conjugates with respect to AB. A similar interpretation may be found for (c).

The construction of (a) depends on the construction of (c). Let $[P_{2p-1}.P_1P_2P_3, \ldots, P_{2p-2}]$ be a $(2p-1)$-ic, and denote by P'_{2p-1} the harmonic conjugate of P_{2p-1} with regard to AB. Then the covariant (a) taken for xy, and the $(2p-1)$-ic is seen to be the harmonic conjugate of P'_{2p-1} with regard to the joint covariant of xy and the $(2p-2)$-ic $[P_1P_2P_3, \ldots, P_{2p-2}]$, which is obtained by writing $\lambda=2$ in the system (c). It is clear that we need only consider one value of λ in order to construct the involution (c). In the case of a quartic and a quadratic the members of (c) are merely the polars of C, the pole of AB, with respect to the 4-point system of conics through the marking points of the quartic.

Hence we might easily deduce the construction of (a) for a quadratic and a quintic. F. Morley gives another method in this case.*

In order to construct the two linear convariants of a quadratic and a septimic we have to indicate how to construct † the covariant

$$s_2' x^2 - \tfrac{1}{2} s_3' xy + s_4' y^2$$

taken for (xy), and the sextic $[P_5 P_6 . P_1 P_2 P_3 P_4]$ given by

$$x^6 - s_1' x^5 y + s_2' x^4 y^2 - \ldots + s_6' y^6,$$

an expression which we write as

$$(x - t_5 y)(x - t_6 y)[x^4 - s_1 x^3 y + s_2 x^2 y^2 - s_3 xy^3 + s_4 y^4].$$

The covariant in question becomes

$$[t_5 + t_6][s_1 x^2 - \tfrac{1}{2} s_2 xy + s_3 y^2]$$
$$+ [(t_5 t_6 + s_2) x^2 - \tfrac{1}{2}(t_5 t_6 s_1 + s_3) xy + (t_5 t_6 s_2 + s_4) y^2].$$

Now $s_1 x^2 - \frac{1}{2} s_2 xy + s_3 y^2 \ldots (a)$ is constructed immediately as a member of the system (c) for AB and $[P_1 P_2 P_3 P_4]$. Again, if M' is the harmonic conjugate with respect to AB of the point M in which AB meets $P_5 P_6$, then the part (β) in the second bracket is the polar of M' with respect to the conic

$$x^2 - s_1 xy - s_2 y^2 + 2 s_2 x - s_3 y + s_4 = 0,$$

which is a member of the system through $(P_1 P_2 P_3 P_4)$ and readily identified by constructing the polar of C with respect to it. The point $[a\beta]$ is then a point on the line required, fifteen such points being obtained by the various separations of $[P_1 P_2 P_3 P_4 P_5 P_6]$.

For the purposes of the next section it is necessary to consider a joint quartic covariant of a quadratic $(AB) = \xi x^2 + \eta y^2$ and a sextic

$$(a_0 a_1 a_2 \ldots a_6)(x, y)^6.$$

This quartic (which may be interpreted as the polar 4-points of B with respect to the quintic consisting of the polar 5-points of A with regard to the sextic) may be written as

$$(a_0 \eta + a_2 \xi) x^4 + 4 (a_1 \eta + a_3 \xi) x^3 y + 6 (a_2 \eta + a_4 \xi) x^2 y^2$$
$$+ 4 (a_3 \eta + a_5 \xi) xy^3 + (a_4 \eta + a_6 \xi) y^4.$$

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* See also Elliott, *Proc. London Math. Soc.*, Ser. 2, Vol. 6, p. 228.

† Our construction applies to the case when AB meets S in imaginary points.

The equation to the connector of the polar 2-points of K_1 $(y=0)$ with respect to this quartic is

$$(t) \qquad \eta[a_0x+2a_1y+a_2]+\xi[a_2x+2a_3y+a_4]=0.$$

The part (α) in the first bracket denotes the join of the polar 2-points of K_1 with respect to the sextic. The part (β) in the second bracket represents the covariant $(c) = s_2x^2-\frac{3}{2}s_3xy+s_4y^2$ in the former notation, taken for the sextic and the quadratic which consists of K_1 and its harmonic conjugate with respect to AB; ($\alpha\beta$) will be a point of (t). Then K_1, ($\alpha\beta$) are a pair of conjugate points with respect to a conic through the marking points of the quartic covariant. By varying the position of K_1 on S we may determine any number of such pairs, and the conic is linearly determined.

V. *The Quadratic Covariants of the Sextic.*

The irreducible system of quadratic covariants of a binary sextic consists of a $C_{3,2}$; $C_{5,2}$; $C_{7,2}$; $C_{8,2}$; $C_{10,2}$; and a $C_{12,2}$. Of these the $C_{3,2}$ is the most important, as, having effected its construction, we can easily deduce constructions for the others. Now we have just shown how to determine linearly a conic through the marking points of a joint quartic covariant linear in the coefficients of a quadratic and a sextic. If we apply this construction to the unique $C_{3,2}$ and the sextic itself we obtain the quartic $C_{4,4}+I_2C_{2,4}$. In Section III it was stated that the polar reciprocal with respect to S of Morley's conic passes through the marking points of this covariant. We proceed to reverse the above process. Since we have determined a conic through $C_{4,4}+I_2C_{2,4}$ we may use the method given in Section III to obtain the join of the polar 2-points of any point $K_1(y=0)$ of S with respect to this quartic. Denote this line by (p). Then $K_2(x=0)$ being any other point of S, we may construct for the quadratic K_1K_2 and the sextic the line (β), which is represented by the coefficient of ξ in equation (t) of the last section. We may also construct (α), the first part of (t), which denotes the join of the polar 2-points of K_1 with regard to the sextic. Now K_1 being fixed, while K_2 moves on S, then $[K_2, \beta]$ will be a pole and polar pair with respect to a certain conic which has also $[K_1\alpha]$ for a pole and polar pair. Let the lines p, α meet in X. Then it is clear that the polar of X with respect to this conic will be a line which passes through K_1 and through the pole of the line that marks the unique $C_{3,2}$ of the sextic. By taking a different position for K_1 we obtain another such line, and the construction required is completed.

Having obtained the $C_{3,2}$ we may at once deduce constructions for any

of the other quadratic covariants. For example, if C is the pole of the $C_{3,2}$, then the polar of C with respect to one of the conics through $C_{2,4}$ is a $C_{5,2}$.

The problem of the remaining quartic covariants of the sextic presents no difficulty, as each may be obtained as the Jacobian of $C_{2,4}$ and a quadratic covariant. It is easy to determine a conic through the marking points of the Jacobian of a quartic and a quadratic.

VI. *A Special Conic associated with the Sextic.*

Consider the joint quartic covariant of $\xi x+\eta y$ and

$$(a_0\, a_1,\, a_2,\, \dots,\, a_6)(x,\, y)^6,$$

which has for leading term

$$[9(a_0a_2-a_1^2)\eta^4-18(a_0a_3-a_1a_2)\eta^3\xi+(15a_0a_4-6a_1a_3-9a_2^2)\eta^2\xi^2$$
$$-6(a_0a_5-a_2a_3)\eta\xi^3+(a_0a_6-a_3^2)\xi^4]x^4.$$

Putting $\xi=0$, the complete expression of the above quartic is

$$9(a_0a_2-a_1^2)x^4+18(a_0a_3-a_1a_2)x^3y+(15a_0a_4-6a_1a_3-9a_2^2)x^2y^2$$
$$+6(a_0a_5-a_2a_3)xy^3+(a_0a_6-a_3^2)y^4.$$

The following equation represents a conic through the marking points of the above quartic

$$9(a_0a_2-a_1^2)X^2+18(a_0a_3-a_1a_2)XY+9(a_0a_4-a_2^2)Y^2$$
$$+6(a_0a_4-a_1a_3)X+6(a_0a_5-a_2a_3)Y+(a_0a_6-a_3^2)=0. \quad [K]$$

Corresponding to each point $P=(\xi x+\eta y)$ of S there will be a conic of the system $[K]$, and it is easy to see that the members of this system are all harmonically inscribed in (apolar to)* some conic Σ'. Now take $\xi x+\eta y$ to be a marking point P_1 of the sextic, and, putting $\xi=0$, write the sextic as $y(b_0,\, b_1,\, b_2,\, \dots,\, b_5)(x,\, y)^5$.

Then $[K]$ becomes the coincident line pair

$$[b_0X+2b_1Y+b_2]^2=0.$$

This line represents the join of the polar 2-points of P_1 with respect to the remaining five points of the sextic, and being apolar to Σ' must touch it.

---

* See Grace and Young, *Algebra of Invariants*, Chap. XIV.

Hence we have the theorem that *the six connectors of the polar 2-points of each of the marking points of a sextic with respect to the remaining five points touch a conic* Σ', *which is apolar to the system* $[K]$.

[When $I_4 = 0$, (K) becomes a pair of conjugate lines with respect to Σ'.]

In concluding we may remark that the four points of contact on S of the common tangents to S and Σ' give the marking points of a $C_{10,4}$.

VII. *Construction for the* $C_{2,2}$ *of a* $(2n+1)$*-ic.*

F. Morley* has given a ruler construction for the line that denotes the $C_{2,2}$ of a quintic $(P_1P_2P_3P_4P_5)$. In essence his construction is a method for finding the point Q_1 which is the conjugate of P_1 with respect to the four point system of conics which pass through the Hessian points of $(P_2P_3P_4P_5)$. The point Q_1 and the corresponding points Q_2, Q_3, Q_4, Q_5 are collinear in the required line.

This method is easily generalised to apply to the case of a $(2n+1)$-ic.

In fact, if P be one of the marking points of a $(2n+1)$-ic, and Q is the conjugate of P with respect to the four point system through the $C_{2,4}$ of the remaining $2n$ points, then Q is on the line that denotes the $C_{2,2}$ of the $(2n+1)$-ic.

For example, since we have determined conics through the $C_{2,4}$ of a sextic, we may construct immediately the $C_{2,2}$ of a septimic.

Further, the construction of the $C_{2,4}$ of a $2n$-ic depends on the following result:—

If P is one of the marking points of a $2n$-ic and (p) denotes the $C_{2,2}$ of the polar $(2n-3)$-points of P with respect to the remaining $2n-1$ points, then $[P, p]$ is a pole and polar pair with respect to a member of the system of conics through the unique $C_{2,4}$ of the $2n$-ic.

Thus in order to treat the case of an octavic, we must show how to construct Morley's line for the quintic which consists of the polar 5-points of P_1, with respect to the remaining seven points.

Take P_1 and P_2 to be $y = 0$ and $x = 0$ respectively, and write the octavic as

$$xy(a_0, a_1, a_2, \ldots, a_6)(x, y)^6.$$

Then the required line will be found to be

$$[21Ax+14By+12C]+12[(a_2^2-a_1a_3)x+(a_2a_3-a_1a_4)y+(a_3^2-a_2a_4)] = 0,$$

* F. Morley, *Math. Ann.*, Bd. XLIX, § 496.

where $Ax+By+C=0$ is the equation of Morley's line for the quintic which consists of the polar 5-points of P_1 with respect to the sextic $(P_3P_4P_5P_6P_7P_8)$. This last line is constructed by Section III, and it is easy to deduce a construction for

$$21Ax+14By+12C=0. \tag{α}$$

Again, the line (β) contained in the second bracket represents the connector of the poles of the two lines

$$a_2x+2a_3y+a_4=0, \tag{i}$$

$$a_1x+2a_2y+a_3=0, \tag{ii}$$

each of which may be readily constructed. Then $[\alpha, \beta]$ is a point on the line required. By taking the pair P_1, P_8 in place of P_1, P_2 we obtain another such point, and the line is determined.

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STUDIES ON THE THEORY OF CONTINUOUS PROBABILITIES, WITH SPECIAL REFERENCE TO ITS BEARING ON NATURAL PHENOMENA OF A PROGRESSIVE NATURE

By A. G. McKendrick, M.B., Ch.B., F.R.S.E., Major, Indian Medical Service.

[Received January 17th, 1914.—Read February 12th, 1914.]

The accompanying analysis had its origin in the following problem :—

An intimate mixture of leucocytes and micro-organisms in a medium of serum is prepared and kept at the heat of the blood. Collisions occur between the two types of cells, with subsequent inclusion of micro-organisms into the substance of the leucocytes. These two stages constitute the phenomenon of phagocytosis. The process may be stopped at any moment, and a specimen preparation obtained, which, after suitable treatment by staining fluids, renders it possible for a complete record to be made of the distribution amongst the leucocytes of such micro-organisms as have been ingested. It is required to find the mathematical law which governs such distributions.

1. If ϕ be the probability that an individual will be affected in a particular manner in an event, then, amongst v_0 individuals which are simultaneously submitted to the event, ϕv_0 will be affected. The system of individuals will, after such an event, comprise two groups, ϕv_0 which have been affected and $v_0-\phi v_0$ which have not been affected. Thus, ϕv_0 measures the decrease in the unaffected class. If we consider an event as occupying an interval of time Δt, and that the events follow each other in succession without intermission, then

$$\frac{\Delta v_0}{\Delta t} = -\phi v_0. \tag{1}$$

If we further consider that these decrements take place continuously

from moment to moment, *i.e.*, if v_0 is a continuous function of the time, then

$$\frac{dv_0}{dt} = -\phi v_0. \tag{2}$$

Taking into consideration the condition that individuals may be affected once, twice, three times, ..., x times, then for each class of individuals above the zero class there is an increase due to incomers from the preceding class, and a decrease due to outgoers from the class under consideration. These are proportional to ϕ and to the numbers of individuals in the respective classes. Consequently

$$\frac{dv_x}{dt} = (v_{x-1} - v_x)\phi, \tag{3}$$

an equation in v, continuous as regards t, but not as regards x; for, from the very nature of the problem, the happenings x are positive integral numbers.

Combining equations (2) and (3) we have

$$\frac{dv_x}{dv_0} = \frac{v_{x-1} - v_x}{-v_0},$$

$$\frac{1}{v_0}\frac{dv_x}{dv_0} - \frac{v_x}{v_0^2} = -\frac{v_{x-1}}{v_0^2},$$

$$\frac{d}{dv_0}\left(\frac{v_x}{v_0}\right) = -\frac{v_{x-1}}{v_0^2},$$

$$\frac{v_x}{v_0} = C_x - \int \frac{v_{x-1}}{v_0^2}\, dv_0 = C_x + \int \frac{v_{x-1}}{v_0}\, d\log\frac{a_0}{v_0},$$

where $v_0 = a_0$, when $t = 0$.

Let $\log\dfrac{a_0}{v_0} = m$, then for $x = 1$, we have

$$\frac{v_1}{v_0} = C_1 + m = \frac{a_1}{a_0} + m,$$

where $v_1 = a_1$, when $t = 0$.

Similarly for $x = 2$,

$$\frac{v_2}{v_0} = \frac{a_2}{a_0} + \frac{a_1}{a_0}m + \frac{m^2}{2!},$$

and generally

$$\frac{v_x}{v_0} = \frac{a_x}{a_0} + \frac{a_{x-1}}{a_0}m + \frac{a_{x-2}}{a_0}\frac{m^2}{2!} + \ldots + \frac{a_1}{a_0}\frac{m^{x-1}}{(x-1)!} + \frac{m^x}{x!}. \tag{4}$$

2. As v_x denotes the number of individuals in the x-th class, the mean class among the total individuals is

$$\sum_{0}^{x=\infty} x v_x \div \sum_{0}^{x=\infty} v_x.$$

The numerator is

$$v_0 \left\{ \begin{aligned} & \frac{a_1}{a_0}+m \\ & +2\left(\frac{a_2}{a_0}+\frac{a_1}{a_0}m+\frac{m^2}{2!}\right) \\ & +3\left(\frac{a_3}{a_0}+\frac{a_2}{a_0}m+\frac{a_1}{a_0}\frac{m^2}{2!}+\frac{m^3}{3!}\right) \\ & \vdots \\ & +x\left(\frac{a_x}{a_0}+\frac{a_{x-1}}{a_0}m+\ldots+\frac{a_1}{a_0}\frac{m^{x-1}}{(x-1)!}+\frac{m^x}{x!}\right) \end{aligned} \right.$$

$$= v_0 \left\{ \begin{aligned} & m\left(1+m+\frac{m^2}{2!}+\ldots+\frac{m^x}{(x-1)!}\right) \\ & +\frac{a_1}{a_0}\left(1+2m+3\frac{m^2}{2!}+\ldots+x\frac{m^{x-1}}{(x-1)!}\right) \\ & +\frac{a_2}{a_0}\left(2+3m+4\frac{m^2}{2!}+\ldots+x\frac{m^{x-2}}{(x-2)!}\right) \\ & \vdots \\ & +\frac{a_x}{a_0}x. \end{aligned} \right.$$

When x is very large, we have in the limit

$$v_0\left\{me^m+\frac{a_1}{a_0}e^m(1+m)+\frac{a_2}{a_0}e^m(2+m)+\ldots+\frac{a_r}{a_0}e^m(r+m)\ldots\right\}$$

$$= v_0\left\{me^m\left(1+\frac{a_1}{a_0}+\frac{a_2}{a_0}+\ldots+\frac{a_r}{a_0}+\ldots\right)\right.$$
$$\left.+e^m\left(\frac{a_1}{a_0}+2\frac{a_2}{a_0}+\ldots+\frac{ra_r}{a_0}+\ldots\right)\right\}$$

$$= m(a_0+a_1+a_2+\ldots)+(a_1+2a_2+3a_3+\ldots).$$

The denominator when similarly treated becomes

$$v_0e^m\left(1+\frac{a_1}{a_0}+\frac{a_2}{a_0}+\ldots+\frac{a_r}{a_0}+\ldots\right) = a_0+a_1+a_2+\ldots+a_r+\ldots,$$

as it should be, since the total number of individuals is unchanged.

Thus the mean class is

$$m+\left(\sum_{0}^{x=\infty} x a_x \div \sum_{0}^{x=\infty} a_x\right),$$

or if μ_t denote the mean at the time t,

$$\mu_t-\mu_0 = m.$$

Equation (4) may be now written in terms of the mean,

$$v_x = e^{-(\mu_t-\mu_0)}\left\{a_x+a_{x-1}(\mu_t-\mu_0)+a_{x-2}\frac{(\mu_t-\mu_0)^2}{2!}+\ldots\right.$$
$$\left.+a_1\frac{(\mu_t-\mu_0)^{x-1}}{(x-1)!}+a_0\frac{(\mu_t-\mu_0)^x}{x!}\right\}. \quad (5)$$

From equation (2) we have $\dfrac{dm}{dt} = \phi$;

therefore $$\mu_t-\mu_0 = \int_0^t \phi dt.$$

Thus the complete solution is

$$v_x = e^{-\int_0^t \phi dt}\left\{a_x+a_{x-1}\int_0^t \phi dt+a_{x-2}\frac{\left[\int_0^t \phi dt\right]^2}{2!}+\ldots\right.$$
$$\left.+a_1\frac{\left[\int_0^t \phi dt\right]^{x-1}}{(x-1)!}+a_0\frac{\left[\int_0^t \phi dt\right]^x}{x!}\right\}. \quad (6)$$

Note.—The square of the "standard deviation," *i.e.*,

$$\sum_{0}^{x=\infty} (\mu_t-x)^2 v_x \div \sum_{0}^{x=\infty} v_x,$$

becomes $$e^{\mu_0}\left\{m+\mu_0+\frac{a_1+4a_2+9a_3+\ldots}{a_0+a_1+a_2+\ldots}\right\},$$

which, when $\mu_0 = 0$, is equal to m.

3. In the particular case of the phagocytic experiment the following conditions occur :—

(*a*) All leucocytes are initially empty, *i.e.*

$$\sum_{0}^{x=\infty} a_x = a_0 \quad \text{and} \quad \mu_0 = 0.$$

(*b*) The probability ϕ is an unknown function of the time. It is dependent on a diminishing concentration of micro-organisms, on a diminishing concentration of specific accelerating substances in the serum, and on the temperature. The distribution is thus defined by equation (4) in terms of v_0, as

$$v_x = v_0 \frac{\left(\log \dfrac{a_0}{v_0}\right)^x}{x!},$$

or, by equation (5), in terms of the mean, as

$$v_x = a_0 e^{-\mu_t} \frac{(\mu_t)^x}{x!}.$$

In the following table are shown results observed by various workers, and figures calculated by equation (4) from observed values of v_0 and a_0.

$v_x \rightarrow$	Obs.	Calc.	Obs.	Calc.	Obs.	Calc.	Obs.	Calc.	Obs.	Calc.
$x=0$	19	(19)	99	(99)	41	(41)	620	(620)	632	(632)
1	59	57·89	227	206·8	126	119·1	282	296·3	282	290
2	98	88·2	208	216·1	154	173·1	79	70·8	65	66·5
3	88	89·7	134	150·5	164	167·7	16	11·29	16	10·1
4	65	68·24	78	78·63	121	121·8	2	1·349	4	1·1
5	37	41·58	34	32·85	62	70·8	1	0·131	1	0·10
6	17	21·12	9	11·44	36	34·3				
7	8	9·192	7	3·415	35	14·2				
8	5	3·501	3	0·8921	5	5·17				
9	2	1·185	0		2	1·67				
10	1	0·361	0		3	0·48				
11	0		0		1					
12	1		1							
Mean	3·005	3·04702	2·0825	2·0832	3·040	2·9065	0·50	0·47804	0·48	0·45887
S.D.	1·777	1·7455	1·4134	1·443	1·8927	1·747	0·74	0·7757	0·74	0·6893

4. As an example of a complete solution in the form of equation (6) the distribution of collisions amongst a_0 molecules of a gas occurring subsequent to an arbitrary point of time $t = 0$ is denoted by the equation

$$\frac{dv_x}{dt} = (v_{x-1} - v_x)\, a_0 \psi,$$

where ψ is constant. The solution by equation (6) is

$$v_x = a_0 e^{-\psi a_0 t} \frac{(\psi a_0 t)^x}{x!}.$$

5. Returning to equation (4): in cases where

$$\sum_{0}^{x=\infty} a_x = a_0 \quad (i.e., \ \mu_0 = 0),$$

let us ascertain the maximum value of v. This will be the x-th term if, of

$$\frac{m^{x-1}}{(x-1)!}, \quad \frac{m^x}{x!}, \quad \frac{m^{x+1}}{(x+1)!}$$

the middle term be the greatest, or if, of

$$x(x+1), \quad (x+1)\,m, \quad m^2$$

the middle term be the greatest.

Thus,

$$x < m < x+1,$$

or x must be the greatest integer not exceeding m. Proceeding to the limit where a_0 is indefinitely large, and the number of classes is indefinitely large, we may put $x\tau = n$, where τ is a fixed small quantity. Thus, in the limit n may be regarded as increasing continuously from class to class, while x increases by successive steps of unity. The mean, for the continuous curve whose ordinates are v_x and abscissæ x, will be at $n = x\tau = m\tau$. The *mode* occurs when $n = (m-f)\,\tau$, where f is a fraction less than unity. Thus n is equal to $m\tau$ in the limit for the *mode* as for the mean.

The form of the curve in the limit may be approximated to, by the aid of Stirling's formula,

$$v_x = \frac{a_0 e^{-m} m^x}{x!} = \frac{a_0 e^{-m} m^x}{\sqrt{2\pi}\, e^{-x} x^{x+\frac{1}{2}}},$$

at the mean, where $x = m$, $\quad v_m = \dfrac{a_0}{\sqrt{2\pi m}}$,

and at a neighbouring point where $x = m+r$,

$$v_{m+r} = \frac{a_0 e^{-m} m^{m+r}}{\sqrt{2\pi}\, e^{-(m+r)} (m+r)^{m+r+\frac{1}{2}}}.$$

Now $$\log\left(1+\frac{r}{m}\right)^{m+r+\frac{1}{2}} = (m+r+\tfrac{1}{2})\left(\frac{r}{m} - \frac{r^2}{2m^2} + \frac{r^3}{3m^3} - \dots\right)$$

$$= \left(1+\frac{2r+1}{2m}\right)\left(r - \frac{r^2}{2m} + \frac{r^3}{3m^2} - \dots\right)$$

$$= r + \frac{r^2+r}{2m} - \frac{2r^3+3r^2}{12m^2} + \dots.$$

Thus $$v_{m+r} = \frac{a_0}{\sqrt{2\pi m}} e^{-(r^2+r)/2m+(2r^3+3r^2)/12m^2-\dots},$$

and if r be large, though small as compared with m,

$$v_{m+r} = \frac{a_0}{\sqrt{2\pi m}} e^{-r^2/2m}. \tag{7}$$

6. In three dimensions equation (3) has the form

$$\frac{dv_{x,y,z}}{dt} = (v_{x-1,y,z} - v_{x,y,z})\phi_1 + (v_{x,y-1,z} - v_{x,y,z})\phi_2 + (v_{x,y,z-1} - v_{x,y,z})\phi_3.$$

Consequently when $\phi_1 = \phi_2 = \phi_3$, we have

$$\frac{dv_{x,y,z}}{dv_{0,0,0}} = \frac{v_{x-1,y,z} + v_{x,y-1,z} + v_{x,y,z-1} - 3v_{x,y,z}}{-3v_{0,0,0}},$$

which, when $\mu_0 = 0$, has for solution

$$v_{x,y,z} = \frac{v_{0,0,0}\left(\frac{m}{3}\right)^{x+y+z}}{x!\,y!\,z!}. \tag{8}$$

7. The theory which has so far been developed refers to cases of an *irreversible* nature, in which the only limitation to ϕ is that it is not a function of x. This restriction may now be removed, and equation (3) may be written

$$\frac{dv_x}{dt} = (f_{x-1}v_{x-1} - f_x v_x)\,\phi, \tag{9}$$

where f_x is a function of x.

Consequently $$\frac{dv_0}{dt} = -f_0 v_0 \phi \quad \text{or} \quad \frac{dm}{dt} = f_0 \phi,$$

where $$m = \log \frac{a_0}{v_0}.$$

Thus $$\frac{dv_x}{dm} + \frac{f_x}{f_0} v_x = \frac{f_{x-1}}{f_0} v_{x-1},$$

whence $$v_x = e^{-(f_x/f_0)m}\left[\int \frac{f_{x-1}}{f_0} v_{x-1} e^{(f_x/f_0)m}\, dm + C_x\right].$$

When $x = 1$, $\quad v_1 = e^{-(f_1/f_0)m}\left[\int v_0 e^{(f_1/f_0)m}\,dm + C_1\right]$;

but $v_0 = a_0 e^{-m}$, hence

$$v_1 = e^{-(f_1/f_0)m}\left[\frac{a_0 e^{(f_1/f_0-1)m}}{\dfrac{f_1-f_0}{f_0}} + C_1\right] = \frac{a_0 f_0 e^{-m}}{f_1-f_0} + C_1 e^{-(f_1/f_0)m}.$$

In cases where $\mu_0 = 0$, *i.e.*, $a_r = 0$ for all values of r other than $r = 0$,

$$C_1 = -\frac{a_0 f_0}{f_1-f_0}.$$

Consequently

$$v_1 = v_0\frac{f_0}{f_1-f_0}\left(1-\frac{a_0}{v_0}\,e^{-(f_1/f_0)m}\right) = v_0\frac{f_0}{f_1-f_0}\left(1-e^{(1-f_1/f_0)m}\right).$$

Similarly,

$$v_2 = v_0\frac{f_0}{(f_1-f_0)}\,\frac{f_1}{(f_2-f_0)}\left[1-\frac{f_2-f_0}{f_2-f_1}\,e^{(1-f_1/f_0)m}+\frac{f_1-f_0}{f_2-f_1}\,e^{(1-f_2/f_0)m}\right].$$

In the case where f_x is a linear function of x, we have, for $f_x = b+cx$,

$$v_1 = v_0\frac{b}{c}\left(1-e^{-c/b\,.\,m}\right),$$

$$v_2 = v_0\frac{b}{c}\left(\frac{b}{c}+1\right)\frac{\left(1-e^{-c/b\,.\,m}\right)^2}{2!},$$

$$\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots$$

$$v_x = v_0\frac{b}{c}\left(\frac{b}{c}+1\right)\cdots\left(\frac{b}{c}+x-1\right)\frac{\left(1-e^{-c/b\,.\,m}\right)^x}{x!}. \tag{10}$$

The mean grade is

$$\sum_0^{x=\infty} x v_x \div \sum_0^{x=\infty} v_x = \frac{b}{c}\left(e^{c/b\,.\,m}-1\right) = \mu.$$

The square of the standard deviation is

$$\sum_0^{x=\infty}(\mu-x)^2 v_x \div \sum_0^{x=\infty} v_x = \sum_0^{x=\infty}\frac{x^2 v_x}{a_0} - \mu^2 = e^{c/b\,.\,m}\mu = \nu.$$

Hence in terms of these moments

$$v_x = a_0\frac{b}{c}\left(\frac{b}{c}+1\right)\cdots\left(\frac{b}{c}+x-1\right)\frac{\left(1-\dfrac{\mu}{\nu}\right)^x}{x!}\left(\frac{\mu}{\nu}\right)^{b/c}. \tag{11}$$

For $f_x = b - cx$, we have similarly

$$v_x = v_0 \frac{b}{c}\left(\frac{b}{c}-1\right)\ldots\left(\frac{b}{c}-x+1\right)\frac{(e^{c/b.m}-1)^x}{x!}, \tag{12}$$

$$\mu = \frac{b}{c}(1-e^{c/b.m}),$$

$$\nu = e^{-(c/b)m}\mu,$$

$$v_x = a_0 \frac{b}{c}\left(\frac{b}{c}-1\right)\ldots\left(\frac{b}{c}-x+1\right)\frac{\left(\frac{\mu}{\nu}-1\right)^x}{x!}\left(\frac{\nu}{\mu}\right)^{b/c}. \tag{13}$$

8. Let us now consider cases in which the phenomenon is of a *reversible* nature, and let us confine ourselves to particular cases in which ϕ is independent of x. Equation (3) takes the form (in one dimension)

$$\frac{dv_x}{dt} = (v_{x-1} - 2v_x + v_{x+1})\,\phi. \tag{14}$$

In order to include cases in which ϕ is a function of the time, let us write t for $\int \phi\, dt$.

By Maclaurin's series we have

$$v_x = (v_x)_0 + \left(\frac{dv_x}{dt}\right)_0 t + \left(\frac{d^2v_x}{dt^2}\right)_0 \frac{t^2}{2!} + \ldots.$$

If when $t=0$, $v_r = a_r$, then for $x = 0$,

$$(v_0)_0 = a_0,$$

$$\left(\frac{dv_0}{dt}\right)_0 = a_{-1} - 2a_0 + a_1,$$

$$\left(\frac{d^2v_0}{dt^2}\right)_0 = a_{-2} - 4a_{-1} + 6a_0 - 4a_1 + a_2,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\left(\frac{d^nv_0}{dt^n}\right)_0 = a_{-n} - 2na_{-n+1} + \frac{2n(2n-1)}{2!}a_{-n+2} - \ldots$$

$$+ \frac{2n(2n-1)}{2!}a_{n-2} - 2na_{n-1} + a_n.$$

Similarly for $x = 1$,

$$(v_1)_0 = a_1,$$

$$\left(\frac{dv_1}{dt}\right)_0 = a_0 - 2a_1 + a_2,$$

$$\left(\frac{d^2v_1}{dt^2}\right)_0 = a_{-1} - 4a_0 + 6a_1 - 4a_2 + a_3,$$

and so on.

In the case of an *instantaneous point source* a_0 at $x = 0$, when $t = 0$, all terms except those in a_0 disappear; and the derivatives in Maclaurin's series for v_0 are equal to a_0 multiplied by the coefficients of the middle terms of binomials of the type $(a-\beta)^{2n}$ (*i.e.* the n-th terms), whilst for $v_{\pm 1}$ the coefficients are those of the $(n \mp 1)$-th terms of the same expansion, and for $v_{\pm x}$ they are those of the $(n \mp x)$-th terms.

Consequently

$$v_x \doteq a_0 \sum_0^{n=\infty} (-)^{n+x} \frac{2n(2n-1)\dots[2n-(n-x)+1]\, t^n}{(n-x)!\; n!}$$

$$= a_0 \sum_0^{n=\infty} (-)^{n+x} \frac{2n!\; t^n}{(n-x)!\; n!\; (n+x)!}. \tag{15}$$

The distribution is obviously a symmetrical one. The mean grade is at $x = 0$.

The square of the standard deviation

$$\nu = \sum_0^{x=\infty} (\mu - x)^2 \nu_x \div \sum_0^{x=\infty} v_x = 2t,$$

where $$t = \int \phi\, dt.$$

9. Cases in which the number of classes is limited are found in the laws governing chemical reactions. Thus, confining ourselves to monomolecular reactions, we have

(a) For an irreversible reaction in which the number of classes is two,

$$\frac{dv_0}{dt} = -\phi v_0, \qquad \frac{dv_1}{dt} = \phi v_0,$$

which, when $v_0 = a_0$ and $v_1 = 0$, when $t = 0$ gives

$$\frac{dv_1}{dt} = \phi(a_0 - v_1).$$

(*b*) For an irreversible reaction in which the number of classes is three or more we have

$$\frac{dv_0}{dt} = -\phi_0 v_0,$$

$$\frac{dv_1}{dt} = \phi_0 v_0 - \phi_1 v_1,$$

$$\cdots \quad \cdots \quad \cdots$$

$$\frac{dv_x}{dt} = \phi_{x-1} v_{x-1} - \phi_x v_x,$$

$$\frac{dv_{x+1}}{dt} = \phi_x v_x.$$

(*c*) For a reversible reaction in which the number of classes is two,

$$\frac{dv_0}{dt} = -\frac{dv_1}{dt} = -\phi_0 v_0 + \phi_1 v_1,$$

or for three classes, $$\frac{dv_0}{dt} = -\phi_0 v_0 + \phi_1 v_1,$$

$$\frac{dv_1}{dt} = \phi_0 v_0 - \phi_1 v_1 - \phi_2 v_1 + \phi_3 v_2,$$

$$\frac{dv_2}{dt} = \phi_2 v_1 - \phi_3 v_2.$$

In the theory of the spread of epidemics by contagion similar cases occur. For instance, where there is no recovery, we have for distribution according to multiplicity of infections,

$$\frac{dv_x}{dt} = (v_{x-1} - v_x)\,\phi \sum_{1}^{x=\infty} v_x,$$

where a_1 cannot be less than unity; and in cases where the community may be divided into three classes, v_0 not yet infected, v_1 affected, v_2 dead or immune, we have

$$\frac{dv_0}{dt} = -\phi v_0 v_1,$$

$$\frac{dv_1}{dt} = \phi v_0 v_1 - \psi v_1,$$

$$\frac{dv_2}{dt} = \psi v_1.$$

10. Cases in which v_x is a continuous function of x. The differences Δx between successive classes have been, in the cases considered in paragraphs 1 to 8, equal to unity. Let us now apply the same reasoning to cases in which Δx may be made as small as we please.

11. *Irreversible Phenomena.*—The flux over a face $x-\frac{1}{2}\Delta x$ of a parallelepipedon Δx, Δy, Δz, whose centre is at x, is proportional to $v_{x-\Delta x}\Delta y\Delta z$, and that over the face $x+\frac{1}{2}\Delta x$ to $-v_x\Delta y\Delta z$. Consequently the excess of influx over efflux in the parallelepipedon is, in this dimension, equal to $-\phi\frac{\partial v_x}{\partial x}\,dx\,dy\,dz$, as Δx, Δy, Δz tend to zero, if ϕ be independent of x, y, z. But this excess can also be considered as the rate of increase in the contents of the parallelepipedon, *i.e.*, by $\frac{\partial q}{dt}$, or by the rate of increase in concentration, *i.e.* by $\frac{\partial v}{\partial t}\,dx\,dy\,dz$.

Hence equation (3) has the form

$$\frac{\partial v}{\partial t} = -\phi\frac{\partial v}{\partial x}, \tag{16}$$

whence $$v = f(x-\phi t).$$

Thus, for any constant value of $(x-\phi t)$, v is constant, that is to say, equation (16) expresses a translation along the x axis.

12. *Reversible Phenomena.*—In this case the flux over the face $x-\frac{1}{2}\Delta x$ is proportional to $\frac{v_{x-\Delta x}-v_x}{\Delta x}\Delta y\Delta z$, and that over the face $x+\frac{1}{2}\Delta x$ to $-\frac{v_x-v_{x+\Delta x}}{\Delta x}\Delta y\Delta z$. Thus, if ϕ be independent of x, and identical in the positive and negative directions, the excess of influx over efflux, as Δx, Δy, Δz tend to zero, tends to

$$\left(\frac{\partial v_{x+\frac{1}{2}dx}}{\partial x}-\frac{\partial v_{x-\frac{1}{2}dx}}{\partial x}\right)dy\,dz,$$

which is equal to $$\frac{\partial^2 v}{\partial x^2}\,dx\,dy\,dz.$$

But, as before, this, with similar terms in y and z, measures the rate

of increase in concentration of the parallelepipedon, *i.e.* $\frac{\partial v}{\partial t}\,dx\,dy\,dz$. Hence

$$\frac{\partial v}{\partial t} = \phi_x \frac{\partial^2 v}{\partial x^2} + \phi_y \frac{\partial^2 v}{\partial y^2} + \phi_z \frac{d^2 v}{\partial z^2}. \tag{17}$$

If ϕ_x, ϕ_y and ϕ_z be equal, that is to say, if there is *randomness* in direction as well as in sense, the equation is that of *Fourier*. For an instantaneous point source of value Q at $x = y = z = 0$ the solution is

$$v = \frac{Q}{8\,(\pi\phi t)^{\frac{3}{2}}}\, e^{-(x^2+y^2+z^2)/4\phi t}. \tag{18}$$

This is the *error function of Gauss* in three dimensions, and is the logical outcome of equation (1) for cases of random reversible progressions, from a point source at the origin, in which v is a continuous function of t, x, y, and z.

13. *Incomplete Reversibility.*—Certain cases occur in which ϕ' in the positive sense is not equal to ϕ in the negative sense. If we divide the greater, say ϕ', into two quantities ϕ and ψ (*i.e.* $\psi = \phi' - \phi$), we may consider the phenomenon as the resultant of two progressions, one of them reversible and proportional to ϕ, the other irreversible and proportional to ψ.

Thus, combining equations (16) and (17), we have in one dimension

$$\frac{\partial v}{\partial t} = \phi \frac{\partial^2 v}{\partial x^2} - \psi \frac{\partial v}{\partial x}. \tag{19}$$

The solution of which for an instantaneous point source at the origin, is

$$v = \frac{Q}{2\sqrt{\pi\phi t}}\, e^{-(x-\psi t)^2/4\phi t}. \tag{20}$$

14. When ϕ is a function of x, the flux over the face $x - \frac{1}{2}\Delta x$ is

$$\phi_{x-\frac{1}{2}\Delta x} \frac{v_{x-\Delta x} - v_x}{\Delta x}\, \Delta y\, \Delta z,$$

that over the face $x + \frac{1}{2}\Delta x$ is

$$-\phi_{x+\frac{1}{2}\Delta x} \frac{v_x - v_{x+\Delta x}}{\Delta x}\, \Delta y\, \Delta z.$$

Hence the excess of influx over efflux, as Δx, Δy, Δz tend to zero, is

$$-\phi_{x-\frac{1}{2}dx} \frac{\partial v_{x-\frac{1}{2}dx}}{\partial x}\, dy\, dz + \phi_{x+\frac{1}{2}dx} \frac{\partial v_{x+\frac{1}{2}dx}}{\partial x}\, dy\, dz,$$

which is equal to $$\left(\phi_x \frac{\partial^2 v}{\partial x^2} + \frac{\partial \phi_x}{\partial x} \frac{\partial v}{\partial x}\right) dx\, dy\, dz.$$

Hence, as before, $$\frac{\partial v}{\partial t} = \phi_x \frac{\partial^2 v}{\partial x^2} + \frac{\partial \phi_x}{\partial x} \frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\, \phi_x \frac{\partial v}{\partial x}. \tag{21}$$

If $\phi_x = \kappa x^2$, we have

$$\frac{\partial v}{\partial t} = \kappa \left(x^2 \frac{\partial^2 v}{\partial x^2} + 2x \frac{\partial v}{\partial x}\right) = \kappa x \left(x \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\right) + \kappa x \frac{\partial v}{\partial x}$$

$$= \kappa x \frac{\partial}{\partial x}\, x \frac{\partial v}{\partial x} + \kappa x \frac{\partial v}{\partial x} = \kappa \frac{\partial^2 v}{\partial \log x^2} + \kappa \frac{\partial v}{\partial \log x};$$

hence $$v = \frac{Q}{2\sqrt{\pi \kappa t}}\, e^{-(\log x/x_0 + \kappa t)^2/4\kappa t}.$$

15. A modification which is of peculiar interest to statisticians occurs in what I may call the *target distribution*. It may be obtained as follows, taking the type of phenomena dealt with in paragraph 13 as an example. Consider a case of random diffusion in three dimensions, combined with translation along the z axis.

$$\frac{\partial v}{\partial t} = \phi \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) - \psi \frac{\partial v}{\partial z}.$$

The solution for an instantaneous point source Q at the origin is

$$v = \frac{Q}{8(\pi \phi t)^{\frac{3}{2}}}\, e^{-[x^2+y^2+(z-\psi t)^2]/4\phi t}.$$

Let a target be introduced in the plane $z = a$, which records the position of particles in transit, but does not interfere with their progress. Writing l^2 for $x^2+y^2+a^2$, and integrating according to the time, we have

$$A = \int_0^\infty v\, dt = \frac{Q e^{\psi a/2\phi}}{8(\pi\phi)^{\frac{3}{2}}} \int_0^\infty e^{-l^2/4\phi t - \psi^2 t/4\phi} \frac{dt}{t^{\frac{3}{2}}},$$

putting $$s = \frac{1}{\sqrt{t}},$$

$$A = \frac{Q e^{\psi a/2\phi}}{8(\pi\phi)^{\frac{3}{2}}} \int_0^\infty e^{-l^2 s^2/4\phi - \psi^2/4\phi s^2}\, ds = \frac{Q e^{\psi/2\phi\,(a - \sqrt{x^2+y^2+a^2})}}{4\pi\phi\sqrt{x^2+y^2+a^2}}. \tag{22}$$

On integrating again according to x and y, from $-\infty$ to $+\infty$, using

polar coordinates, we have

$$B = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A\,dx\,dy = \frac{Qe^{\psi a/2\phi}}{4\pi\phi}\int_0^{\infty}\int_0^{2\pi}\frac{e^{-\psi/2\phi\sqrt{r^2+a^2}}}{\sqrt{r^2+a^2}}\,r\,dr\,d\theta$$

$$= \frac{Qe^{\psi a/2\phi}}{2\phi}\int_0^{\infty}\frac{e^{-\psi/2\phi\sqrt{r^2+a^2}}}{\sqrt{r^2+a^2}}\,r\,dr.$$

Putting $s^2 = r^2+a^2,$

$$B = \frac{Qe^{\psi a/2\phi}}{2\phi}\int_a^{\infty} e^{-\psi s/2\phi}\,ds = \frac{Q}{\psi}.$$

Now A represents the length of time during which any point whose coordinates are x and y has been occupied by particles in transit across the plane $z=a$. It is thus directly proportional to the number of individuals which have crossed, and inversely to the time occupied in their transit. It will be noted that in the differential equation the translation due to the factor $-\psi\frac{\partial v}{\partial z}$ does not deviate from the direction *z plus*, and that, since by hypothesis ϕ is random in direction and sense at any moment of time, momentum is excluded; thus, any lengthening of individual occupation due to obliquity of transit is equally probable at any point in the plane $z = a$.

Hence if $w_{x,y}$ represent the number of transits at any point x, y in the plane $z = a$,

$$w_{x,y} = A\psi = \frac{Q\psi}{4\pi\phi}\,\frac{e^{\psi/2\phi(a-\sqrt{x^2+y^2+a^2})}}{\sqrt{x^2+y^2+a^2}}, \tag{23}$$

and the total number of transits is

$$B\psi = Q.$$

The approximation of the *target distribution* expressed by equation (23) to the error function, when ψ is great as compared with ϕ, and x^2+y^2 is less than a^2, may be shown as follows:—

$$w = \frac{Q\psi}{4\pi\phi}\,\frac{e^{\psi/2\phi(a-\sqrt{r^2+a^2})}}{\sqrt{r^2+a^2}}$$

$$= \frac{Q\psi}{4\pi\phi a}\,e^{\psi a/2\phi(1-\sqrt{1+r^2/a^2})-\frac{1}{2}\log(1+r^2/a^2)}$$

$$= \frac{Q\psi}{4\pi\phi a}\,e^{\psi a/2\phi(-r^2/2a^2+r^4/8a^4-r^6/16a^6+\dots)-(r^2/2a^2-r^4/4a^4+r^6/6a^6-\dots)}$$

$$= \frac{Q\psi}{4\pi\phi a}\,e^{-\psi r^2/4\phi a\,[(1+2\phi/\psi a)-r^2/4a^2(1+4\phi/\psi a)+r^4/8a^4(1+16\phi/3\psi a)-\dots]}.$$

16. In conclusion, I trust that I have in these notes directed attention to methods of attack which will be of service in the solution of problems chiefly of a biological nature. Statisticians have not, I think, recognized the applicability of Fourier's theorem to many of the problems in the investigation of which they are engaged. If diffusion be considered as progression, random in sense and direction, and uniform as regards time, its applicability is apparent. For instance, the broad lines of mosquito distribution may be resolved into a problem of "sources," and, if destructive methods be employed, it may take the form of a problem of "sources" and "sinks." The range of applicability is further extended when the dimensions are not those of space, but of degree of certain characteristics. In paragraphs 13 and 15 I have broken ground for further investigation into the processes of *Evolution;* for evolution, with its accompanying degeneration, is in its essence a phenomenon of incomplete reversibility according to a number of characteristics. A collection of fossils is a target distribution, though of a much more complicated nature than the simple case I have instanced. The target may be considered as a fixed degree in the development of some characteristic z; the target distribution will classify this selected population according to degree of characteristics x and y. Some of the population $(z = a)$ will be early comers and some late comers, and, according as they come early or late, so will they tend to take up different positions according to the characteristics x and y.

GENERALISATIONS OF THE HERMITE FUNCTIONS AND THEIR CONNEXION WITH THE BESSEL FUNCTIONS

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Introductory.

IN a preceding paper I have discussed a connexion between the Hermite functions and the Legendre functions analogous to the connexion between the Legendre functions and the Bessel functions.* Prof. Hill suggested to me the problem of searching for a similar relation between the Hermite functions and the Bessel functions. The present paper arises out of that suggestion. A connexion is established between the Bessel functions and certain functions whose properties have recently been discussed by Prof. Karl Pearson and Mr. Cunningham. These functions may be regarded as important generalisations of the Hermite functions, and in the course of the paper it will be seen that if we are restricted to searching for a relation between the Bessel functions and the (non-generalised) Hermite functions the problem evades solution.

In a memoir † read before the Royal Society in June 1908, Mr. Cunningham discusses the generalized form of certain functions occurring in statistics and called by Prof. Karl Pearson by the name of ω-functions. The function $\omega_{n,m}$ discussed by Mr. Cunningham arises in searching for all the solutions of the differential equation occurring in the theory of the conduction of heat,

$$\nabla^2 u = \frac{\partial u}{\partial t},$$

that have the form $f(t)\,\phi\left(\frac{r^2}{t}\right)\Theta$, where Θ depends only upon angular

* *Proc. London Math. Soc.*, Ser. 2, Vol. 12, pp. 236 *et seq.*

† "The ω-Functions, a Class of Normal Functions occurring in Statistics," E. Cunningham, *Proc. Roy. Soc.*, A, Vol. 81, 1908. [For particular values of m the function $\omega_{n,m}$ is very closely related to the function $T_m^{n-\frac{1}{2}m}$ occurring in Sonine's "Memoir on the Bessel Functions," *Math. Ann.*, Vol. XVI, §§ 40, 41.]

coordinates. The characteristic property of these functions which renders them of interest is that they all vanish at infinity like $e^{-r^2/t}$, so that they are specially adapted to deal with problems of the cooling of infinite bodies, more so in some ways than the highly developed Fourier analysis. But their special advantage lies in the possibility of fitting approximately a given initial distribution of temperature by taking a few terms of an expansion, the coefficients being found in terms of the successive moments about the coordinate axes, a method of very general adoption in statistics. The problem of fitting surfaces to frequency distributions in two qualities (see the example in Mr. Cunningham's paper of the ages of wives and husbands at marriage) is, unfortunately, made very heavy by the enormous labour in finding moments in two variables. It may be noticed that, for $t = 0$, the functions vanish everywhere except at $r = 0$, so that they seem to suggest a means of dealing with problems of the cooling of a body which is originally strongly heated in a small neighbourhood of $r = 0$.

In concluding these introductory remarks I have to very gratefully acknowledge the debt I owe to Mr. Cunningham for his valuable suggestions concerning the types of problems to which these functions may be applied, and for his kindness in allowing me to embody these suggestions in my preface.

1. *The fundamental differential equation and its solution by two definite integrals. Definitions of $H_{\nu,\mu}(x)$ and $K_{\nu,\mu}(x)$.*

The differential equation for $\omega_{n,m}(\xi)$ is

$$\xi \frac{d^2y}{d\xi^2} + (\xi+1+m)\frac{dy}{d\xi} + \left(n+1+\frac{m}{2}\right)y = 0 \quad \text{(p. 313, Cunningham).}$$

On putting

$$y = e^{-\xi} z, \quad \xi = x^2, \quad n = \frac{\nu}{2} - \tfrac{1}{4}, \quad m = -\tfrac{1}{2} + \mu,$$

the equation that arises is

$$\frac{d^2z}{dx^2} + \frac{2\mu}{x}\frac{dz}{dx} - 2x\frac{dz}{dx} + 2(\nu-\mu)z = 0. \tag{I}$$

When $\mu = 0$ this deforms into Hermite's differential equation. It forms the fundamental differential equation in this paper.

It is easily seen that, on making the necessary modifications in

Mr. Cunningham's paper, this equation is satisfied by

$$\int_L e^{x^2 t}(t-1)^{\frac{1}{2}(\nu+\mu-1)}\, t^{\frac{1}{2}(\mu-\nu)-1}\, dt,$$

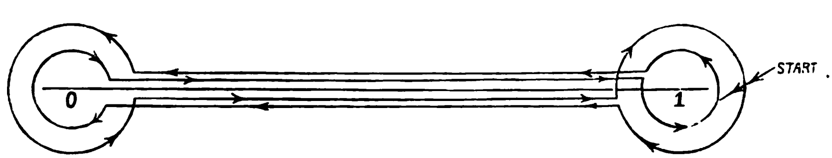
if the integrand has the same value at each extremity of L.

Let $\Gamma_{0,1}$ denote a contour starting where the real part of $x^2 t$ is infinitely great and negative, circulating positively round the origin by means of a circle of radius *greater* than unity and returning to its starting point. Then the function $H_{\nu,\mu}(x)$ will be defined by

$$H_{\nu,\mu}(x) = \frac{1}{2\pi i}\int_{\Gamma_{0,1}} e^{x^2 t}(t-1)^{\frac{1}{2}(\nu+\mu-1)}\, t^{\frac{1}{2}(\mu-\nu)-1}\, dt, \qquad \text{(II)}$$

where that branch of $(t-1)^{\frac{1}{2}(\nu+\mu-1)}$ is taken which assumes the value 1 when the modulus of $t-1$ is unity and its argument zero, and that branch of $t^{\frac{1}{2}(\mu-\nu)}$ is taken which assumes the value 1 when the modulus of t is unity and its argument zero. $H_{\nu,\mu}(x)$ is a solution of (I).

Secondly, let $\gamma_{0,1}$ denote the double-loop contour that begins at a point $t = 1+c$, where c is small and real and positive, makes a small semicircle of radius c above the real axis round $t = 1$, proceeds to near the origin along the real axis, circulates positively round the origin by means of a small circle, then proceeds along the real axis to near the point $t = 1$, circulates round $t = 1$ by a small negative circuit, proceeds along the real axis to near the origin, circulates round the origin by a small negative circuit, finally proceeds along the real axis to near $t = 1$, and describes a small semicircle under the real axis to finish at its starting point.



Then the function $K_{\nu,\mu}(x)$ will be defined by

$$K_{\nu,\mu}(x) = \frac{e^{\frac{1}{2}(\nu-\mu)\pi i}}{4\pi\cos\dfrac{\nu+\mu}{2}\pi}\int_{\gamma_{0,1}} e^{x^2 t}(t-1)^{\frac{1}{2}(\nu+\mu-1)}\, t^{\frac{1}{2}(\mu-\nu)-1}\, dt, \qquad \text{(III)}$$

the same branches of $(t-1)^{\frac{1}{2}(\nu+\mu-1)}$ and $t^{\frac{1}{2}(\mu-\nu)}$ being taken as before. $K_{\nu,\mu}(x)$ is (in general) a second solution of (I).

These two functions $H_{\nu,\mu}(x)$ and $K_{\nu,\mu}(x)$ may be regarded as generalisations of the Hermite functions.

2. *Developments of $H_{\nu,\mu}(x)$ and $K_{\nu,\mu}(x)$ in ascending power series in x.*

Let $u = x^2 t$. Then

$$H_{\nu,\mu}(x) = \frac{1}{2\pi i}\int_{\Gamma_{0,x^2}} e^u \left(\frac{u}{x^2}-1\right)^{\frac{1}{2}(\nu+\mu-1)} u^{\frac{1}{2}(\mu-\nu)-1}\, x^{\nu-\mu}\, du$$

$$= \frac{x^{1-2\mu}}{2\pi i}\int_{\Gamma_{0,x^2}} e^u \left(1-\frac{x^2}{u}\right)^{\frac{1}{2}(\nu+\mu-1)} u^{\mu-\frac{3}{2}}\, du,$$

where Γ_{0,x^2} circulates round the origin by means of a circle which *encloses* $u = x^2$, and starts and finishes where the real part of u is infinitely great and negative.

Thus at all points along the path of integration,

$$\left|\frac{x^2}{u}\right| < 1.$$

Thus

$$H_{\nu,\mu}(x) = \frac{x^{1-2\mu}}{2\pi i}\sum_{r=0}^{\infty}\binom{\frac{\nu+\mu-1}{2}}{r}(-1)^r x^{2r}\int_{\Gamma_0} e^u u^{\mu-\frac{3}{2}-r}\, du,$$

where $\binom{n}{r}$ denotes $\dfrac{n(n-1)\ldots(n-r+1)}{r!}$, and where Γ_0 denotes a contour that begins at the negative end of the real axis, circulates positively round the origin, and returns to its starting point. Hence

$$H_{\nu,\mu}(x) = \sum_{r=0}^{\infty}(-1)^r\binom{\frac{\nu+\mu-1}{2}}{r}\frac{x^{1-2\mu+2r}}{\Gamma(\frac{3}{2}+r-\mu)}. \tag{IV}$$

In particular,

$$H_{\nu,0}(x) = \sum_{r=0}^{\infty}\frac{\Gamma\left(\frac{\nu+1}{2}\right)(-1)^r x^{2r+1}}{r!\,\Gamma\left(\frac{\nu+1}{2}-r\right)\Gamma(r+\frac{3}{2})} = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\pi^{\frac{1}{2}}}\sum_{r=0}^{\infty}\frac{(-1)^r(2x)^{2r+1}}{(2r+1)!\,\Gamma\left(\frac{\nu+1}{2}-r\right)}$$

$$= \frac{\Gamma\left(\frac{\nu+1}{2}\right)e^{\nu\pi i}}{\pi^{\frac{1}{2}}\Gamma(\nu+1)}\,\frac{U_\nu(x)-U_\nu(-x)}{2\sin\frac{\nu\pi}{2}} \tag{Va}$$

(see p. 240, preceding paper*).

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* In the formula (v) of p. 240, there is an error. The terms x^{2s} and x^{2s+1} should be replaced by $(2x)^{2s}$ and $(2x)^{2s+1}$. This error is isolated and does not affect subsequent work.

If $\nu+\mu$ is an odd positive integer, then $H_{\nu,\mu}(x)$ is a terminating series. The case where $(\frac{1}{2}-\mu)$ is a negative integer receives special attention in the next section.

Before developing $K_{\nu,\mu}(x)$ as a power series, we have first to consider the integral

$$I_{\nu,\mu,r} = \int_{\gamma_{0,1}} (t-1)^{\frac{1}{2}(\nu+\mu-1)}\, t^{\frac{1}{2}(\mu-\nu)+r-1}\, dt.$$

Suppose first that the real parts of $(\nu+\mu+1)$ and of $(\mu-\nu)$ are each greater than zero; and that r is a positive integer.

Then

$$\begin{aligned}
I_{\nu,\mu,r} = {} & e^{\frac{1}{2}(\nu+\mu-1)\pi i}\int_1^0 (1-x)^{\frac{1}{2}(\nu+\mu-1)} x^{\frac{1}{2}(\mu-\nu)+r-1}\,dx \\
& + e^{[\frac{1}{2}(\nu+\mu-1)+\mu-\nu]\pi i}\int_0^1 (1-x)^{\frac{1}{2}(\nu+\mu-1)}\, x^{\frac{1}{2}(\mu-\nu)+r-1}\,dx \\
& + e^{-\frac{1}{2}(\nu+\mu-1)\pi i+(\mu-\nu)\pi i}\int_1^0 (1-x)^{\frac{1}{2}(\nu+\mu-1)}\, x^{\frac{1}{2}(\mu-\nu)+r-1}\,dx \\
& + e^{-\frac{1}{2}(\nu+\mu-1)\pi i}\int_0^1 (1-x)^{\frac{1}{2}(\nu+\mu-1)}\, x^{\frac{1}{2}(\mu-\nu)+r-1}\,dx \\
= {} & 4e^{\frac{1}{2}(\mu-\nu)\pi i}\cos\frac{\nu+\mu}{2}\pi\,\sin\frac{\mu-\nu}{2}\pi\,\frac{\Gamma\left(\frac{\nu+\mu+1}{2}\right)\Gamma\left(\frac{\mu-\nu}{2}+r\right)}{\Gamma(\mu+r+\frac{1}{2})}.
\end{aligned}$$

The restrictions upon μ, ν may now be removed by means of the reduction formulæ for the beta functions.

Hence

$$\begin{aligned}
K_{\nu,\mu}(x) &= \frac{e^{\frac{1}{2}(\nu-\mu)\pi i}}{4\pi\cos\frac{\nu+\mu}{2}\pi}\sum_{r=0}^{\infty} I_{\nu,\mu,r}\frac{x^{2r}}{r!} \quad \text{[by (III)]} \\
&= \Gamma\left(\frac{\nu+\mu+1}{2}\right)\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{r!\,\Gamma\left(\frac{\nu-\mu}{2}-r+1\right)\Gamma(\mu+r+\frac{1}{2})}. \qquad \text{(VI)}
\end{aligned}$$

In particular

$$K_{\nu,0}(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)e^{\nu\pi i}}{\pi^{\frac{1}{2}}\Gamma(\nu+1)}\,\frac{U_\nu(x)+U_\nu(-x)}{2\cos\frac{\nu\pi}{2}}. \qquad \text{(V}b\text{)}$$

Combining (Va) and (Vb), we obtain

$$U_\nu(x) = \frac{\pi^{\frac{1}{2}}\Gamma(\nu+1)}{\Gamma\left(\frac{\nu+1}{2}\right)e^{\nu\pi i}}\left\{\cos\frac{\nu\pi}{2}K_{\nu,0}(x)+\sin\frac{\nu\pi}{2}H_{\nu,0}(x)\right\}. \qquad \text{(VII)}$$

This leads us to construct the following generalisation of the U-function, viz.,

$$U_{\nu,\mu}(x) = \frac{\pi^{\frac{1}{2}}\Gamma(\nu+\mu+1)}{e^{\frac{1}{2}(\nu+\mu)\pi i}\,\Gamma\left(\frac{\nu+\mu+1}{2}\right)}\left\{\frac{\cos\frac{\nu+\mu}{2}\pi\,.\,K_{\nu,\mu}(x)+\sin\frac{\nu-\mu}{2}\pi\,.\,H_{\nu,\mu}(x)}{e^{\frac{1}{2}(\nu-\mu)\pi i}\cos\mu\pi}\right\}. \tag{VIII}$$

It will presently be found that this function is intimately connected with Cunningham's generalised ω function.

Returning to $K_{\nu,\mu}(x)$, it will be seen that the series terminates if $\nu-\mu$ is an even positive integer.

The exceptional case that arises when $\mu+\frac{1}{2}$ is a positive integer or zero is treated in the next section.

(α) If $\nu+\mu$ is an odd positive integer, and at the same time $\mu-\nu$ is an even positive integer (not zero), then $H_{\nu,\mu}(x)$ becomes zero. In this case we may take

$$\bar{H}_{\nu,\mu}(x) = \frac{1}{\pi}\int_0^\infty e^{-x^2 t}(t+1)^{\frac{1}{2}(\nu+\mu-1)}t^{\frac{1}{2}(\mu-\nu)-1}\,dt,$$

as a solution of (I).

(β) If $\nu+\mu$ is an odd negative integer (μ not being half an odd integer, a case dealt with in the next section), then $K_{\nu,\mu}(x)$ becomes infinite for all values of x. In this case we may take

$$L_{\nu,\mu}(x) = \frac{\pi}{\Gamma\left(\frac{1-\nu-\mu}{2}\right)}\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{r!\,\Gamma\left(\frac{\nu-\mu}{2}-r+1\right).\Gamma(\mu+r+\frac{1}{2})},$$

as a solution.

(γ) If $\nu-\mu$ is an even negative integer (not zero), then $K_{\nu,\mu}(x)$ vanishes, here we may take as a second solution,

$$\bar{K}_{\nu,\mu}(x) = \frac{\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\pi}\sum_{r=0}^{\infty}\frac{\Gamma\left(\frac{\mu-\nu}{2}+r\right)x^{2r}}{r!\,\Gamma(\mu+r+\frac{1}{2})}.$$

(δ) In particular

$$K_{\nu,\nu}(x) = 1 \quad \text{and} \quad H_{\nu-1,\nu+1}(x) = \frac{e^{x^2}}{\Gamma(\frac{1}{2}-\nu)\,x^{2\nu+1}}.$$

3. *Discussion of the case when $\mu+\frac{1}{2}$ is integral. The $W_{\nu,\mu}$ function.*

It follows at once from (IV) and (VI) that

$$K_{\nu,\frac{1}{2}}(x) = H_{\nu,\frac{1}{2}}(x).$$

To proceed further, let $\mu = \frac{1}{2}+\rho$, where ρ is a positive integer.

Then

$$H_{\nu,\frac{1}{2}+\rho}(x) = x^{-2\rho}\,\Gamma\left(\frac{\nu+\rho+\frac{3}{2}}{2}\right)\sum_{r=\rho}^{r=\infty}\frac{(-1)^r x^{2r}}{r!\,\Gamma\left(\frac{\nu+\rho+\frac{3}{2}}{2}-r\right)\Gamma(1+r-\rho)}$$

$$= (-1)^\rho\,\Gamma\left(\frac{\nu+\rho+\frac{3}{2}}{2}\right)\sum_{s=0}^{\infty}\frac{(-1)^s x^{2s}}{\Gamma\left(\frac{\nu-\rho+\frac{3}{2}}{2}-s\right)\Gamma(\rho+s+1)\,s!}$$

$$= (-1)^\rho K_{\nu,\frac{1}{2}+\rho}(x) \quad \text{[by (VI)]}.$$

Similarly it can be shewn that when $\mu = \frac{1}{2}-\rho$, where ρ is a positive integer, that

$$K_{\nu,\frac{1}{2}-\rho}(x) = (-1)^\rho H_{\nu,\frac{1}{2}-\rho}(x).$$

Thus, in general, if ρ is integral or zero, then

$$H_{\nu,\frac{1}{2}+\rho}(x) = (-1)^\rho K_{\nu,\frac{1}{2}+\rho}(x), \tag{IX}$$

and a second solution of (I) must be found in this case in order to complete the solution of (I).

Adopting the same procedure as in dealing with the Bessel equation we form a second solution $W_{\nu,\frac{1}{2}+\rho}(x)$ by taking

$$W_{\nu,\frac{1}{2}+\rho}(x) = \operatorname*{Lt}_{\epsilon\to 0}\frac{(-1)^\rho H_{\nu,\frac{1}{2}+\rho-\epsilon}(x) - K_{\nu,\frac{1}{2}+\rho-\epsilon}(x)}{\epsilon}.$$

The expansion of this in the form of a series is not required in the subsequent work.

The preceding argument needs closer investigation when ν, as well as μ, is half an odd integer. Let $\nu = \sigma-\frac{1}{2}$, and $\mu = \rho+\frac{1}{2}$, where σ and ρ are integral. Then

$$H_{\sigma-\frac{1}{2},\rho+\frac{1}{2}}(x) = \Gamma\left(\frac{\sigma+\rho+1}{2}\right)\sum_{r=0}^{\infty}\frac{(-1)^r x^{-2\rho+2r}}{r!\,\Gamma\left(\frac{\sigma+\rho+1}{2}-r\right)\Gamma(1+r-\rho)} \quad \text{[by (IV)]},$$

$$K_{\sigma-\frac{1}{2},\rho+\frac{1}{2}}(x) = \Gamma\left(\frac{\sigma+\rho+1}{2}\right)\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{r!\,\Gamma\left(\frac{\sigma-\rho+1}{2}-r\right)\Gamma(1+r+\rho)} \quad \text{[by (VI)]},$$

and further discussion becomes necessary when $\sigma+\rho$ is odd and negative. If in this case ρ is positive, no difficulty arises, for the first series merely merges into the second. But suppose ρ is negative; then an examination of the second series shows that a distinctly exceptional case arises if $\sigma-\rho$ is positive, so that $\frac{\sigma-\rho+1}{2}$ is a positive integer and necessarily less than

$-\rho$. In this case the theorem of formula (IX) must be modified as follows :—

$$\underset{\epsilon_m \to 0,\, \epsilon_n \to 0}{\mathrm{Lt}} K_{\frac{1}{2}-2(m+\epsilon_m)+(n+\epsilon_n),\, \frac{1}{2}-(n+\epsilon_n)}$$

$$= (-1)^{n-m} \underset{\epsilon_n \to 0,\, \epsilon_m \to 0}{\mathrm{Lt}} \left(\frac{\epsilon_n}{\epsilon_m}\right) \sum_{r=0}^{r=n-m} \frac{(n-r-1)!\, x^{2r}}{r!\,(n-m-r)!\,(m-1)!}$$

$$+(-1)^n \underset{\epsilon_n \to 0,\, \epsilon_m \to 0}{\mathrm{Lt}} \left(1-\frac{\epsilon_n}{\epsilon_m}\right) H_{\frac{1}{2}-2m+n,\, \frac{1}{2}-n}(x),$$

where m and n are positive (non-zero) integers, such that $m \not< n$.

4. *Formulæ of transformation required in subsequent work.*

On applying the transformation $t = 1-u$ to each of the definite integrals (II) and (III), it is seen that $e^{x^2} H_{-\nu-1,\mu}(ix)$, and $e^{x^2} K_{-\nu-1,\mu}(ix)$ are constant multiples of $H_{\nu,\mu}(x)$ and $K_{\nu,\mu}(x)$ respectively.

On examining the initial terms in the expansions of each of these functions, we obtain

$$e^{x^2} H_{-\nu-1,\mu}(ix) = e^{\frac{1}{2}\pi i(1-2\mu)} H_{\nu,\mu}(x), \tag{XIa}$$

and

$$e^{x^2} K_{-\nu-1,\mu}(ix) = \frac{\cos\dfrac{\mu+\nu}{2}\pi}{\sin\dfrac{\mu-\nu}{2}\pi} K_{\nu,\mu}(x). \tag{XIb}$$

Further, by using (IV),

$$H_{\nu,1-\mu}(x) = x^{2\mu-1}\Gamma\left(\frac{\nu-\mu}{2}+1\right) \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{r!\,\Gamma\left(\dfrac{\nu-\mu}{2}-r+1\right)\Gamma(\frac{1}{2}+r+\mu)}$$

$$= x^{2\mu-1} \frac{\Gamma\left(\dfrac{\nu-\mu}{2}+1\right)}{\Gamma\left(\dfrac{\nu+\mu+1}{2}\right)} K_{\nu,\mu}(x). \tag{XII}$$

5. *Recurrence formulæ.*

$$x\frac{d}{dx} H_{\nu,\mu}(x) = \frac{2}{2\pi i}\int_{\Gamma_{0,1}} (t-1)^{\frac{1}{2}(\nu+\mu-1)}\, t^{\frac{1}{2}(\mu-\nu)} \frac{d}{dt} e^{x^2 t}\, dt \quad \text{[using (II)]}$$

$$= -\frac{1}{2\pi i}\int_{\Gamma_{0,1}} e^{x^2 t}(t-1)^{\frac{1}{2}(\nu+\mu-3)}\, t^{\frac{1}{2}(\mu-\nu)-1} \{(\nu+\mu-1)t+(\mu-\nu)(t-1)\}\, dt.$$

Therefore

$$x\frac{d}{dx} H_{\nu,\mu}(x) + (\mu-\nu) H_{\nu,\mu}(x) + (\nu+\mu-1) H_{\nu-2,\mu}(x) = 0, \tag{XIII}$$

On differentiating (XIII), and then using (XIII) along with (I), we obtain

$$(\mu-\nu)\,H_{\nu,\mu}(x)+(2\nu-3-2x^2)\,H_{\nu-2,\mu}(x)+(3-\mu-\nu)\,H_{\nu-4,\mu}(x)=0. \quad \text{(XIV)}$$

These relations also hold good for $K_{\nu,\mu}(x)$, since $\dfrac{e^{\frac{1}{2}(\nu-\mu)\pi i}}{\cos\dfrac{\nu+\mu}{2}\pi}$ is unaltered, of increasing ν by an even integer.

Another set of relations is obtained as follows:—

$$\frac{d}{dx}H_{\nu,\mu}(x)=2xH_{\nu-1,\mu+1}(x) \quad \text{[on using (II)].} \qquad \text{(XV}a\text{)}$$

From this is obtained the relation

$$2x^2H_{\nu-2,\mu+2}(x)+(2\mu+1-2x^2)\,H_{\nu-1,\mu+1}(x)+(\nu-\mu)\,H_{\nu,\mu}(x)=0. \quad \text{(XV}b\text{)}$$

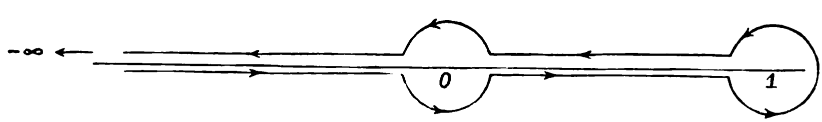
On turning to (III), it will be seen that these relations (XVa) and (XVb) also hold good for

$$\frac{\cos\dfrac{\nu+\mu}{2}\pi}{e^{\frac{1}{2}(\nu-\mu)\pi i}}\,K_{\nu,\mu}(x).$$

6. *The connexion between Cunningham's ω-function and the H and K functions.*

First suppose that the real parts of $\nu+\mu+1$ and of $\mu-\nu$ are each positive (and suppose for instance that x is real).

Let the contour $\Gamma_{0,1}$ take the form indicated in the accompanying diagram.



Then

$$H_{\nu,\mu}(x)=\frac{1}{2\pi i}\int_{\infty}^{0}e^{-x^2u}e^{-\frac{1}{2}(\nu+\mu-1)\pi i}(u+1)^{\frac{1}{2}(\nu+\mu-1)}e^{-[\frac{1}{2}(\mu-\nu)-1]\pi i}u^{\frac{1}{2}(\mu-\nu)-1}(-du)$$

$$+\frac{1}{2\pi i}\int_{0}^{1}e^{x^2t}e^{-\frac{1}{2}(\nu+\mu-1)\pi i}(1-t)^{\frac{1}{2}(\nu+\mu-1)}t^{\frac{1}{2}(\mu-\nu)-1}\,dt$$

$$+\frac{1}{2\pi i}\int_{1}^{0}e^{x^2t}e^{\frac{1}{2}(\nu+\mu-1)\pi i}(1-t)^{\frac{1}{2}(\nu+\mu-1)}t^{\frac{1}{2}(\mu-\nu)-1}\,dt$$

$$+\frac{1}{2\pi i}\int_{0}^{\infty}e^{-x^2u}e^{\frac{1}{2}(\nu+\mu-1)\pi i}(u+1)^{\frac{1}{2}(\nu+\mu-1)}e^{[\frac{1}{2}(\mu-\nu)-1]\pi i}u^{\frac{1}{2}(\mu-\nu)-1}(-du).$$

Thus $$H_{\nu,\mu}(x) = -\frac{\sin(\mu-\frac{3}{2})\pi}{\pi}\int_0^\infty e^{-x^2u}(u+1)^{\frac{1}{2}(\nu+\mu-1)}\,u^{\frac{1}{2}(\mu-\nu)-}\,du$$

$$-\frac{\sin\dfrac{\nu+\mu-1}{2}\pi}{\pi}\int_0^1 e^{x^2t}(1-t)^{\frac{1}{2}(\nu+\mu-1)}\,t^{\frac{1}{2}(\mu-\nu)-1}\,dt.$$

Now applying the same method as that used in § 2 for evaluating $I_{\nu,\mu,r}$, we obtain

$$K_{\nu,\mu}(x)$$

$$= \frac{e^{\frac{1}{2}(\nu-\mu)\pi i}}{4\pi\cos\dfrac{\nu+\mu}{2}\pi}\,4e^{\frac{1}{2}\pi i(\mu-\nu)}\sin\frac{\mu-\nu}{2}\pi\cos\frac{\nu+\mu}{2}\pi\int_0^1 e^{x^2t}(1-t)^{\frac{1}{2}(\nu+\mu-1)}\,t^{\frac{1}{2}(\mu-\nu)-1}\,dt$$

$$= \frac{\sin\dfrac{\mu-\nu}{2}\pi}{\pi}\int_0^1 e^{x^2t}(1-t)^{\frac{1}{2}(\nu+\mu-1)}\,t^{\frac{1}{2}(\mu-\nu)-1}\,dt.$$

Using the notation of § 2 in Mr. Cunningham's paper

$$e^{x^2}\omega_{\frac{1}{2}\nu-\frac{1}{2},\,\mu-\frac{1}{2}}(x^2) = \frac{1}{2\pi i}\int_{\gamma_0} e^{-x^2u}(u+1)^{\frac{1}{2}(\nu+\mu-1)}\,u^{\frac{1}{2}(\mu-\nu)-1}\,du,$$

where γ_0 denotes a contour beginning where the real part of x^2u is infinitely great and positive, circulating round the origin by means of a circle of radius *less* than unity, and returning to its initial point. Thus with the present restrictions upon ν and μ,

$$e^{x^2}\omega_{\frac{1}{2}\nu-\frac{1}{2},\,\mu-\frac{1}{2}}(x^2) = \frac{e^{\frac{1}{2}(\mu-\nu)\pi i}\sin\dfrac{\mu-\nu}{2}\pi}{\pi}\int_0^\infty e^{-x^2u}(u+1)^{\frac{1}{2}(\nu+\mu-1)}\,u^{\frac{1}{2}(\mu-\nu)-1}\,du.$$

Therefore, if the real parts of $\nu+\mu+1$ and $\mu-\nu$ are each positive,

$$H_{\nu,\mu}(x) = \frac{-\cos\mu\pi}{e^{\frac{1}{2}(\mu-\nu)\pi i}\sin\dfrac{\mu-\nu}{2}\pi}\,e^{x^2}\omega_{\frac{1}{2}\nu-\frac{1}{2},\,\mu-\frac{1}{2}}(x^2) + \frac{\cos\dfrac{\nu+\mu}{2}\pi}{\sin\dfrac{\mu-\nu}{2}\pi}\,K_{\nu,\mu}(x).$$

That is,

$$e^{x^2}\omega_{\frac{1}{2}\nu-\frac{1}{2},\,\mu-\frac{1}{2}}(x^2) = \frac{\cos\dfrac{\nu+\mu}{2}\pi\,.\,K_{\nu,\mu}(x)+\sin\dfrac{\nu-\mu}{2}\pi\,.\,H_{\nu,\mu}(x)}{e^{\frac{1}{2}(\nu-\mu)\pi i}\cos\mu\pi}. \qquad \text{(XVI)}$$

And the reduction formulæ (XIV) and (XV) hold good, as can easily be

shewn, for each of the functions

$$\cos\mu\pi\,.\,e^{x^2}\omega_{\frac{1}{2}\nu-\frac{1}{4},\,\mu-\frac{1}{2}}(x^2),\qquad \frac{\cos\dfrac{\nu+\mu}{2}\pi}{e^{\frac{1}{2}(\nu-\mu)\pi i}}K_{\nu,\mu}(x),\qquad \frac{\sin\dfrac{\nu-\mu}{2}\pi}{e^{\frac{1}{2}(\nu-\mu)\pi i}}H_{\nu,\mu}(x).$$

Hence the restrictions laid upon ν and μ may be removed.

On applying formula (VIII), formula (XVI) takes the form

$$e^{x^2}\omega_{\frac{1}{2}\nu-\frac{1}{4},\,\mu-\frac{1}{2}}(x^2)=\frac{e^{\frac{1}{2}(\nu+\mu)\pi i}\,\Gamma\left(\dfrac{\nu+\mu+1}{2}\right)}{\pi^{\frac{1}{2}}\,\Gamma(\nu+\mu+1)}U_{\nu,\mu}(x).$$

The most important case in Mr. Cunningham's paper occurs when $\nu-\mu$ is an even positive integer. In this case formula (XVI) becomes

$$e^{x^2}\omega_{\frac{1}{2}\nu-\frac{1}{4},\,\mu-\frac{1}{2}}(x^2)=K_{\nu,\mu}(x). \tag{XVIa}$$

Another important case occurs when $\nu+\mu$ is an odd positive integer. In this case

$$e^{x^2}\omega_{\frac{1}{2}\nu-\frac{1}{4},\,\mu-\frac{1}{2}}(x^2)=e^{(\mu-\frac{1}{2})\pi i}\,H_{\nu,\mu}(x). \tag{XVIb}$$

An interesting case arises when μ takes the form $\rho+\frac{1}{2}$, where ρ is integral. For, on putting $\mu=\rho+\frac{1}{2}-\epsilon$ in (XVI),

$$e^{x^2}\omega_{\frac{1}{2}\nu-\frac{1}{4},\,\rho-\epsilon}(x^2)$$

$$=\frac{\cos\dfrac{\nu+\rho+\frac{1}{2}-\epsilon}{2}\pi\,.\,K_{\nu,\,\rho+\frac{1}{2}-\epsilon}(x)+\sin\dfrac{\nu-\rho-\frac{1}{2}+\epsilon}{2}\pi\,.\,H_{\nu,\,\rho+\frac{1}{2}-\epsilon}(x)}{e^{(\nu-\rho-\frac{1}{2}+\epsilon)\frac{1}{2}\pi i}\cos(\rho+\frac{1}{2}-\epsilon)\pi}$$

$$=\frac{\sin\dfrac{\nu-\rho-\frac{1}{2}+\epsilon}{2}\pi\,.\,H_{\nu,\,\rho+\frac{1}{2}-\epsilon}(x)-(-1)^\rho\sin\dfrac{\nu-\rho-\frac{1}{2}-\epsilon}{2}\pi\,.\,K_{\nu,\,\rho+\frac{1}{2}-\epsilon}(x)}{(-1)^\rho\,e^{(\nu-\rho-\frac{1}{2}+\epsilon)\frac{1}{2}\pi i}\sin\epsilon\pi}$$

$$=\sin(\nu-\rho-\tfrac{1}{2})\frac{\pi}{2}\cos\frac{\epsilon\pi}{2}\left[\frac{(-1)^\rho H_{\nu,\,\rho+\frac{1}{2}-\epsilon}(x)-K_{\nu,\,\rho+\frac{1}{2}-\epsilon}(x)}{e^{(\nu-\rho-\frac{1}{2}+\epsilon)\frac{1}{2}\pi i}\sin\epsilon\pi}\right]$$

$$+\cos(\nu-\rho-\tfrac{1}{2})\frac{\pi}{2}\sin\frac{\epsilon\pi}{2}\left[\frac{(-1)^\rho H_{\nu,\,\rho+\frac{1}{2}-\epsilon}(x)+K_{\nu,\,\rho+\frac{1}{2}-\epsilon}(x)}{e^{(\nu-\rho-\frac{1}{2}+\epsilon)\frac{1}{2}\pi i}\sin\epsilon\pi}\right].$$

Hence, on making ϵ tend to zero, and on using (IX) and (X), we obtain if ρ is integral,

$$e^{x^2}\omega_{\frac{1}{2}\nu-\frac{1}{4},\,\rho}(x^2)=\frac{\sin(\nu-\rho-\frac{1}{2})\dfrac{\pi}{2}}{\pi\,.\,e^{(\nu-\rho-\frac{1}{2})\frac{1}{2}\pi i}}W_{\nu,\,\rho+\frac{1}{2}}(x)+\frac{\cos(\nu-\rho-\frac{1}{2})\dfrac{\pi}{2}}{e^{(\nu-\rho-\frac{1}{2})\frac{1}{2}\pi i}}K_{\nu,\,\rho+\frac{1}{2}}(x). \tag{XVIc}$$

7. *A connexion between the Bessel functions and the functions* $H_{-\frac{1}{2},\mu}(x)$, $K_{-\frac{1}{2},\mu}(x)$ *respectively.*

In the fundamental differential equation (I), let $\nu = -\frac{1}{2}$, so that we have to deal with the differential equation

$$\frac{d^2y}{dx^2}+\frac{2\mu}{x}\frac{dy}{dx}-2x\frac{dy}{dx}-(2\mu+1)\,y = 0. \qquad (a)$$

Now let $y = e^{\frac{1}{2}x^2}x^{\frac{1}{2}-\mu}t$, so that the equation for t is

$$\frac{d^2t}{dx^2}+\frac{1}{x}\frac{dt}{dx}-\left(x^2+\frac{(\mu-\frac{1}{2})^2}{x^2}\right)t = 0.$$

On writing $x^2 = -2iv$, this becomes

$$\frac{d^2t}{dv^2}+\frac{1}{v}\frac{dt}{dv}+\left(1-\frac{\left(\frac{\mu}{2}-\frac{1}{4}\right)^2}{v^2}\right)t = 0.$$

This is the differential equation for the Bessel function of order $\frac{\mu}{2}-\frac{1}{4}$. Thus the general solution of (a) may be written in the form

$$y = Ae^{\frac{1}{2}x^2}x^{\frac{1}{2}-\mu}J_{-\frac{1}{2}\mu+\frac{1}{4}}\left(\frac{x^2i}{2}\right)+Be^{\frac{1}{2}x^2}x^{\frac{1}{2}-\mu}J_{\frac{1}{2}\mu-\frac{1}{4}}\left(\frac{x^2i}{2}\right),$$

provided $\frac{\mu}{2}-\frac{1}{4}$ is not integral.

Since $H_{-\frac{1}{2},\mu}(x)$ and $K_{-\frac{1}{2},\mu}(x)$ begin with terms in $x^{1-2\mu}$ and x^0 respectively, it follows that

$$H_{-\frac{1}{2},\mu}(x) = Ae^{\frac{1}{2}x^2}x^{\frac{1}{2}-\mu}J_{-\frac{1}{2}\mu+\frac{1}{4}}\left(\frac{x^2i}{2}\right) \quad\text{and}\quad K_{-\frac{1}{2},\mu}(x) = Be^{\frac{1}{2}x^2}x^{\frac{1}{2}-\mu}J_{\frac{1}{2}\mu-\frac{1}{4}}\left(\frac{x^2i}{2}\right),$$

where A and B are constants, provided μ is not half an odd integer (see § 3).

On an examination of the initial terms in each expansion these relations become

$$H_{-\frac{1}{2},\mu}(x) = \frac{\pi^{\frac{1}{2}}.\,e^{(\mu-\frac{1}{2})\frac{1}{4}\pi i}}{\Gamma\left(\frac{3}{4}-\frac{\mu}{2}\right)}e^{\frac{1}{2}x^2}x^{\frac{1}{2}-\mu}J_{-\frac{1}{2}\mu+\frac{1}{4}}\left(\frac{x^2i}{2}\right). \qquad (\text{XVII}a)$$

$$K_{-\frac{1}{2},\mu}(x) = \frac{\pi^{\frac{1}{2}}.\,e^{-(\mu-\frac{1}{2})\frac{1}{4}\pi i}}{\Gamma\left(\frac{3}{4}-\frac{\mu}{2}\right)}e^{\frac{1}{2}x^2}x^{\frac{1}{2}-\mu}J_{\frac{1}{2}\mu-\frac{1}{4}}\left(\frac{x^2i}{2}\right). \qquad (\text{XVII}b)$$

The case where $\frac{\mu}{2}-\frac{1}{4}$ is integral calls for special attention. Here μ is of the form $2\sigma+\frac{1}{2}$, where σ is integral or zero. Then, because of (IX) and because of the relation between $J_n(z)$ and $J_{-n}(z)$ when n is integral, the relations (XVIIa) and (XVIIb) become equivalent.

Now, by using (XVIIa) and (XVIIb),

$$\frac{(-1)^{2\sigma}H_{-\frac{1}{2},\,2\sigma+\frac{1}{2}-2\epsilon}(x)-K_{-\frac{1}{2},\,2\sigma+\frac{1}{2}-2\epsilon}(x)}{2\epsilon}$$

$$=\frac{\pi^{\frac{1}{2}}e^{\frac{1}{2}x^2}x^{-2(\sigma-\epsilon)}}{\Gamma(\frac{1}{2}-\sigma+\epsilon)}\left\{\frac{e^{\frac{1}{2}\pi i(\sigma-\epsilon)}J_{-(\sigma-\epsilon)}\left(\frac{x^2i}{2}\right)-e^{-\frac{1}{2}\pi i(\sigma-\epsilon)}J_{\sigma-\epsilon}\left(\frac{x^2i}{2}\right)}{2\epsilon}\right\}$$

$$=\frac{\pi^{\frac{1}{2}}e^{\frac{1}{2}x^2}x^{-2(\sigma-\epsilon)}}{\Gamma(\frac{1}{2}-\sigma+\epsilon)}\left\{\frac{\cos\frac{\epsilon\pi}{2}\left\{e^{\frac{1}{2}\pi i\sigma}J_{-(\sigma-\epsilon)}\left(\frac{x^2i}{2}\right)-e^{-\frac{1}{2}\pi i\sigma}J_{(\sigma-\epsilon)}\left(\frac{x^2i}{2}\right)\right\}}{2\epsilon}\right.$$

$$\left.-\frac{i\sin\frac{\epsilon\pi}{2}\left\{e^{\frac{1}{2}\pi i\sigma}J_{-(\sigma-\epsilon)}\left(\frac{x^2i}{2}\right)+e^{-\frac{1}{2}\pi i\sigma}J_{(\sigma-\epsilon)}\left(\frac{x^2i}{2}\right)\right\}}{2\epsilon}\right\}.$$

Hence when σ is integral, by using (X), we obtain

$$W_{-\frac{1}{2},\,2\sigma+\frac{1}{2}}(x)=\frac{e^{-\frac{1}{2}\pi i\sigma}\pi^{\frac{1}{2}}e^{\frac{1}{2}x^2}}{2\Gamma(\frac{1}{2}-\sigma)x^{2\sigma}}\left\{\bar{Y}_\sigma\left(\frac{x^2i}{2}\right)-i\pi J_\sigma\left(\frac{x^2i}{2}\right)\right\},\qquad\text{(XVII}c\text{)}$$

where $\bar{Y}_\sigma$ is Hankel's form of the Bessel function of the second kind, viz.,

$$\bar{Y}_\sigma(z)=\underset{\epsilon\to0}{\mathrm{Lt}}\frac{(-1)^\sigma J_{-(\sigma-\epsilon)}(z)-J_{(\sigma-\epsilon)}(z)}{\epsilon}.$$

It is to be noticed that the case where μ is of the form $2\lambda+\frac{3}{2}$, where λ is a negative integer, comes under the case, investigated in the footnote to § 3, where formula (IX) breaks down. The analogous theorem to do with the Bessel functions is as follows:—

$$J_{\frac{1}{2}-\lambda}(t)-(-1)^\lambda iJ_{\lambda-\frac{1}{2}}(t)=\left(\frac{2}{\pi t}\right)^{\frac{1}{2}}e^{it}\sum_{s=0}^{s=\lambda-1}\frac{(\lambda+s-1)!\,i^{\lambda-1+s}}{(\lambda-1-s)!\,s!\,(2t)^s}.$$

8. *The expression of $H_{\nu,\mu}(x)$ and $K_{\nu,\mu}(x)$ in terms of definite integrals involving Bessel functions.*

Let us search for a function $\phi(v)$ and an index ρ, so that

$$\int_{G_t}e^{-t^2}t^\rho\phi(xt)\,dt$$

satisfies the fundamental differential equation (I), G_t being a contour that

begins at the positive end of the real axis, avoids the origin by means of a semicircle above the real axis, and proceeds to the negative end of the real axis.

We must have

$$\int_{G_t} e^{-t^2} t^{\rho+2} \left[\phi''(xt) + \frac{2\mu}{xt}\phi'(xt) - \frac{2x}{t}\phi'(xt) + \frac{2(\nu-\mu)}{t^2}\phi(xt) \right] dt = 0.$$

Now put $\rho = -\nu+\mu-1$, so that we have to secure that

$$\int_{G_t} e^{-t^2} \left[t^{-\nu+\mu-1} \left\{ \phi''(xt) + \frac{2\mu}{xt}\phi'(xt) \right\} \right.$$
$$\left. -2xt^{-\nu+\mu}\phi'(xt) + 2(\nu-\mu)t^{-\nu+\mu-1}\phi(xt) \right] dt = 0.$$

But

$$2\frac{d}{dt}\left\{e^{-t^2}t^{-\nu+\mu}\phi(xt)\right\}$$
$$= e^{-t^2}\left\{-4t^{-\nu+\mu+1}\phi(xt) + 2xt^{-\nu+\mu}\phi'(xt) - 2(\nu-\mu)t^{-\nu+\mu-1}\phi(xt)\right\}.$$

Therefore we require to find a function $\phi(v)$ such that

$$-2\left[e^{-t^2}t^{-\nu+\mu}\phi(xt)\right]_{G_t}$$
$$+\int_{G_t} e^{-t^2}t^{-\nu+\mu-1}\left\{\phi''(xt) + \frac{2\mu}{xt}\phi'(xt) - 4\phi(xt)\right\} dt = 0.$$

Now
$$\frac{d^2u}{dz^2} + \frac{2\mu}{z}\frac{du}{dz} - 4u = 0$$

reduces to the Bessel differential equation

$$\frac{d^2v}{d\zeta^2} + \frac{1}{\zeta}\frac{dv}{d\zeta} + \left\{1 - \frac{(\mu-\frac{1}{2})^2}{\zeta^2}\right\} v = 0,$$

on putting
$$z = \frac{\zeta i}{2}, \quad \text{and} \quad u = \zeta^{\frac{1}{2}-\mu} v.$$

Thus we obtain

$$\phi(xt) = Ax^{\frac{1}{2}-\mu}t^{\frac{1}{2}-\mu}J_{\mu-\frac{1}{2}}(-2xti) + Bx^{\frac{1}{2}-\mu}t^{\frac{1}{2}-\mu}J_{\frac{1}{2}-\mu}(-2xti),$$

and, clearly,
$$\left[e^{-t^2}t^{-\nu+\mu}\phi(xt)\right]_{G_t} = 0.$$

Thus, provided μ is not half an odd integer, it follows that $H_{\nu,\mu}(x)$ may be put into the form

$$Ax^{\frac{1}{2}-\mu}\int_{G_t} e^{-t^2}t^{-\nu-\frac{1}{2}}J_{\frac{1}{2}-\mu}(-2xti)\, dt,$$

and $K_{\nu,\mu}(x)$ may be put into the form

$$Bx^{\frac{1}{2}-\mu}\int_{G_t} e^{-t^2}t^{-\nu-\frac{1}{2}}J_{\mu-\frac{1}{2}}(-2xti)\,dt,$$

since $H_{\nu,\mu}(x)$ and $K_{\nu,\mu}(x)$ begin with terms in $x^{1-2\mu}$ and x^0 respectively.

After determining A and B by examining the initial terms in each expansion, we obtain, in the general case,

$$H_{\nu,\mu}(x) = \frac{e^{\frac{1}{2}\pi i(\nu-\frac{1}{2})}\,\Gamma\left(\dfrac{\nu+\mu+1}{2}\right)}{\pi x^{-\frac{1}{2}+\mu}}\int_{G_t} e^{-t^2}t^{-\nu-\frac{1}{2}}J_{\frac{1}{2}-\mu}(-2xti)\,dt, \quad \text{(XIX}a\text{)}$$

$$K_{\nu,\mu}(x) = \frac{e^{\frac{1}{2}\pi i(\nu-\frac{1}{2})}\,\Gamma\left(\dfrac{\nu+\mu+1}{2}\right)}{\pi x^{-\frac{1}{2}+\mu}}\int_{G_t} e^{-t^2}t^{-\nu-\frac{1}{2}}J_{\mu-\frac{1}{2}}(-2xti)\,dt. \quad \text{(XIX}b\text{)}$$

If μ is half an odd integer, then by using (X) along with (XIXa) and (XIXb), we obtain

$$W_{\nu,\mu}(x) = \frac{e^{\frac{1}{2}\pi i(\nu-\frac{1}{2})}\,\Gamma\left(\dfrac{\nu+\mu+1}{2}\right)}{\pi x^{-\frac{1}{2}+\mu}}\int_{G_t} e^{-t^2}t^{-\nu-\frac{1}{2}}\,\bar{Y}_{\mu-\frac{1}{2}}(-2xti)\,dt, \quad \text{(XIX}c\text{)}$$

where $\bar{Y}_n(z)$ denotes Hankel's second Bessel function when n is integral.

On applying formula (VII) to (XIXa) and (XIXb), we find

$$U_\nu(x) = \frac{\Gamma(\nu+1)\,x^{\frac{1}{2}}}{\pi^{\frac{1}{2}}.e^{(\nu+\frac{1}{2})\frac{1}{2}\pi i}}\int_{G_t} e^{-t^2}t^{-\nu-\frac{1}{2}}\left\{\cos\frac{\nu\pi}{2}J_{-\frac{1}{2}}(-2xti)+\sin\frac{\nu\pi}{2}J_{\frac{1}{2}}(-2xti)\right\}dt.$$

Now
$$J_{-\frac{1}{2}}(-2xti) = \frac{e^{\frac{1}{4}\pi i}}{\pi^{\frac{1}{2}}.x^{\frac{1}{2}}.t^{\frac{1}{2}}}\cosh(2xt),$$

and
$$J_{\frac{1}{2}}(-2xti) = \frac{-e^{+\frac{1}{4}\pi i}}{\pi^{\frac{1}{2}}.x^{\frac{1}{2}}.t^{\frac{1}{2}}}\sinh(2xt).$$

Hence

$$\begin{aligned}U_\nu(x) &= \frac{\Gamma(\nu+1)}{2\pi i\, e^{\frac{1}{2}\nu\pi i}}\int_{G_t} e^{-t^2}t^{-\nu-1}\left(e^{-\frac{1}{2}\nu\pi i}e^{2xt}+e^{\frac{1}{2}\nu\pi i}e^{-2xt}\right)dt\\
&= \frac{\Gamma(\nu+1)}{2\pi i}\int_{G_t} e^{-t^2-2xt}t^{-\nu-1}\,dt+\frac{e^{-\nu\pi i}\Gamma(\nu+1)}{2\pi i}\int_{G_t} e^{-t^2+2xt}t^{-\nu-1}\,dt\\
&= \frac{\Gamma(\nu+1)}{2\pi i}\int_{\Gamma_t} e^{-t^2-2xt}t^{-\nu-1}\,dt,\end{aligned}$$

where Γ_t denotes a contour beginning at the positive end of the real axis,

circulating positively round the origin, and returning again to the positive end of the real axis. This is the definition of $U_\nu(x)$ used in the preceding paper. Thus, the attempt to use formulæ (XIX) in order to obtain formulæ expressing connexions between the (*non-generalised*) Hermite functions and the Bessel functions fails in its object. This had an important bearing upon the genesis of the present paper.

9. *Relations converse to those of the preceding paragraph.*

Consider the integral

$$\int_{G_t} e^{-t^2} t^\rho H_{\nu,\mu}\left(\frac{x}{t}\right) dt,$$

where G_t has the same meaning as in the preceding paragraph, and where ρ, at present, is an undetermined index.

This integral is equal to

$$\sum_{r=0}^{\infty} \frac{(-1)^r \Gamma\left(\frac{\nu+\mu+1}{2}\right) x^{1-2\mu+2r}}{r!\,\Gamma\left(\frac{\nu+\mu+1}{2}-r\right)\Gamma(\frac{3}{2}+r-\mu)} \int_{G_t} e^{-t^2} t^{-1+2\mu-2r+\rho}\, dt \quad \text{[by (IV)]}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma\left(\frac{\nu+\mu+1}{2}\right) x^{1-2\mu+2r}}{r!\,\Gamma\left(\frac{\nu+\mu+1}{2}-r\right)\Gamma(\frac{3}{2}+r-\mu)} (e^{(2\mu+\rho)\pi i}-1)\,\tfrac{1}{2}\Gamma\left(\mu-r+\frac{\rho}{2}\right).$$

Now choose $$\rho = \nu-\mu+1.$$

Then the above expression reduces to

$$\sum_{r=0}^{\infty} \frac{(-1)^r \Gamma\left(\frac{\nu+\mu+1}{2}\right) x^{1-2\mu+2r}}{r!\,\Gamma(\frac{3}{2}+r-\mu)\, 2} (e^{(\nu+\mu+1)\pi i}-1)$$

$$= -e^{\frac{1}{2}(\nu+\mu)\pi i} \cos\frac{\nu+\mu}{2}\pi \,.\, \Gamma\left(\frac{\nu+\mu+1}{2}\right) x^{\frac{1}{2}-\mu} J_{\frac{1}{2}-\mu}(2x).$$

Thus

$$J_{\frac{1}{2}-\mu}(2x) = \frac{\Gamma\left(\frac{1-\nu-\mu}{2}\right) x^{\mu-\frac{1}{2}}}{e^{\frac{1}{2}(\nu+\mu+2)\pi i}\,\pi} \int_{G_t} e^{-t^2} t^{\nu-\mu+1} H_{\nu,\mu}\left(\frac{x}{t}\right) dt. \quad \text{(XXa)}$$

On substituting $1-\mu$ for μ in (XXa) and using (XII) (one of the formulæ

of transformation found in § 4), we obtain

$$J_{\mu-\frac{1}{2}}(2x) = \frac{x^{\mu-\frac{1}{2}}}{e^{\frac{1}{2}(\nu-\mu-1)\pi i} \sin\frac{\mu-\nu}{2}\pi . \Gamma\left(\frac{\nu+\mu+1}{2}\right)} \int_{G_t} e^{-t^2} t^{\nu-\mu+1} K_{\nu,\mu}\left(\frac{x}{t}\right) dt. \tag{XXb}$$

Formulæ (XXa) and (XXb) become equivalent when $\mu+\frac{1}{2}$ is integral. Before proceeding to obtain the formula in that case involving the Y function, it will be found convenient to apply formulæ (XIa) and (XIb) to formulæ (XXa) and (XXb) respectively, the result of this operation being

$$J_{\frac{1}{2}-\mu}(2x) = \frac{e^{-\frac{1}{2}\pi i(1-2\mu)}\Gamma\left(\frac{1-\nu-\mu}{2}\right) x^{\mu-\frac{1}{2}}}{e^{\frac{1}{2}(\nu+\mu+2)\pi i}\, \pi} \int_{G_t} e^{-t^2+x^2/t^2} t^{\nu-\mu+1} H_{-\nu-1,\mu}\left(\frac{ix}{t}\right) dt,$$

and

$$J_{\mu-\frac{1}{2}}(2x) = \frac{\sin\frac{\mu-\nu}{2}\pi . x^{\mu-\frac{1}{2}}}{e^{\frac{1}{2}(\nu-\mu-1)\pi i}\cos\frac{\nu+\mu}{2}\pi \sin\frac{\mu-\nu}{2}\pi . \Gamma\left(\frac{\nu+\mu+1}{2}\right)}$$
$$\times \int_{G_t} e^{-t^2+x^2/t^2} t^{\nu-\mu+1} K_{-\nu-1,\mu}\left(\frac{ix}{t}\right) dt.$$

And now the factors outside each integral are not merely equal when μ is half an odd integer, but are identically equal. Thus, if we now apply formula (X), we obtain

$$\bar{Y}_{\mu-\frac{1}{2}}(2x) = \frac{\Gamma\left(\frac{1-\nu-\mu}{2}\right) x^{\mu-\frac{1}{2}}}{e^{\frac{1}{2}(\nu-\mu-1)\pi i}\, \pi} \int_{G_t} e^{-t^2+x^2/t^2} t^{\nu-\mu+1} W_{-\nu-1,\mu}\left(\frac{ix}{t}\right) dt, \tag{XXc}$$

when μ is half an odd integer.

This completes the solution of the problem suggested in the introductory remarks to this paper. The results established in the following two paragraphs seem, however, to be of importance in their bearing upon the theory of these generalised Hermite functions.

10. *The expression of the preceding generalisations of the Hermite functions in terms of definite integrals involving the non-generalised Hermite functions.*

Consider the integral

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} K_{\nu-\mu,0}(x\sin\theta)\cos^{2\mu-1}\theta\,d\theta,$$

where the real part of μ is positive. This is equal to

$$\sum_{r=0}^{\infty}\frac{\Gamma\left(\frac{\nu-\mu+1}{2}\right)(-1)^r x^{2r}}{r!\,\Gamma\left(\frac{\nu-\mu}{2}-r+1\right)\Gamma(r+\frac{1}{2})}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\sin^{2r}\theta\cos^{2\mu-1}\theta\,d\theta \quad \text{[by (VI)]}$$

$$=\Gamma\left(\frac{\nu-\mu+1}{2}\right)\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{r!\,\Gamma\left(\frac{\nu-\mu}{2}-r+1\right)\Gamma(r+\frac{1}{2})}\,\frac{\Gamma(r+\frac{1}{2})\Gamma(\mu)}{\Gamma(\mu+r+\frac{1}{2})}$$

$$=\frac{\Gamma\left(\frac{\nu-\mu+1}{2}\right)\Gamma(\mu)}{\Gamma\left(\frac{\nu+\mu+1}{2}\right)}K_{\nu,\mu}(x),$$

while
$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} H_{\nu-\mu,0}(x\sin\theta)\cos^{2\mu-1}\theta\,d\theta=0,$$

since $H_{\nu-\mu,0}(z)$ is an odd function of z. Hence

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} U_{\nu-\mu}(x\sin\theta)\cos^{2\mu-1}\theta\,d\theta$$

$$=\frac{\pi^{\frac{1}{2}}.\Gamma(\nu-\mu+1)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)e^{(\nu-\mu)\pi i}}\cos\frac{\nu-\mu}{2}\pi\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}K_{\nu-\mu,0}(x\sin\theta)\cos^{2\mu-1}\theta\,d\theta \quad \text{[by (VII)]}$$

$$=\frac{\pi^{\frac{1}{2}}.\Gamma(\nu-\mu+1)\cos\frac{\nu-\mu}{2}\pi.\Gamma(\mu)}{e^{(\nu-\mu)\pi i}\,\Gamma\left(\frac{\nu+\mu+1}{2}\right)}K_{\nu,\mu}(x).$$

Thus

$$K_{\nu,\mu}(x)=\frac{e^{(\nu-\mu)\pi i}\,\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\pi^{\frac{1}{2}}.\Gamma(\nu-\mu+1)\,\Gamma(\mu)\cos\frac{\nu-\mu}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}U_{\nu-\mu}(x\sin\theta)\cos^{2\mu-1}\theta\,d\theta, \tag{XXIa}$$

provided the real part of μ is positive.

To generalise this relation so as to remove the restriction upon μ, consider

$$\int_{\gamma_{0,1}} K_{\nu-\mu,0}(xt^{\frac{1}{2}})(t-1)^{\mu-1}\frac{dt}{t^{\frac{1}{2}}},$$

where $\gamma_{0,1}$ has the same significance as in § 1.

This integral is equal to

$$\Gamma\left(\frac{\nu-\mu+1}{2}\right)\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{r!\,\Gamma\left(\frac{\nu-\mu}{2}-r+1\right)\Gamma(r+\frac{1}{2})}\int_{\gamma_{0,1}} t^{r-\frac{1}{2}}(t-1)^{\mu-1}\,dt \qquad \text{[by (VI)]}$$

$$=\Gamma\left(\frac{\nu-\mu+1}{2}\right)\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{r!\,\Gamma\left(\frac{\nu-\mu}{2}-r+1\right)\Gamma(r+\frac{1}{2})}I_{\mu-1,\mu,r} \qquad \text{(see § 2)}$$

$$=\Gamma\left(\frac{\nu-\mu+1}{2}\right)\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{r!\,\Gamma\left(\frac{\nu-\mu}{2}-r+1\right)\Gamma(r+\frac{1}{2})}$$

$$\times 4e^{\frac{1}{2}\pi i}\cos(\mu-\tfrac{1}{2})\pi\sin\frac{\pi}{2}\,\frac{\Gamma(\mu)\,\Gamma(r+\frac{1}{2})}{\Gamma(\mu+r+\frac{1}{2})}$$

$$=\Gamma\left(\frac{\nu-\mu+1}{2}\right)4i\sin\mu\pi\,\Gamma(\mu)\frac{1}{\Gamma\left(\frac{\nu+\mu+1}{2}\right)}K_{\nu,\mu}(x),$$

while $\displaystyle\int_{\gamma_{0,1}} H_{\nu-\mu,0}(xt^{\frac{1}{2}})(t-1)^{\mu-1}\frac{dt}{t^{\frac{1}{2}}}$

$$=\sum_{r=0}^{\infty}\binom{\frac{\nu-\mu-1}{2}}{r}(-1)^r\frac{x^{1+2r}}{\Gamma(\frac{3}{2}+r)}\int_{\gamma_{0,1}} t^r(t-1)^{\mu-\frac{1}{2}}\,dt,$$

and the integral in each of these terms is a zero factor owing to the nature of $\gamma_{0,1}$, and the fact that r is an integer.

Thus (XXIa) may be extended to take the form

$$K_{\nu,\mu}(x)=\frac{e^{(\nu-\mu)\pi i}\,\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{4i\sin\mu\pi.\pi^{\frac{1}{2}}\cos\frac{\nu-\mu}{2}\pi.\Gamma(\nu-\mu+1)\,\Gamma(\mu)}$$

$$\times\int_{\gamma_{0,1}} U_{\nu-\mu}(xt^{\frac{1}{2}})(t-1)^{\mu-1}\frac{dt}{t^{\frac{1}{2}}}. \qquad \text{(XXI}b\text{)}$$

On applying formula (XII), we have

$$H_{\nu,\mu}(x) = \frac{x^{1-2\mu} e^{(\nu+\mu-1)\pi i}\,\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{4i \sin \mu\pi . \pi^{\frac{1}{2}} \sin \frac{\nu+\mu}{2}\pi . \Gamma(\nu+\mu)\,\Gamma(1-\mu)}$$

$$\times \int_{\gamma_{0,1}} U_{\nu+\mu-1}(xt^{\frac{1}{2}})(t-1)^{-\mu}\frac{dt}{t^{\frac{1}{2}}}$$

$$= \frac{x^{1-2\mu} e^{(\nu+\mu-1)\pi i}\,\Gamma\left(\frac{\nu+\mu+1}{2}\right)\Gamma(\mu)}{4\pi i . \pi^{\frac{1}{2}} \sin \frac{\nu+\mu}{2}\pi . \Gamma(\nu+\mu)}\,\frac{(-1)}{2(\nu+\mu)}$$

$$\times \int_{\gamma_{0,1}} \left\{\frac{d}{d(xt^{\frac{1}{2}})} U_{\nu+\mu}(xt^{\frac{1}{2}})\right\}(t-1)^{-\mu}\frac{dt}{t^{\frac{1}{2}}}$$

[p. 238 (III) preceding paper]

$$= \frac{x^{-2\mu} e^{(\nu+\mu)\pi i}\,\Gamma\left(\frac{\nu+\mu+1}{2}\right)\Gamma(\mu)}{4\pi i . \pi^{\frac{1}{2}} \sin \frac{\nu+\mu}{2}\pi . \Gamma(\nu+\mu+1)} \int_{\gamma_{0,1}} \left\{\frac{d}{dt} U_{\nu+\mu}(xt^{\frac{1}{2}})\right\}(t-1)^{-\mu}\,dt$$

$$= \frac{e^{(\nu+\mu)\pi i}\,\Gamma\left(\frac{\nu+\mu+1}{2}\right)\Gamma(\mu+1)\,x^{-2\mu}}{4\pi i . \pi^{\frac{1}{2}} \sin \frac{\nu+\mu}{2}\pi . \Gamma(\nu+\mu+1)} \int_{\gamma_{0,1}} U_{\nu+\mu}(xt^{\frac{1}{2}})(t-1)^{-\mu-1}\,dt. \tag{XXIc}$$

11. *The expression of the generalised Hermite polynomials as differential coefficients. Integral properties.*

The Hermite function $U_n(x)$ is a polynomial when n is a positive integer, and is defined by the relation

$$U_n(x) = e^{x^2}\frac{d^n}{dx^n}e^{-x^2}.$$

Now when $\nu-\mu$ is an even positive integer $K_{\nu,\mu}(x)$ is a polynomial, and when $\nu+\mu$ is an odd positive integer $H_{\nu,\mu}(x)$ is a polynomial. Further, in consequence of relation (VIII), when these functions are polynomials, they may be regarded as generalised Hermite polynomials (multiplied by

constant factors). The object of this paragraph is to show that these generalised polynomials may be defined by means of relations similar to that defining $U_n(x)$ when $U_n(x)$ is a polynomial.

Let ρ be a positive integer and let $\nu = 2\rho+\mu$.

Then

$$K_{\mu+2\rho,\,\mu}(x^{\frac{1}{2}}) = \frac{1}{4\pi\cos\mu\pi}\int_{\gamma_{0,1}} e^{xt}(t-1)^{\mu+\rho-\frac{1}{2}}t^{-\rho-1}\,dt \quad \text{[by (III)]}.$$

Let T denote $e^{xt}(t-1)^{\mu+\rho-\frac{1}{2}}t^{-\rho-1}$, and T' denote $e^{xt}(1-t)^{\mu+\rho-\frac{1}{2}}t^{-\rho-1}$. Now, in the figure associated with $\gamma_{0,1}$, let C denote a small circle of radius c, described positively about the origin. Let A denote a small semicircle of radius b, described positively about $t=1$, above the real axis. Let B describe a similar semicircle below the real axis.

Then

$4\pi\cos\mu\pi K_{\mu+2\rho,\,\mu}(x^{\frac{1}{2}})$

$$= \int_A T\,dt + e^{(\mu+\rho-\frac{1}{2})\pi i}\int_{1-b}^{c} T'\,dt + e^{(\mu+\rho-\frac{1}{2})\pi i}\int_C T'\,dt$$

$$+e^{(\mu+\rho-\frac{1}{2})\pi i}\int_c^{1-b} T'\,dt - \int_A T\,dt - \int_B T\,dt + e^{-\pi i(\mu+\rho-\frac{1}{2})}\int_{1-b}^{c} T'\,dt$$

$$-e^{-\pi i(\mu+\rho-\frac{1}{2})}\int_C T'\,dt + e^{-\pi i(\mu+\rho-\frac{1}{2})}\int_c^{1-b} T'\,dt + \int_B T\,dt$$

$$= 2i\sin(\mu+\rho-\tfrac{1}{2})\pi\int_C e^{xt}(1-t)^{\mu+\rho-\frac{1}{2}}t^{-\rho-1}\,dt,$$

where C is a simple contour enclosing the origin but not $t=1$.

Now let $t = 1-v/x$, then

$$K_{\mu+2\rho,\,\mu}(x^{\frac{1}{2}}) = \frac{(-1)^\rho e^x}{2\pi i}\int_D \frac{e^{-v}x^{-\mu-\rho+\frac{1}{2}}v^{\mu+\rho-\frac{1}{2}}x^{\rho+1}(-1)\,dv}{x(x-v)^{\rho+1}}$$

$$= \frac{e^x x^{\frac{1}{2}-\mu}}{2\pi i}\int_D \frac{e^{-v}v^{\mu+\rho-\frac{1}{2}}}{(v-x)^{\rho+1}}\,dv,$$

where D is a similar contour enclosing $v=x$ but not $v=0$.

Hence, if ρ is a positive integer,

$$K_{\mu+2\rho,\,\mu}(x^{\frac{1}{2}}) = e^x x^{\frac{1}{2}-\mu} \frac{1}{\rho!} \frac{d^\rho}{dx^\rho} e^{-x} x^{\mu+\rho-\frac{1}{2}}. \qquad \text{(XXII}a\text{)}$$

And, on applying formula (XII), we see that if ρ is a positive integer,

$$H_{2\rho+1-\mu,\,\mu}(x^{\frac{1}{2}}) = \frac{e^x}{\Gamma(\frac{3}{2}+\rho-\mu)} \frac{d^\rho}{dx^\rho} e^{-x} x^{-\mu+\rho+\frac{1}{2}}. \qquad \text{(XXII}b\text{)}$$

Integral properties of the generalised polynomials.

Now consider

$$\int_{\Gamma_u} e^{-u} u^{\mu-\frac{1}{2}} K_{\mu+2\rho,\,\mu}(u^{\frac{1}{2}})\, K_{\mu+2\sigma,\,\mu}(u^{\frac{1}{2}})\, du,$$

where ρ and σ are positive integers and where Γ_u denotes a contour that begins at positive infinity, circulates once positively round the origin, and returns to its starting point.

Using (XXIIa) this integral takes the form

$$\frac{1}{\rho!} \int_{\Gamma_u} \frac{d^\rho}{du^\rho} (e^{-u} u^{\mu+\rho-\frac{1}{2}})\, K_{\mu+2\sigma,\,\mu}(u^{\frac{1}{2}})\, du.$$

Now suppose ρ is greater than σ, then, since $K_{\mu+2\sigma,\,\mu}(u^{\frac{1}{2}})$ is a polynomial of degree σ, it is at once seen that, on using the process of repeated partial integration, this integral may be proved equal to zero.

If ρ is equal to σ, then the integral becomes

$$\frac{(-1)^\rho}{\rho!} \int_{\Gamma_u} e^{-u} u^{\mu+\rho-\frac{1}{2}} \frac{d^\rho}{du^\rho} K_{\mu+2\rho,\,\mu}(u^{\frac{1}{2}})\, du$$

$$= \frac{(-1)^\rho}{\rho!} \Gamma(\mu+\rho+\tfrac{1}{2}) \frac{(-1)^\rho \rho!}{\rho!\,\Gamma(1)\,\Gamma(\mu+\rho+\frac{1}{2})} \int_{\Gamma_u} e^{-u} u^{\mu+\rho-\frac{1}{2}} du \qquad \text{[on using (VI)]}$$

$$= \frac{1}{\rho!} (e^{(2\mu-1)\pi i} - 1)\, \Gamma(\mu+\rho+\tfrac{1}{2}).$$

Thus $$\int_{\Gamma_u} e^{-u} u^{\mu-\frac{1}{2}} K_{\mu+2\rho,\,\mu}(u^{\frac{1}{2}})\, K_{\mu+2\sigma,\,\mu}(u^{\frac{1}{2}})\, du$$

$$= 0 \quad (\text{if } \rho \neq \sigma)$$

$$= \frac{-2e^{\mu\pi i} \cos \mu\pi\, \Gamma(\mu+\rho+\frac{1}{2})}{\rho!} \quad (\text{if } \rho = \sigma). \qquad \text{(XXIII}a\text{)}$$

On applying formula (XII) this takes the more compact form

$$\int_{\Gamma_u} e^{-u} H_{\mu+2\rho,\,1-\mu}(u^{\frac{1}{2}})\, K_{\mu+2\sigma,\,\mu}(u^{\frac{1}{2}})\,du = 0 \quad (\text{if } \rho \neq \sigma)$$

$$= -2e^{\mu\pi i}\cos\mu\pi \quad (\text{if } \rho = \sigma), \qquad (\text{XXIII}b)$$

and this admits of further simplification if the real part of $(\mu+\frac{1}{2})$ is positive. This formula is not new, however; it is merely a generalisation of (26) and (27) in Mr. Cunningham's memoir, the method of approach being different.

We shall finally prove the relation

$$\frac{1}{2\pi i}\int_{\Gamma_t} \frac{e^{-t} H_{\mu+2\rho,\,1-\mu}(t^{\frac{1}{2}})\,dt}{(t-x)} = e^{(\mu-\frac{1}{2})\pi i} e^{-x} K_{\mu+2\rho,\,1-\mu}(x^{\frac{1}{2}}),$$

where Γ_t denotes a contour that begins at positive infinity on the real axis, circulates round the origin by means of a circle of radius greater than $|x|$ and returns again to its starting point, ρ being a positive integer.

$$\int_{\Gamma_t} \frac{e^{-t} H_{\mu+2\rho,\,1-\mu}(t^{\frac{1}{2}})\,dt}{(t-x)}$$

$$= \sum_{r=0}^{\infty} x^r \int_{\Gamma_t} e^{-t} t^{-r-1} H_{\mu+2\rho,\,1-\mu}(t^{\frac{1}{2}})\,dt$$

$$= \sum_{r=0}^{\infty} x^r \int_{\Gamma_t} e^{-t} t^{-r-1} \frac{e^t}{\Gamma(\mu+\rho+\frac{1}{2})} \frac{d^\rho}{dt^\rho}(t^{\mu+\rho-\frac{1}{2}} e^{-t})\,dt \quad [\text{by (XXII}b)]$$

$$= \frac{1}{\Gamma(\mu+\rho+\frac{1}{2})} \sum_{r=0}^{\infty} x^r \frac{\Gamma(r+\rho+1)}{\Gamma(r+1)} \int_{\Gamma_t} t^{-r-1-\rho+\mu+\rho-\frac{1}{2}} e^{-t}\,dt$$

(by repeated partial integration)

$$= \frac{1}{\Gamma(\mu+\rho+\frac{1}{2})} \sum_{r=0}^{\infty} x^r \frac{\Gamma(r+\rho+1)}{r!} e^{(\mu-\frac{1}{2})\pi i}\, 2i \sin(\mu-\tfrac{1}{2})\pi\, \Gamma(-r+\mu-\tfrac{1}{2})$$

$$= \frac{2ie^{(\mu-\frac{1}{2})\pi i}}{\Gamma(\mu+\rho+\frac{1}{2})} \pi \sum_{r=0}^{\infty} \frac{(-1)^r x^r . \Gamma(r+\rho+1)}{\Gamma(r+1)\,\Gamma(\frac{3}{2}+r-\mu)}$$

$$= \frac{2\pi i\,\Gamma(\rho+1)\, e^{(\mu-\frac{1}{2})\pi i}}{\Gamma(\mu+\rho+\frac{1}{2})} \frac{H_{-2\rho-1-\mu,\,\mu}(ix^{\frac{1}{2}})}{(ix^{\frac{1}{2}})^{1-2\mu}} \quad [\text{by (IV)}].$$

Now, from (XII) and (XIb),

$$\frac{H_{\nu,\mu}(ix^{\frac{1}{2}})}{(ix^{\frac{1}{2}})^{1-2\mu}} = \frac{\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\Gamma\left(\frac{\nu-\mu}{2}+1\right)} K_{\nu,\,1-\mu}(ix^{\frac{1}{2}})$$

$$= \frac{\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\Gamma\left(\frac{\nu-\mu}{2}+1\right)} e^{-x} \frac{\cos\frac{\mu+\nu}{2}\pi}{\sin\frac{\mu-\nu}{2}\pi} K_{-\nu-1,\,1-\mu}(x^{\frac{1}{2}})$$

$$= e^{-x} \frac{\Gamma\left(\frac{\mu-\nu}{2}\right)}{\Gamma\left(\frac{1-\mu-\nu}{2}\right)} K_{-\nu-1,\,1-\mu}(x^{\frac{1}{2}})\,;$$

therefore

$$\frac{1}{2\pi i}\int_{\Gamma_t} \frac{e^{-t} H_{\mu+2\rho,\,1-\mu}(t^{\frac{1}{2}})\,dt}{(t-x)} = \frac{\Gamma(\rho+1)\,e^{(\mu-\frac{1}{2})\pi i}}{\Gamma(\mu+\rho+\frac{1}{2})}\, e^{-x}\, \frac{\Gamma(\mu+\rho+\frac{1}{2})}{\Gamma(\rho+1)} K_{\mu+2\rho,\,1-\mu}(x^{\frac{1}{2}})$$

$$= e^{(\mu-\frac{1}{2})\pi i}\, e^{-x} K_{\mu+2\rho,\,1-\mu}(x^{\frac{1}{2}}). \qquad \text{(XXIV)}$$

ON BINARY FORMS

By A. YOUNG.

[Read January 22nd, 1914.]

THE object of this paper is to develop a method of attacking some of the problems in the theory of binary forms. Problems connected with the enumeration of complete systems are particularly in view.

Every method introduced requires some justification for its existence; its utility needs to be judged by results. In this case the method is at once applied to covariant types of degree four of the binary form of order n, and the complete irreducible set of these is obtained.

The preliminary analysis is concerned with the theory of perpetuants, and incidentally the complete system of perpetuant syzygies for every degree and weight is obtained. It appears that all perpetuant syzygies of the first kind can be obtained symbolically from those due to Stroh, and that consequently the extension to any degree of the work * of Mr. Wood and myself, for the first eight degrees, depends solely on accurate enumeration, and does not require the introduction of any new principle or the discovery of a different type of syzygy.

I. *Explanation of Method.*

1. We are concerned here entirely with the symbolical notation. Its introduction by Aronhold at once gave a method by which all covariants could be mathematically expressed. At the same time in the calculus it provides every form considered has the covariant property. But it has the drawback that a great many unnecessary forms appear in any discussion. Various methods have or can be suggested by which the forms considered may be limited to a linearly independent set. But such methods cannot avail much in most problems unless it is possible to express the product of two forms so expressed in terms of the corresponding forms.

* *Proc. London Math. Soc.*, Ser. 2, Vol. 2.

Grace,* in applying the symmetrical notation to MacMahon's theory of perpetuants, has succeeded in doing this for the case when the order of every quantic considered is infinite. In this case he selected one quantic $a_{1_x}^{\infty}$ for particular attention, introducing the symbol a_1 into every determinant factor, by means of the equation

$$(a_2 a_3)\, a_{1_x} = (a_1 a_3)\, a_{2_x} - (a_1 a_2)\, a_{3_x}.$$

Thus the only symbolical products he had to consider were of the form (omitting factors a_x)

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \dots (a_1 a_\delta)^{\lambda_\delta}.$$

These, when perpetuant *types* are under consideration, are all linearly independent. There are no superfluous forms.

Now, when we come to forms of finite order, we cannot, as a rule, apply this method as it stands, for the reason that there are not a sufficient number of factors a_{1_x} in order to be able to introduce the letter a_1 into every determinant factor. In fact, if we can do so, n_1, the order of the corresponding quantic, must be equal to or greater than the weight of the covariant considered.

Let w be the weight of the covariant C, then if we multiply C symbolically by $a_{1_x}^{w-n_1}$, we can express $a_{1_x}^{w-n_1} C$ in the form

$$\Sigma N (a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \dots (a_1 a_\delta)^{\lambda_\delta}\, a_{2_x}^{n_2-\lambda_2}\, a_{3_x}^{n_3-\lambda_3} \dots a_{\delta_x}^{n_\delta-\lambda_\delta},$$

where N is numerical.

We have thus, as in the case of perpetuants, a linearly independent set of symbolical products

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \dots (a_1 a_\delta)^{\lambda_\delta}$$

to consider. But there is this difference: separate products do not represent actual covariants, but only certain linear functions of such products. We shall proceed to shew how every such product may be made to represent a covariant or else a form which we shall call a *fundamental form.*

After that we shall proceed to shew how products of covariants may be dealt with, as in the case of perpetuants.

2. Let us consider covariant types of degree δ; that is, covariants

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* *Proc. London Math. Soc.*, Vol. xxxv, p. 107.

linear in the coefficients of each of the quantics

$$a_{1_x}^{n_1},\ a_{2_x}^{n_2},\ \ldots,\ a_{\delta_x}^{n_\delta}.$$

It is supposed, to start with, that these quantics are arranged in a fixed sequence.

Let us fix our attention on some covariant type expressed in the ordinary manner as a single symbolical product. We say that this covariant is a term of the continued transvectant

$$((\ldots((a_1 a_2)^{\lambda_2},\ a_3)^{\lambda_3},\ a_4)^{\lambda_4},\ \ldots,\ a_\delta)^{\lambda_\delta}$$

(using the single symbolical letter to denote the corresponding quantic). This statement is nearly obvious. An immediate proof is obtained by induction. Assume it true for degree δ; then, if C be a symbolical product representing a covariant of degree $\delta+1$, C is a term of a transvectant

$$(P,\ a_{\delta+1})^{\lambda_{\delta+1}},$$

and, since P is a symbolical product representing a covariant of degree δ, the theorem in question is true for P, and therefore it is also true for C.

Now the fact that every term of a transvectant differs from the whole transvectant, by a linear function of transvectants of lower index, leads us at once to the fact that any term of the continued transvectant

$$((\ldots((a_1 a_2)^{\lambda_2},\ a_3)^{\lambda_3},\ a_4)^{\lambda_4},\ \ldots,\ a_\delta)^{\lambda_\delta}$$

differs from the whole transvectant by a linear function of forms

$$((\ldots((a_1 a_2)^{\mu_2},\ a_3)^{\mu_3},\ a_4)^{\mu_4},\ \ldots,\ a_\delta)^{\mu_\delta},$$

which are such that the first of the differences

$$\lambda_\delta-\mu_\delta,\ \lambda_{\delta-1}-\mu_{\delta-1},\ \ldots,\ \lambda_2-\mu_2,$$

which does not vanish is positive.

We are then at liberty to express every covariant type of degree δ in terms of continued transvectants of the above form.

3. Let us now retnrn to the consideration of a single symbolical product which represents a covariant type C of degree δ. Let the weight of C be w.

The symbolical product $a_{1_x}^{w-n} C$ can be expressed in the form

$$\Sigma N (a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \ldots (a_1 a_\delta)^{\lambda_\delta},$$

where N is numerical: by repeated use of the equation

$$(a_r a_s)\, a_{1_x} = (a_1 a_s)\, a_{r_x} - (a_1 a_r)\, a_{s_x}.$$

We shall arrange the products in a definite sequence by saying that

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \ldots (a_1 a_\delta)^{\lambda_\delta}$$

precedes $$(a_1 a_2)^{\mu_2} (a_1 a_3)^{\mu_3} \ldots (a_1 a_\delta)^{\mu_\delta},$$

provided that the first of the differences

$$\lambda_\delta - \mu_\delta,\ \lambda_{\delta-1} - \mu_{\delta-1},\ \ldots,\ \lambda_2 - \mu_2,$$

which does not vanish is positive.

The continued transvectants will be supposed arranged in sequence according to the same law.

Now it is to be observed that a continued transvectant is defined by the same set of numbers $\lambda_2, \lambda_3, \ldots, \lambda_\delta$, as a product

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \ldots (a_1 a_\delta)^{\lambda_\delta}.$$

If the continued transvectant be expressed as a sum of the products considered (by multiplying it by $a_{1_x}^{w-n_1}$), the first of the products in our sequence to appear will be that which is defined by the same numbers.

Now every continued transvectant represents a covariant type; but only certain linear functions of the products (viz., such as are divisible by $a_{1_x}^{w-n_1}$) represent actual covariants. The difference between the two cases being accounted for by the fact that there are certain limitations to be imposed on the indices of the transvectant; whilst the only limitations to the indices of the product are those expressed by the inequalities

$$\lambda_2 \ngtr n_2,\ \lambda_3 \ngtr n_3,\ \ldots,\ \lambda_\delta \ngtr n_\delta.$$

These limitations are also necessary for the transvectant, but in addition we must have

(i) $\lambda_2 \ngtr n_1,\ 2\lambda_2 + \lambda_3 \ngtr n_1 + n_2,\ 2\lambda_2 + 2\lambda_3 + \lambda_4 \ngtr n_1 + n_2 + n_3,\ \ldots,$

$$2\lambda_2 + 2\lambda_3 + \ldots + 2\lambda_{\delta-1} + \lambda_\delta \ngtr n_1 + n_2 + n_3 + \ldots + n_{\delta-1}.$$

In the case of products we shall use the term *fundamental forms* to denote products for which the set of inequalities (i) is not satisfied.

4. We proceed to shew that corresponding to every other product, that is to every product for which the inequalities (i) are satisfied, there is

a unique covariant which can be represented as a linear function of that product and of fundamental forms. We have seen that the transvectant

$$((\ldots((a_1a_2)^{\lambda_2},\ a_3)^{\lambda_3},\ a_4)^{\lambda_4},\ \ldots,\ a_\delta)^{\lambda_\delta}$$

can be expressed as a linear function of our products of which the first term is

$$(a_1a_2)^{\lambda_2}(a_1a_3)^{\lambda_3}\ldots(a_1a_\delta)^{\lambda_\delta}.$$

Let

$$N(a_1a_2)^{\mu_2}(a_1a_3)^{\mu_3}\ldots(a_1a_\delta)^{\mu_\delta}$$

be the next term in the order of our sequence to appear; if it is not a fundamental form we may subtract the covariant

$$N((\ldots((a_1a_2)^{\mu_2},\ a_3)^{\mu_3},\ a_4)^{\mu_4},\ \ldots,\ a_\delta)^{\mu_\delta}$$

from both sides of our equation.

Proceeding thus step by step, we arrive at the truth of the above statement. That the covariant is unique is evident from the fact that every covariant can be expressed in terms of the transvectants considered, and that these transvectants can be expressed in terms of the covariants found, and *vice versa*.

5. Let us use the notation

$$(\lambda_2,\ \lambda_3,\ \ldots,\ \lambda_\delta)$$

to denote the covariant corresponding to

$$(a_1a_2)^{\lambda_2}(a_1a_3)^{\lambda_3}\ldots(a_1a_\delta)^{\lambda_\delta},$$

i.e. the covariant obtained from this product by the addition of a linear function of fundamental forms.

Then we have a set of linearly independent covariant types of degree δ in terms of which every such covariant type may be linearly expressed. And this set is composed of the forms

$$(\lambda_2,\ \lambda_3,\ \ldots,\ \lambda_\delta),$$

where

$$\lambda_2 \ngtr n_2,\ \lambda_3 \ngtr n_3,\ \ldots,\ \lambda_\delta \ngtr n_\delta,$$

and the λ's further satisfy conditions (i).

It will be convenient to have a notation for the covariant

$$(\lambda_2,\ \lambda_3,\ \ldots,\ \lambda_\delta),$$

in which the letters corresponding to the different quantics appear; we

shall for this purpose use the notation

$$\left(\frac{a_2^{\lambda_2} a_3^{\lambda_3} \dots a_\delta^{\lambda_\delta}}{a_1}\right) \equiv (\lambda_2, \lambda_3, \dots, \lambda_\delta).$$

In order to discover what forms are reducible, or to find relations between products of forms, it is necessary to be able to express the product of any two of our forms as a linear function of the forms of a higher degree.

Thus, for example, the product

$$\left(\frac{a_2^{\lambda_2} a_3^{\lambda_3} \dots a_\delta^{\lambda_\delta}}{a_1}\right) (a_{\delta+1} a_{\delta+2})^\lambda = \Sigma (-)^i \binom{\lambda}{i} \left(\frac{a_2^{\lambda_2} a_3^{\lambda_3} \dots a_\delta^{\lambda_\delta} a_{\delta+1}^i a_{\delta+2}^{\lambda-i}}{a_1}\right).$$

The case of perpetuants is much simpler than that of forms of finite order, and the analysis in this case is a necessary preliminary to that of the more difficult case.

II. *Perpetuants.*

6. Grace proved that the perpetuants

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \dots (a_1 a_\delta)^{\lambda_\delta}$$

can be expressed in terms of products of perpetuants and of forms of this kind for which

$$\lambda_2 \geqslant 2^{\delta-2}, \ \lambda_3 \geqslant 2^{\delta-3}, \ \dots, \ \lambda_\delta \geqslant 2^0.$$

This is the result. The method by which the result was obtained (by means of certain relations due to Stroh) is not the method we require here. We shall therefore proceed to establish the same result by a slightly different method for the sake of the analysis. The analysis will be capable of application to forms of finite order.

7. It is our aim at the outset to express every possible product of two forms as a linear function of forms

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \dots (a_1 a_\delta)^{\lambda_\delta}.$$

In order to do this we must separate the letters $a_1, a_2, \dots, a_\delta$ into two sets. We may write them

$$a_1, \ a_{r_2}, \ \dots, \ a_{r_\epsilon},$$

$$a_{s_1}, \ a_{s_2}, \ \dots, \ a_{s_\eta}.$$

Then we consider the product of any covariant type of the one set by any covariant type of the other set.

The product to be considered is of the form

$$(a_1 a_{r_2})^{\lambda_{r_2}} (a_1 a_{r_3})^{\lambda_{r_3}} \dots (a_1 a_{r_\epsilon})^{\lambda_{r_\epsilon}} (a_{s_1} a_{s_2})^{\lambda_{s_2}} (a_{s_1} a_{s_3})^{\lambda_{s_3}} \dots (a_{s_1} a_{s_\eta})^{\lambda_{s_\eta}}$$

$$= \Sigma(-)^{i_2+i_3+\dots+i_\eta} \binom{\lambda_{s_2}}{i_2} \dots \binom{\lambda_{s_\eta}}{i_\eta} (a_1 a_{r_2})^{\lambda_{r_2}} \dots (a_1 a_{r_\epsilon})^{\lambda_{r_\epsilon}}$$

$$\times (a_1 a_{s_1})^{i_2+\dots+i_\eta} (a_1 a_{s_2})^{\lambda_{s_2}-i_2} \dots (a_1 a_{s_\eta})^{\lambda_{s_\eta}-i_\eta}$$

$$= e^{-(a_1 a_{s_1})\partial/[\partial(a_1 a_{s_2})]-(a_1 a_{s_1})\partial/[\partial(a_1 a_{s_3})]-\dots-(a_1 a_{s_1})\partial/[\partial(a_1 a_{s_\eta})]}$$

$$\times (a_1 a_{r_2})^{\lambda_{r_2}} \dots (a_1 a_{r_\epsilon})^{\lambda_{r_\epsilon}} (a_1 a_{s_2})^{\lambda_{s_2}} \dots (a_1 a_{s_\eta})^{\lambda_{s_\eta}}.$$

Let us suppose that $s_1 = 2$, and let us use the notation

$$D_s \equiv \frac{\partial}{\partial(a_1 a_s)}.$$

Then without fear of ambiguity we may write our result [replacing $(a_1 a_2)$ by a_2 in the exponential index]

$$e^{-a_2 D_{s_2} - a_2 D_{s_3} - \dots - a_2 D_{s_\eta}} (0, \lambda_3, \lambda_4, \dots, \lambda_\delta)$$

$$= \left(\frac{a_{r_2}^{\lambda_{r_2}} a_{r_3}^{\lambda_{r_3}} \dots a_{r_\epsilon}^{\lambda_{r_\epsilon}}}{a_1}\right) \left(\frac{a_{s_2}^{\lambda_{s_2}} a_{s_3}^{\lambda_{s_3}} \dots a_{s_\eta}^{\lambda_{s_\eta}}}{a_2}\right);$$

since $\qquad (\lambda_2, \lambda_3, \dots, \lambda_\delta) \equiv (a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \dots (a_1 a_\delta)^{\lambda_\delta}$

for perpetuants.

8. We thus have a set of equations

$$e^{-a_2 D_{s_2} - a_2 D_{s_3} - \dots - a_2 D_{s_\eta}} (0, \lambda_3, \lambda_4, \dots, \lambda_\delta) = R$$

to consider, where $\qquad s_2, s_3, \dots, s_\eta$

are any, all or none of the numbers

$$3, 4, \dots, \delta.$$

Since each of the $\delta - 2$ numbers may be taken or left we obtain $2^{\delta-2}$ equations. We shall shew that the $2^{\delta-2}$ equations are, in general, independent and are just sufficient to express every form

$$(\lambda_2, \lambda_3, \dots, \lambda_\delta),$$

for which $\lambda_2 < 2^{\delta-2}$ in terms of similar forms for which $\lambda_2 \geqslant 2^{\delta-2}$ and of products of forms of lower order.

In order to prove this we must arrange our equations in a particular manner. We begin with the equation

$$(0,\ \lambda_3,\ \dots,\ \lambda_\delta) = R,$$

representing the fact that this form has the quantic $a_{2_x}^{\infty}$ for a factor.

The next equation will be

$$e^{-a_2 D_\delta}(0,\ \lambda_3,\ \dots,\ \lambda_\delta) = R,$$

or $(0,\ \lambda_3,\ \dots,\ \lambda_\delta) - \lambda_\delta(1,\ \lambda_3,\ \dots,\ \lambda_\delta - 1) + \binom{\lambda_\delta}{2}(2,\ \lambda_3,\ \dots,\ \lambda_\delta - 2) - \dots = R.$

This equation with the help of that already used reduces $(1,\ \lambda_3,\ \dots,\ \lambda_\delta - 1)$; *i.e.*, it expresses this form in terms of earlier forms in the sequence and of products of forms.

We next consider

$$e^{-a_2 D_{\delta-1}}(0,\ \lambda_3,\ \dots,\ \lambda_{\delta-1},\ \lambda_\delta) = R,$$

and it is easy to see that this reduces the form

$$(2,\ \lambda_3,\ \dots,\ \lambda_{\delta-1} - 2,\ \lambda_\delta).$$

When we come to our next equation

$$e^{-a_2 D_{\delta-1} - a_2 D_\delta}(0,\ \lambda_3,\ \dots,\ \lambda_{\delta-1},\ \lambda_\delta) = R,$$

it is necessary to take it in conjunction with the last. We have, on subtracting,

$$\begin{aligned}
&\left[e^{-a_2 D_{\delta-1} - a_2 D_\delta} - e^{-a_2 D_{\delta-1}}\right](0,\ \lambda_3,\ \dots,\ \lambda_{\delta-1},\ \lambda_\delta) \\
&\quad = \lambda_\delta(1,\ \lambda_3,\ \dots,\ \lambda_{\delta-1},\ \lambda_\delta - 1) - \lambda_\delta\lambda_{\delta-1}(2,\ \lambda_3,\ \dots,\ \lambda_{\delta-1} - 1,\ \lambda_\delta - 1) \\
&\qquad + \lambda_\delta\binom{\lambda_{\delta-1}}{2}(3,\ \lambda_3,\ \dots,\ \lambda_{\delta-1} - 2,\ \lambda_\delta - 1) - \dots \\
&\qquad + \text{terms in which the last argument is less than } \lambda_\delta - 1 \\
&\quad = R.
\end{aligned}$$

Also

$$\begin{aligned}
&\left[e^{-a_2 D_{\delta-1}} - 1\right](0,\ \lambda_3,\ \dots,\ \lambda_{\delta-1} + 1,\ \lambda_\delta - 1) \\
&\quad = -(\lambda_{\delta-1} + 1)(1,\ \lambda_3,\ \dots,\ \lambda_{\delta-1},\ \lambda_\delta - 1) + \binom{\lambda_{\delta-1} + 1}{2}(2,\ \lambda_3,\ \dots,\ \lambda_{\delta-1} - 1,\ \lambda_\delta - 1) \\
&\qquad - \binom{\lambda_{\delta-1} + 1}{3}(3,\ \lambda_3,\ \dots,\ \lambda_{\delta-1} - 2,\ \lambda_\delta - 1) + \dots \\
&\quad = R.
\end{aligned}$$

Using the results of our first two equations we may write these two equations

$$\lambda_{\delta-1}(2, \lambda_3, \ldots, \lambda_{\delta-1}-1, \lambda_\delta-1)-\binom{\lambda_{\delta-1}}{2}(3, \lambda_3, \ldots, \lambda_{\delta-1}-2, \lambda_\delta-1) = R,$$

$$\binom{\lambda_{\delta-1}+1}{2}(2, \lambda_3, \ldots, \lambda_{\delta-1}-1, \lambda_\delta-1)$$
$$-\binom{\lambda_{\delta-1}+1}{3}(3, \lambda_3, \ldots, \lambda_{\delta-1}-2, \lambda_\delta-1) = R.$$

These two equations are proved to be independent by calculating the determinant formed by the coefficients—its value is $\frac{1}{2}\lambda_{\delta-1}\binom{\lambda_{\delta-1}+1}{3}$.

Thus we can express

$$(2, \lambda_3, \ldots, \lambda_{\delta-1}, \lambda_\delta) \quad \text{and} \quad (3, \lambda_3, \ldots, \lambda_{\delta-1}, \lambda_\delta)$$

in terms of forms $\quad (\mu_2, \lambda_3, \ldots, \lambda_{\delta-2}, \mu_{\delta-1}, \mu_\delta),$

and of products of forms; where $\mu_2 \nless 4$ and the first of the differences

$$\mu_\delta-\lambda_\delta, \quad \mu_{\delta-1}-\lambda_{\delta-1}$$

which does not vanish is negative.

In general we shall consider the equation

$$e^{-a_2D_{r_1}-a_2D_{r_2}-\ldots-a_2D_{r_\epsilon}}(0, \lambda_3, \ldots, \lambda_\delta) = R \quad (r_1 < r_2 < \ldots < r_\epsilon)$$

before the equation

$$e^{-a_2D_{s_1}-a_2D_{s_2}-\ldots-a_2D_{s_\eta}}(0, \lambda_3, \ldots, \lambda_\delta) = R \quad (s_1 < s_2 < \ldots < s_\eta),$$

if $r_1 > s_1$.

If $r_1 = s_1$ we consider the two equations simultaneously. In fact, we have a set of $2^{\delta-r_1}$ simultaneous equations in which the first operator in the exponential index is D_{r_1}.

9. THEOREM.—*The $2^{\delta-r}$ equations*

$$e^{-a_2D_{s_1}-a_2D_{s_2}-\ldots-a_2D_{s_\eta}}(0, \lambda_3, \ldots, \lambda_\delta) = R,$$

where $s_1, s_2, \ldots, s_\eta$ are all, any or none of the numbers $r+1, r+2, \ldots, \delta$ are just sufficient to express all forms

$$(\lambda_2, \lambda_3, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_\delta),$$

for which $\lambda_2 < 2^{\delta-r}$ in terms of products of forms, and of forms

$$(\mu_2, \lambda_3, \ldots, \lambda_r, \mu_{r+1}, \ldots, \mu_\delta),$$

where $\mu_2 \geqslant 2^{\delta-r}$, *and the first of the differences*

$$\lambda_\delta-\mu_\delta,\ \lambda_{\delta-1}-\mu_{\delta-1},\ \ldots,\ \lambda_{r+1}-\mu_{r+1},$$

which does not vanish is positive.

Let us assume the theorem to be true as it stands for a particular value of r. We proceed to show then that it is true when r is changed to $r-1$.

Consider the equations

$$e^{-a_2D_r-a_2D_{s_1}-a_2D_{s_2}-\ldots-a_2D_{s_\eta}}\quad (0,\ \lambda_3,\ \ldots,\ \lambda_\delta)=R,$$

for which $s_1, s_2, \ldots, s_\eta$ are all, any or none of the numbers

$$r+1,\ r+2,\ \ldots,\ \delta.$$

The equations may be written

$$e^{-a_2D_{s_1}-a_2D_{s_2}-\ldots-a_2D_{s_\eta}}\left[e^{-a_2D_r}(0,\ \lambda_3,\ \ldots,\ \lambda_\delta)\right]=R,$$

and when they are written in this way they are identical in form with the set of equations for which we have just assumed our theorem true. Hence, on making use of the assumption, we find that

$$e^{-a_2D_r}(\lambda_2,\ \lambda_3,\ \ldots,\ \lambda_r,\ \lambda_{r+1},\ \ldots,\ \lambda_\delta)=R,$$

if $\lambda_2 < 2^{\delta-r}$; and that the symbol R here stands for products of forms and numerical multiples of forms

$$(\mu_2,\ \lambda_3,\ \ldots,\ \lambda_r,\ \mu_{r+1},\ \ldots,\ \mu_\delta),$$

where $\mu_2 \geqslant 2^{\delta-r}$ and the first of the differences

$$\lambda_\delta-\mu_\delta,\ \lambda_{\delta-1}-\mu_{\delta-1},\ldots,\ \lambda_{r+1}-\mu_{r+1}$$

which does not vanish is positive.

We thus have $2^{\delta-r}$ equations to consider of a simplified form, in which the covariants we consider differ only in the arguments λ_2 and λ_r, the general equation of the set being

$$\Sigma(-)^r\binom{\lambda_r}{\xi}(\lambda_2+\xi,\ \lambda_3,\ \ldots,\ \lambda_{r-1},\ \lambda_r-\xi,\ \lambda_{r+1},\ \ldots,\ \lambda_\delta)=R.$$

Using our assumption again we see that we have a reduction for all those terms for which $\lambda_2+\xi < 2^{\delta-r}$, and, in fact, we may suppose that these reductions are inserted, taken over to the other side of the equation, and included in the general symbol R. Taking then the first

$2^{\delta-r}$ terms of each of our equations, we have a set of $2^{\delta-r}$ linear equations to solve for the $2^{\delta-r}$ variables

$$(2^{\delta-r}+\xi,\ \lambda_3,\ \ldots,\ \lambda_{r-1},\ \lambda_r-\xi-2^{\delta-r},\ \lambda_{r+1},\ \ldots,\ \lambda_\delta) \quad (\xi = 0,\ 1,\ \ldots,\ 2^{\delta-r}-1).$$

If the determinant formed by the coefficients of these $2^{\delta-r}$ variables in the several equations is not zero, then the equations give a reduction for every one of these covariants.

The determinant in question is

$$\begin{vmatrix} \binom{\lambda_r}{2^{\delta-r}} & \binom{\lambda_r}{2^{\delta-r}+1} & \cdots & \binom{\lambda_r}{2^{\delta-r+1}-1} \\ \binom{\lambda_r-1}{2^{\delta-r}-1} & \binom{\lambda_r-1}{2^{\delta-r}} & \cdots & \binom{\lambda_r-1}{2^{\delta-r+1}-2} \\ \cdots & \cdots & \cdots & \cdots \\ \binom{\lambda_r-m}{2^{\delta-r}-m} & \binom{\lambda_r-m}{2^{\delta-r}+1-m} & \cdots & \binom{\lambda_r-m}{2^{\delta-r+1}-1-m} \\ \cdots & \cdots & \cdots & \cdots \\ \binom{\lambda_r-2^{\delta-r}+1}{1} & \binom{\lambda_r-2^{\delta-r}+1}{2} & \cdots & \binom{\lambda_r-2^{\delta-r}+1}{2^{\delta-r}} \end{vmatrix}$$

$$= \frac{\lambda_r!\,(\lambda_r-1)!\ \ldots\ (\lambda_r-2^{\delta-r}+1)!}{(\lambda_r-2^{\delta-r})!\,(\lambda_r-2^{\delta-r}-1)!\ \ldots\ (\lambda_r-2^{\delta-r+1}+1)!}$$

$$\times \frac{1!\,2!\ \ldots\ (2^{\delta-r}-1)!}{2^{\delta-r}!\,(2^{\delta-r}+1)!\ \ldots\ (2^{\delta-r+1}-1)!}$$

$$\times \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \binom{2^{\delta-r}}{1} & \binom{2^{\delta-r}+1}{1} & \cdots & \binom{2^{\delta-r+1}-1}{1} \\ \cdots & \cdots & \cdots & \cdots \\ \binom{2^{\delta-r}}{m} & \binom{2^{\delta-r}+1}{m} & \cdots & \binom{2^{\delta-r+1}-1}{m} \\ \cdots & \cdots & \cdots & \cdots \\ \binom{2^{\delta-r}}{2^{\delta-r}-1} & \binom{2^{\delta-r}+1}{2^{\delta-r}-1} & \cdots & \binom{2^{\delta-r+1}-1}{2^{\delta-r}-1} \end{vmatrix}$$

$$= \frac{\binom{\lambda_r}{2^{\delta-r}}\binom{\lambda_r}{2^{\delta-r}+1}\cdots\binom{\lambda_r}{2^{\delta-r+1}-1}}{\binom{\lambda_r}{1}\binom{\lambda_r}{2}\cdots\binom{\lambda_r}{2^{\delta-r}-1}}.$$

This is not zero unless $\lambda_r < 2^{\delta-r+1}-1$; but in this case our equations only involve $\lambda_r - 2^{\delta-r}+1$ *variables* of the form

$$(2^{\delta-r}+\xi,\ \lambda_3,\ \ldots,\ \lambda_{r-1},\ \lambda_r-\xi-2^{\delta-r},\ \lambda_{r+1},\ \ldots,\ \lambda_\delta),$$

i.e., those for which ξ has the values $0, 1, 2, \ldots, \lambda_r - 2^{\delta-r}$. (If $\lambda_r < 2^{\delta-r}$ none of these forms occur.)

To solve our equations for these, we take the first $\lambda_r - 2^{\delta-r}+1$ equations and calculate the determinant formed by the coefficients. Its value, obtained as above, is

$$\frac{\binom{\lambda_r}{2^{\delta-r}}\binom{\lambda_r}{2^{\delta-r}+1}\cdots\binom{\lambda_r}{\lambda_r}}{\binom{\lambda_r}{1}\binom{\lambda_r}{2}\cdots\binom{\lambda_r}{\lambda_r-2^{\delta-r}}}.$$

Thus in any case the solution of our equations gives

$$(2^{\delta-r}+\xi,\ \lambda_3,\ \ldots,\ \lambda_{r-1},\ \lambda_r-\xi-2^{\delta-r},\ \lambda_{r+1},\ \ldots,\ \lambda_\delta) = R,$$

when $$\xi < 2^{\delta-r} \quad \text{and} \quad \lambda_r \nless \xi+2^{\delta-r}.$$

The terms included in the symbol R are either products or forms

$$(\mu_2,\ \lambda_3,\ \ldots,\ \lambda_{r-1},\ \mu_r,\ \mu_{r+1},\ \ldots,\ \mu_\delta),$$

which occur later in our sequence than the term on the left, and for which $\mu_2 \nless 2^{\delta-r}$. By repeated application of this result to all terms on the right for which $\mu_2 < 2^{\delta-r+1}$ we find that we may restrict μ_2 to be equal to or greater than $2^{\delta-r+1}$.

Thus, if the theorem is true for any particular value of r, it is true for $r-1$; but we have seen that it is true when $r=\delta$ or $r=\delta-1$. Hence it is true in general.

In particular we deduce that the form $(\lambda_2, \lambda_3, \ldots, \lambda_\delta)$ can be expressed in terms of products of later forms in the sequence when $\lambda_2 < 2^{\delta-2}$.

10. The equations

$$e^{-a_2 D_{s_1} - a_2 D_{s_2} - \ldots - a_2 D_{s_\eta}}\,(0,\ \lambda_3,\ \ldots,\ \lambda_\delta) = R$$

result in establishing reductions which depend solely on the value of λ_2.

We have another set of equations

$$e^{-a_r D_{t_1} - a_r D_{t_2} - \ldots - a_r D_{t_\zeta}}\,(\lambda_2,\ \lambda_3,\ \ldots,\ \lambda_{r-1},\ 0,\ \lambda_{r+1},\ \ldots,\ \lambda_\delta) = R,$$

where $$t_\zeta > t_{\zeta-1} > \ldots t_2 > t_1 > r;$$

which establish reductions dependent on the value of λ_r. They give a reduction when $\lambda_r < 2^{\delta-r}$.

We shall consider all our equations in regular sequence, and those equations which affect the value of λ_r will be considered *before* those which affect the value of λ_s when $r > s$.

Thus, when we examine any form

$$(\lambda_2, \lambda_3, \ldots, \lambda_\delta),$$

we may find that it is reducible because $\lambda_r < 2^{\delta-r}$ and also because $\lambda_s < 2^{\delta-s}$. Then, if $r > s$, we shall suppose that the form is reduced by the λ_r equations; it is then necessary for a complete discussion of these forms to discover what the λ_s equations may mean. In the case of perpetuants we know from the well known facts of the subject that these λ_s equations cannot introduce any new reductions, for all reducible forms have been reduced, and that therefore they must lead to syzygies. But, so far as the present investigation has gone, it might happen that they lead to new reductions. Indeed, in the case of forms of finite order the discussion may be carried on on precisely similar lines, and then it will frequently be found that these λ_s equations lead to new reductions and not to syzygies. We have shewn (§ 7) that every possible product of perpetuants of total degree s can be expressed in the form

$$e^{-a_rD_{s_1}-a_rD_{s_2}-\ldots-a_rD_{s_\eta}}(\lambda_2, \lambda_3, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_\delta),$$

where $$r < s_1 < s_2 < \ldots < s_\eta.$$

Hence a complete discussion of our equations involves not only a complete discussion of the question of reducibility, but also of that of syzygies as well.

We shall proceed to prove the following theorem:

The equation

$$e^{-a_rD_{s_1}-a_rD_{s_2}-\ldots-a_rD_{s_\eta}}(\lambda_2, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_\delta) = R,$$

where $$r < s_1 < s_2 < \ldots < s_\eta,$$

reduces to a syzygy when $\lambda_\sigma < 2^{\delta-\sigma+1}$, *or when* $\lambda_\tau < 2^{\delta-\tau}$, *where* σ *is any one of the numbers* $s_1, s_2, \ldots, s_\eta$, *and* τ *is one of the numbers* $r+1, r+2, \ldots, \delta$, *which is not included in the set* $s_1, s_2, \ldots, s_\eta$.

11. Let us first consider the equation ($r < s$)

$$\begin{aligned} e^{-a_rD_s}&(\lambda_2, \lambda_3, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_\delta) = R \\ &= (a_r a_s)^{\lambda_s}(a_1a_2)^{\lambda_2}(a_1a_3)^{\lambda_3}\ldots(a_1a_{r-1})^{\lambda_{r-1}}(a_1a_{r+1})^{\lambda_{r+1}}\ldots \\ &\qquad (a_1a_{s-1})^{\lambda_{s-1}}(a_1a_{s+1})^{\lambda_{s+1}}\ldots(a_1a_\delta)^{\lambda} \\ &\equiv [\lambda_2, \lambda_3, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_{s-1}, \underline{\lambda_s}, \lambda_{s+1}, \ldots, \lambda_\delta], \text{ say.} \end{aligned}$$

Consider the identity

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \dots (a_1 a_{r-1})^{\lambda_{r-1}} \{(a_1 a_{r+2}) - (a_1 a_{r+1})\}^{\lambda_{r+2}} (a_1 a_{r+3})^{\lambda_{r+3}} \dots$$
$$(a_r a_s)^{\lambda_s} (a_1 a_{s+1})^{\lambda_{s+1}} \dots (a_1 a_\delta)^{\lambda_\delta}$$
$$= (a_1 a_2)^{\lambda_2} \dots (a_1 a_{r-1})^{\lambda_{r-1}} (a_{r+1} a_{r+2})^{\lambda_{r+2}} (a_1 a_{r+3})^{\lambda_{r+3}} \dots (a_1 a_{s-1})^{\lambda_{s-1}}$$
$$\times \{(a_1 a_s) - (a_1 a_r)\}^{\lambda_s} (a_1 a_{s+1})^{\lambda_{s+1}} \dots (a_1 a_\delta)^{\lambda_\delta}.$$

Expanding the braces on each side by the binomial theorem, we obtain a syzygy.

The syzygy at once gives us the relation between the equations

$$\Sigma(-)^i \binom{\lambda_{r+2}}{i} e^{-a_r D_s} (\lambda_2, \dots, \lambda_{r-1}, 0, i, \lambda_{r+2}-i, \lambda_{r+3}, \dots, \lambda_\delta)$$

$$= \Sigma(-)^j \binom{\lambda_s}{j} e^{-a_{r+1} D_{r+2}} (\lambda_2, \dots, \lambda_{r-1}, j, 0, \lambda_{r+2}, \dots, \lambda_{s-1}, \lambda_s - j, \lambda_{s+1}, \dots, \lambda_\delta).$$

Now every equation on the right-hand side is discussed before any of those on the left since $r+1 > r$. Hence this syzygy yields the relation

$$e^{-a_{r+1} D_{r+2}} [\lambda_2, \dots, \lambda_{r-1}, 0, 0, \lambda_{r+2}, \dots, \lambda_{s-1}, \underline{\lambda_s}, \lambda_{s+1}, \dots, \lambda_\delta] = R.$$

And in general when $\sigma_2 > \sigma_1 > r$, and neither σ_1 or σ_2 is equal to s, we obtain just such another syzygy which yields the relation

$$e^{-a_{\sigma_1} D_{\sigma_2}} ([\lambda_2, \dots, \lambda_{r-1}, 0, \lambda_{r+1}, \dots, \lambda_{s-1}, \underline{\lambda_s}, \lambda_{s+1}, \dots, \lambda_\delta]_{\lambda_{\sigma_1}=0}) = R.$$

The result may be at once extended to a slightly more general syzygy to which the relation

$$e^{-a_\sigma D_{\sigma_1} - a_\sigma D_{\sigma_2} - \dots - a_\sigma D_{\sigma_\kappa}} ([\lambda_2, \dots, \lambda_{r-1}, 0, \lambda_{r+1}, \dots, \underline{\lambda_s}, \dots, \lambda_\delta]_{\lambda_\sigma=0}) = R$$

(where $r < \sigma < \sigma_1 < \dots < \sigma_\kappa$, and none of the σ's which here appear is equal to s) corresponds.

Let us call these syzygies the perpetuant syzygies of the type A.

12. Consider the identity $(r < s)$

$$(a_1 a_2)^{\lambda_2} \dots (a_1 a_{r-1})^{\lambda_{r-1}} \{(a_r a_s) - (a_1 a_{r+1})\}^{\lambda_s} (a_1 a_{r+2})^{\lambda_{r+2}} \dots$$
$$(a_1 a_{s-1})^{\lambda_{s-1}} (a_1 a_{s+1})^{\lambda_{s+1}} \dots (a_1 a_\delta)^{\lambda_\delta}$$
$$= (a_1 a_2)^{\lambda_2} \dots (a_1 a_{r-1})^{\lambda_{r-1}} \{(a_{r+1} a_s) - (a_1 a_r)\}^{\lambda_s} (a_1 a_{r+2})^{\lambda_{r+2}} \dots$$
$$(a_1 a_{s-1})^{\lambda_{s-1}} (a_1 a_{s+1})^{\lambda_{s+1}} \dots (a_1 a_\delta)^{\lambda_\delta}.$$

Expanding the braces on both sides we obtain a Stroh syzygy, and this at once gives the relation between our equations

$$\Sigma(-)^i \binom{\lambda_s}{i} e^{-a_r D_s} (\lambda_2, \ldots, \lambda_{r-1}, 0, i, \lambda_{r+2}, \ldots, \lambda_{s-1}, \lambda_s - i, \lambda_{s+1}, \ldots, \lambda_\delta)$$

$$= \Sigma(-)^i \binom{\lambda_s}{i} e^{-a_{r+1} D_s} (\lambda_2, \ldots, \lambda_{r-1}, i, 0, \lambda_{r+2}, \ldots, \lambda_{s-1}, \lambda_s - i, \lambda_{s+1}, \ldots, \lambda_\delta).$$

Every equation represented on the right is considered before any of those on the left of this relation: hence we may write it

$$\Sigma(-)^i \binom{\lambda_s}{i} [\lambda_2, \ldots, \lambda_{r-1}, 0, i, \lambda_{r+2}, \ldots, \lambda_{s-1}, \underline{\lambda_s - i}, \lambda_{s+1}, \ldots, \lambda_\delta] = R.$$

And although a slightly different meaning must be attached to the operator, we may, without fear of ambiguity, write this equation

$$e^{-a_{r+1} D_s} [\lambda_2, \ldots, \lambda_{r-1}, 0, 0, \lambda_{r+2}, \ldots, \lambda_{s-1}, \underline{\lambda_s}, \lambda_{s+1}, \ldots, \lambda_\delta] = R.$$

In the same way we obtain, whenever $s > \sigma$,

$$e^{-a_\sigma D_s} ([\lambda_2, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_{s-1}, \underline{\lambda_s}, \lambda_{s+1}, \ldots, \lambda_\delta]_{\lambda_\sigma = 0}) = R,$$

and whenever $\sigma > s$,

$$e^{-a_s D_\sigma} [\lambda_2, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_{s-1}, \underline{0}, \lambda_{s+1}, \ldots, \lambda_\delta] = R.$$

That is, we obtain syzygies which yield these relations.

Now combining one of these syzygies with one of those of the last paragraph, we have a syzygy expressed by

$$(a_1 a_2)^{\lambda_2} \ldots (a_1 a_{r-1})^{\lambda_{r-1}} \{(a_1 a_{r+2}) - (a_1 a_{r+1})\}^{\lambda_{r+2}} \{(a_r a_s) - (a_1 a_{r+1})\}^{\lambda_s}$$
$$(a_1 a_{r+3})^{\lambda_{r+3}} \ldots (a_1 a_{s-1})^{\lambda_{s-1}} (a_1 a_{s+1})^{\lambda_{s+1}} \ldots (a_1 a_\delta)^{\lambda_\delta}$$
$$= (a_1 a_2)^{\lambda_2} \ldots (a_1 a_{r-1})^{\lambda_{r-1}} (a_{r+1} a_{r+2})^{\lambda_{r+2}} \{(a_{r+1} a_s) - (a_1 a_r)\}^{\lambda_s} (a_1 a_{r+3})^{\lambda_{r+3}} \ldots$$
$$(a_1 a_{s-1})^{\lambda_{s-1}} (a_1 a_{s+1})^{\lambda_{s+1}} \ldots (a_1 a_\delta)^{\lambda_\delta},$$

which yields a relation

$$e^{-a_{r+1} D_{r+2} - a_{r+1} D_s} [\lambda_2, \ldots, \lambda_{r-1}, 0, \lambda_{r+2}, \ldots, \lambda_{s-1}, \underline{\lambda_s}, \lambda_{s+1}, \ldots, \lambda_\delta] = R.$$

In this way we obtain syzygies to give each of the relations

$$e^{-a_\sigma D_{\sigma_1} - a_\sigma D_{\sigma_2} - \ldots - a_\sigma D_{\sigma_\kappa}} ([\lambda_2, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_{s-1}, \underline{\lambda_s}, \lambda_{s+1}, \ldots, \lambda_\delta]_{\lambda_\sigma = 0}) = R,$$

when $\quad r < \sigma < \sigma_1 < \sigma_2 < \ldots < \sigma_\kappa \quad$ and $\quad \sigma \neq s.$

These relations have already been fully discussed in § 9, when dis-

cussing the question of reducibility: we obtain from them at once the result

$$[\lambda_2, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_{s-1}, \underline{\lambda_s}, \lambda_{s+1}, \ldots, \lambda_\delta] = R,$$

when $\lambda_\sigma < 2^{\delta-\sigma}$, where $r < \sigma \neq s$.

We will call the syzygies of this paragraph the perpetuant syzygies of the type B.

13. In obtaining the limitations to the value of λ_s, and the corresponding syzygies, for the equation

$$e^{-a_r D_s}(\lambda_2, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_\delta) = R.$$

We shall simplify the work and not lose anything in generality if we suppose $r = 2$ and $s = 3$. Thus we consider

$$e^{-a_2 D_3}(0, \lambda_3, \lambda_4, \ldots, \lambda_\delta) \equiv [0, \underline{\lambda_3}, \lambda_4, \ldots, \lambda_\delta] = R.$$

If $\lambda_3 = 0$, our equation becomes

$$(0, 0, \lambda_4, \ldots, \lambda_\delta) = a_{2_x}^\infty \left(\frac{a_3^0 a_4^{\lambda_4} \ldots a_\delta^{\lambda_\delta}}{a_1}\right);$$

but we already know from a previous equation that

$$(0, 0, \lambda_4, \ldots, \lambda_\delta) = a_{3_x}^\infty \left(\frac{a_2^0 a_4^{\lambda_4} \ldots a_\delta^{\lambda_\delta}}{a_1}\right).$$

Thus the equation simply gives the obvious syzygy

$$a_{2_x}^\infty \left(\frac{a_3^0 a_4^{\lambda_4} \ldots a_\delta^{\lambda_\delta}}{a_1}\right) = a_{3_x}^\infty \left(\frac{a_3^0 a_4^{\lambda_4} \ldots a_\delta^{\lambda_\delta}}{a_1}\right).$$

If $\lambda_3 = 1$, our equation becomes

$$(0, 1, \lambda_4, \ldots, \lambda_\delta) - (1, 0, \lambda_4, \ldots, \lambda_\delta)$$
$$= (a_2 a_3)\left(\frac{a_4^{\lambda_4} \ldots a_\delta^{\lambda_\delta}}{a_1}\right) = a_{2_x}^\infty \left(\frac{a_3^1 a_4^{\lambda_4} \ldots a_\delta^{\lambda_\delta}}{a_1}\right) - a_{3_x}^\infty \left(\frac{a_2^1 a_4^{\lambda_4} \ldots a_\delta^{\lambda_\delta}}{a_1}\right);$$

giving again a syzygy. This syzygy is the Jacobian syzygy.

Consider the two identities

$$\{(a_1 a_4) - (a_2 a_3)\}^{\lambda_4} (a_1 a_5)^{\lambda_5} \ldots (a_1 a_\delta)^{\lambda_\delta} = \{(a_1 a_2) + (a_3 a_4)\}^{\lambda_4} (a_1 a_5)^{\lambda_5} \ldots (a_1 a_\delta)^{\lambda_\delta},$$

and

$$\{(a_1 a_4) + (a_2 a_3)\}^{\lambda_4} (a_1 a_5)^{\lambda_5} \ldots (a_1 a_\delta)^{\lambda_\delta} = \{(a_1 a_3) + (a_2 a_4)\}^{\lambda_4} (a_1 a_5)^{\lambda_5} \ldots (a_1 a_\delta)^{\lambda_\delta},$$

when these are expanded they yield Stroh syzygies. These syzygies give us the relations

$$e^{-a_3 D_4}[0, \underline{0}, \lambda_4, \ldots, \lambda_\delta] = R,$$

and
$$e^{+a_3 D_4}[0, \underline{0}, \lambda_4, \ldots, \lambda_\delta] = R.$$

And in general we find in this way syzygies which give the relations

$$e^{-a_3 D_\sigma}[0, \underline{0}, \lambda_4, \ldots, \lambda_\delta] = R,$$

and
$$e^{+a_3 D_\sigma}[0, \underline{0}, \lambda_4, \ldots, \lambda_\delta] = R,$$

for
$$\sigma = 4, 5, \ldots, \delta.$$

Further, from the syzygies

$$\{(a_1 a_4)-(a_2 a_3)\}^{\lambda_4}\{(a_1 a_5)-(a_2 a_3)\}^{\lambda_5}(a_1 a_6)^{\lambda_6}\ldots(a_1 a_\delta)^{\lambda_\delta}$$
$$= \{(a_1 a_2)+(a_3 a_4)\}^{\lambda_4}\{(a_1 a_2)+(a_3 a_5)\}^{\lambda_5}(a_1 a_6)^{\lambda_6}\ldots(a_1 a_\delta)^{\lambda_\delta},$$

and $\{(a_1 a_4)+(a_2 a_3)\}^{\lambda_4}\{(a_1 a_5)+(a_2 a_3)\}^{\lambda_5}(a_1 a_6)^{\lambda_6}\ldots(a_1 a_\delta)^{\lambda_\delta}$

$$= \{(a_1 a_3)+(a_2 a_4)\}^{\lambda_4}\{(a_1 a_3)+(a_2 a_5)\}^{\lambda_5}(a_1 a_6)^{\lambda_6}\ldots(a_1 a_\delta)^{\lambda_\delta},$$

we obtain the relations

$$e^{\pm(a_3 D_4 + a_3 D_5)}[0, \underline{0}, \lambda_4, \ldots, \lambda_\delta] = R.$$

Proceeding thus we can write down a set of syzygies which give us the relations
$$e^{\pm(a_3 D_{s_1} + a_3 D_{s_2} + \ldots + a_3 D_{s_\eta})}[0, \underline{0}, \lambda_4, \ldots, \lambda_\delta] = R,$$

where $s_1, s_2, \ldots, s_\eta$ are all, any or none of $4, 5, \ldots, \delta$.

These syzygies we shall refer to as the perpetuant syzygies of the type C.

14. It is necessary to discuss the equations just found.

We shall arrange them in a sequence as we have done the other equations:

Thus the equations

$$e^{\pm(a_3 D_{r_1} + a_3 D_{r_2} + \ldots + a_3 D_{r_\epsilon})}[0, \underline{0}, \lambda_4, \ldots, \lambda_\delta] = R \quad (r_1 < r_2 < \ldots < r_\epsilon),$$

will be discussed before the equations

$$e^{\pm(a_3 D_{s_1} + a_3 D_{s_2} + \ldots + a_3 D_{s_\eta})}[0, \underline{0}, \lambda_4, \ldots, \lambda_\delta] = R \quad (s_1 < s_2 < \ldots < s_\eta),$$

when
$$r_1 > s_1.$$

But, if $r_1 = s_1$, the equations are discussed simultaneously.

Thus the first pair of equations to be discussed is

$$e^{\pm a_3 D_\delta}[0, \underline{0}, \lambda_4, \ldots, \lambda_\delta] = R.$$

Whence $[0, \underline{0}, \lambda_4, \ldots, \lambda_{\delta-1}, \lambda_\delta] \pm \lambda_\delta [0, \underline{1}, \lambda_4, \ldots, \lambda_{\delta-1}, \lambda_\delta - 1]$

$$+ \binom{\lambda_\delta}{2}[0, \underline{2}, \lambda_4, \ldots, \lambda_{\delta-1}, \lambda_\delta - 2] \pm \binom{\lambda_\delta}{3}[0, \underline{3}, \lambda_4, \ldots, \lambda_{\delta-1}, \lambda_\delta - 3] + \ldots = R,$$

giving immediate reductions for

$$[0, \underline{2}, \lambda_4, \ldots, \lambda_{\delta-1}, \lambda_\delta - 2],$$

and

$$[0, \underline{3}, \lambda_4, \ldots, \lambda_{\delta-1}, \lambda_\delta - 3].$$

The forms $[0, \underline{\lambda_3}, \lambda_4, \ldots, \lambda_\delta]$ being arranged in sequence according to the same rules as the forms $(\lambda_2, \lambda_3, \ldots, \lambda_\delta)$.

15. LEMMA.—*The* $2^{\delta-r+1}$ *equations*

$$e^{\pm(a_3 D_{s_1} + a_3 D_{s_2} + \ldots + a_3 D_{s_\eta})}[0, \underline{0}, \lambda_4, \ldots, \lambda_\delta] = 0,$$

where $s_1, s_2, \ldots, s_\eta$ *are all, any or none of the numbers* $r+1, r+2, \ldots, \delta$ *are just sufficient to express all forms*

$$[0, \underline{\lambda_3}, \lambda_4, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_\delta],$$

for which $\lambda_3 < 2^{\delta-r+1}$, *in terms of forms*

$$[0, \underline{\mu_3}, \lambda_4, \ldots, \lambda_r, \mu_{r+1}, \ldots, \mu_\delta],$$

where $\mu_3 \nless 2^{\delta-r+1}$ *and the first of the differences*

$$\lambda_\delta - \mu_\delta, \lambda_{\delta-1} - \mu_{\delta-1}, \ldots, \lambda_{r+1} - \mu_{r+1},$$

which does not vanish is positive.

Let us assume the truth of this proposition for a particular value of r, and then consider the $2^{\delta-r+1}$ equations

$$e^{\pm(a_3 D_r + a_3 D_{s_1} + \ldots + a_3 D_{s_\eta})}[0, \underline{0}, \lambda_4, \ldots, \lambda_\delta] = 0,$$

where $s_1, s_2, \ldots, s_\eta$ are all, any or none of the numbers $r+1, r+2, \ldots, \delta$.

Let us write

$$e^{-a_3 D_r}[0, \underline{0}, \lambda_4, \ldots, \lambda_{r-1}, \lambda_r, \lambda_{r+1}, \ldots, \lambda_\delta] \equiv [0, \underline{0}, \lambda_4, \ldots, \lambda_{r-1}, \underline{\lambda_r}, \lambda_{r+1}, \ldots, \lambda_\delta],$$

and

$$e^{+a_3D_r}[0, \underline{0}, \lambda_4, \ldots, \lambda_{r-1}, \lambda_r, \lambda_{r+1}, \ldots, \lambda_\delta] \equiv [0, \underline{0}, \lambda_4, \ldots, \lambda_{r-1}, \underline{\lambda_r'}, \lambda_{r+1}, \ldots, \lambda_\delta].$$

Then we have two sets of equations

$$e^{-a_3D_{s_1}-a_3D_{s_2}-\ldots-a_3D_{s_\eta}}[0, \underline{0}, \lambda_4, \ldots, \lambda_{r-1}, \underline{\lambda_r}, \lambda_{r+1}, \ldots, \lambda_\delta] = 0,$$

and $$e^{+a_3D_{s_1}+a_3D_{s_2}+\ldots+a_3D_{s_\eta}}[0, \underline{0}, \lambda_4, \ldots, \lambda_{r-1}, \underline{\lambda_r'}, \lambda_{r+1}, \ldots, \lambda_\delta] = 0.$$

From the theorem of § 9 we know that the solution of these equations expresses all forms ($\lambda_3 < 2^{\delta-r}$)

$$[0, \underline{\lambda_3}, \lambda_4, \ldots, \lambda_{r-1}, \underline{\lambda_r}, \lambda_{r+1}, \ldots, \lambda_\delta],$$

in terms of forms $[0, \underline{\mu_3}, \lambda_4, \ldots, \lambda_{r-1}, \underline{\lambda_r}, \mu_{r+1}, \ldots, \mu_\delta]$;

and all forms ($\lambda_3 < 2^{\delta-r}$)

$$[0, \underline{\lambda_3}, \lambda_4, \ldots, \lambda_{r-1}, \underline{\lambda_r'}, \lambda_{r+1}, \ldots, \lambda_\delta],$$

in terms of forms $[0, \underline{\mu_3}, \lambda_4, \ldots, \lambda_{r-1}, \underline{\lambda_r'}, \mu_{r+1}, \ldots, \mu_\delta]$;

where in both cases $\mu_3 \nless 2^{\delta-r}$, and the first of the differences

$$\lambda_\delta-\mu_\delta, \lambda_{\delta-1}-\mu_{\delta-1}, \ldots, \lambda_{r+1}-\mu_{r+1},$$

which does not vanish is positive.

We thus obtain two sets of equations

$$e^{\pm a_3D_r}[0, \underline{\lambda_3}, \lambda_4, \ldots, \lambda_{r-1}, \lambda_r, \ldots, \lambda_\delta] = R,$$

where $$\lambda_3 = 0, 1, \ldots, 2^{\delta-r}-1.$$

Expanding them out, we have

$$\Sigma(-)^i \binom{\lambda_r}{i}[0, \underline{\lambda_3+i}, \lambda_4, \ldots, \lambda_{r-1}, \lambda_r-i, \lambda_{r+1}, \ldots, \lambda_\delta] = R,$$

and $$\Sigma \binom{\lambda_r}{i}[0, \underline{\lambda_3+i}, \lambda_4, \ldots, \lambda_{r-1}, \lambda_r-i, \lambda_{r+1}, \ldots, \lambda_\delta] = R.$$

Now using the assumption made we see that these equations may be regarded as equations to give the values of

$$[0, \underline{2^{\delta-r+1}+\xi}, \lambda_4, \ldots, \lambda_{r-1}, \lambda_r-2^{\delta-r+1}-\xi, \lambda_{r+1}, \ldots, \lambda_\delta] = R,$$

$$\xi = 0, 1, 2, \ldots, 2^{\delta-r+1}-1.$$

Adding and subtracting our equations in pairs, we obtain two new sets;

one of which connects those forms for which ξ is even, and the other those forms for which ξ is odd.

They may be written

$$\cosh a_3 D_r [0, \underline{\lambda_3}, \lambda_4, \ldots, \lambda_\delta] = R,$$

and

$$\sinh a_3 D_r [0, \underline{\lambda_3}, \lambda_4, \ldots, \lambda_\delta] = R.$$

We desire to prove the linear independence of each set.

For this purpose we must calculate the determinants formed by the coefficients. In the first case the determinant is

$$\begin{vmatrix} \binom{\lambda_r}{2^{\delta-r+1}} & \binom{\lambda_r}{2^{\delta-r+1}+2} & \cdots & \binom{\lambda_r}{2^{\delta-r+1}+2\sigma} & \cdots & \binom{\lambda_r}{2^{\delta-r+2}-2} \\ \binom{\lambda_r-1}{2^{\delta-r+1}-1} & \binom{\lambda_r-1}{2^{\delta-r+1}+1} & \cdots & \binom{\lambda_r-1}{2^{\delta-r+1}+2\sigma-1} & \cdots & \binom{\lambda_r-1}{2^{\delta-r+2}-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \binom{\lambda_r-\tau}{2^{\delta-r+1}-\tau} & \binom{\lambda_r-\tau}{2^{\delta-r+1}+2-\tau} & \cdots & \binom{\lambda_r-\tau}{2^{\delta-r+1}+2\sigma-\tau} & \cdots & \binom{\lambda_r-1}{2^{\delta-r+2}-2-\tau} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \binom{\lambda_r-2^{\delta-r}+1}{2^{\delta-r}+1} & \binom{\lambda_r-2^{\delta-r}+1}{2^{\delta-r}+3} & \cdots & \binom{\lambda_r-2^{\delta-r}+1}{2^{\delta-r}+1+2\sigma} & \cdots & \binom{\lambda_r-2^{\delta-r}+1}{2^{\delta-r+2}-1-2^{\delta-r}} \end{vmatrix}$$

$$= \frac{\binom{\lambda_r}{2^{\delta-r+1}}\binom{\lambda_r}{2^{\delta-r+1}+2}\cdots\binom{\lambda_r}{2^{\delta-r+2}-2}}{\binom{\lambda_r}{1}\binom{\lambda_r}{2}\cdots\binom{\lambda_r}{2^{\delta-r}-1}}\,\Delta.$$

Where, on changing columns into rows and rows into columns,

$$\Delta = \begin{vmatrix} 1 & \binom{2^{\delta-r+1}}{1} & \binom{2^{\delta-r+1}}{2} & \cdots & \binom{2^{\delta-r+1}}{2^{\delta-r}-1} \\ 1 & \binom{2^{\delta-r+1}+2}{1} & \binom{2^{\delta-r+1}+2}{2} & \cdots & \binom{2^{\delta-r+1}+2}{2^{\delta-r}-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \binom{2^{\delta-r+1}+2\sigma}{1} & \binom{2^{\delta-r+1}+2\sigma}{2} & \cdots & \binom{2^{\delta-r+1}+2\sigma}{2^{\delta-r}-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \binom{2^{\delta-r+2}-2}{1} & \binom{2^{\delta-r+2}-2}{2} & \cdots & \binom{2^{\delta-r+2}-2}{2^{\delta-r}-1} \end{vmatrix}.$$

We shall now consider the more general determinant

$$\Delta_k = \begin{vmatrix} 1 & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{k-1} \\ 1 & \binom{n+2}{1} & \binom{n+2}{2} & \cdots & \binom{n+2}{k-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \binom{n+2k-2}{1} & \binom{n+2k-2}{2} & \cdots & \binom{n+2k-2}{k-1} \end{vmatrix}.$$

Subtract each row from that immediately below it, then the $(\sigma+1)$-th row becomes

$$0,\ \binom{n+2\sigma-1}{0}+\binom{n+2\sigma-2}{0},\ \ldots,\ \binom{n+2\sigma-1}{\tau-1}+\binom{n+2\sigma-2}{\tau-1},\ \ldots,$$
$$\binom{n+2\sigma-1}{k-2}+\binom{n+2\sigma-2}{k-2};$$

since
$$\binom{n+2\sigma}{\tau} = \binom{n+2\sigma-1}{\tau}+\binom{n+2\sigma-1}{\tau-1}$$
$$= \binom{n+2\sigma-2}{\tau}+\binom{n+2\sigma-2}{\tau-1}+\binom{n+2\sigma-1}{\tau-1}.$$

Next we repeat the process of subtracting each row from the next below, leaving the first two rows unaltered. The $(\tau+1)$-th element of the $(\sigma+1)$-th row becomes now

$$\binom{n+2\sigma-2}{\tau-2}+2\binom{n+2\sigma-3}{\tau-2}+\binom{n+2\sigma-4}{\tau-2}.$$

We keep on repeating the process, each time leaving one more row unchanged. After t subtractions the $(\tau+1)$-th element of the $(\sigma+1)$-row becomes

$$\binom{n+2\sigma-t}{\tau-t}+\binom{t}{1}\binom{n+2\sigma-t-1}{\tau-t}+\binom{t}{2}\binom{n+2\sigma-t-2}{\tau-t}+\ldots$$
$$+\binom{t}{t}\binom{n+2\sigma-2t}{\tau-t}.$$

This $(\sigma+1)$-th row is not left unchanged until $t=\sigma$, and so its final form will be obtained by giving t the value σ. The $(\tau+1)$-th element is then

zero when $\tau < \sigma$, and its value when $\tau = \sigma$ is

$$1+\binom{\sigma}{1}+\binom{\sigma}{2}+\ldots+\binom{\sigma}{\sigma} = 2^{\sigma}.$$

Thus we eventually transform Δ_k into a determinant in which every element below the leading diagonal is zero, and where the elements of this diagonal are

$$1,\ 2,\ 2^2,\ \ldots,\ 2^{k-1}.$$

Hence

$$\Delta_k = 2^{0+1+2+\ldots+k-1} = 2^{\binom{k}{2}}.$$

Hence the determinant formed by the coefficients of our equations which we wished to calculate

$$= \frac{\binom{\lambda_r}{2^{\delta-r+1}}\binom{\lambda_r}{2^{\delta-r+1}+2}\cdots\binom{\lambda_r}{2^{\delta-r+2}-2}}{\binom{\lambda_r}{1}\binom{\lambda_r}{2}\cdots\binom{\lambda_r}{2^{\delta-r}-1}}\, 2^{\binom{2^{\delta-r}}{2}}.$$

The determinant of the coefficients of the other set of equations can, in a similar manner, be shewn to be

$$\frac{\binom{\lambda_r}{2^{\delta-r+1}+1}\binom{\lambda_r}{2^{\delta-r+1}+3}\cdots\binom{\lambda_r}{2^{\delta-r+2}-1}}{\binom{\lambda_r}{1}\binom{\lambda_r}{2}\cdots\binom{\lambda_r}{2^{\delta-r}-1}}\, 2^{\binom{2^{\delta-r}}{2}}.$$

Thus we obtain

$$[0,\ \underline{2^{\delta-r+1}+\xi},\ \lambda_4,\ \ldots,\ \lambda_{r-1},\ \lambda_r-2^{\delta-r+1}-\xi,\ \lambda_{r+1},\ \ldots,\ \lambda_\delta] = R,$$

for all values $\xi = 0, 1, 2, \ldots, 2^{\delta-r+1}-1$; provided $\lambda_r \not< 2^{\delta-r+2}-1$. If λ_r is less than this value, we can remove some of our equations, for there are fewer forms to solve for. The determinants, when we take the same number of equations (starting from the beginning), as there are forms, can easily be calculated, and are found not to be zero.

Hence the equations give

$$[0,\ \underline{\lambda_3},\ \lambda_4,\ \ldots,\ \lambda_{r-1},\ \lambda_r,\ \ldots,\ \lambda_\delta] = R,$$

provided

$$\lambda_3 < 2^{\delta-r+2}.$$

Where R consists of forms

$$[0,\ \underline{\mu_3},\ \lambda_4,\ \ldots\ \lambda_{r-1},\ \mu_r,\ \ldots,\ \mu_\delta],$$

for which $\mu_3 \nless 2^{\delta-r+1}$, and where the first of the differences

$$\lambda_\delta - \mu_\delta,\ \lambda_{\delta-1} - \mu_{\delta-1},\ \ldots,\ \lambda_r - \mu_r,$$

which does not vanish is positive.

We may apply this result again to all forms on the right-hand side for which $\mu_3 < 2^{\delta-r+2}$, and thus ultimately we obtain the condition $\mu_3 < 2^{\delta-r+2}$. Thus, if the lemma is true for a particular value of r, it is true when we replace r by $r-1$. Now it is true when $r = \delta - 1$; hence it is always true. Thus the truth of the lemma is established.

16. We may now apply the lemma to the equations of § 14. We find at once that the syzygies obtained in § 13 are sufficient to express the equation ($\lambda_3 < 2^{\delta-2}$),

$$e^{-a_2 D_3}(0, \lambda_3, \lambda_4, \ldots, \lambda_\delta) = R$$

in terms of equations already considered and of equations

$$e^{-a_2 D_3}(0, \mu_3, \mu_4, \ldots, \mu_\delta) = R,$$

where $\mu_3 \nless 2^{\delta-2}$, and the first of the differences

$$\lambda_\delta - \mu_\delta,\ \lambda_{\delta-1} - \mu_{\delta-1},\ \ldots,\ \lambda_4 - \mu_4,$$

which does not vanish is positive.

The equation $\qquad e^{-a_2 D_3}(0, \lambda_3, \lambda_4, \ldots, \lambda_\delta) = R,$

then may be said to yield a syzygy when

$$\lambda_3 < 2^{\delta-2}, \text{ or } \lambda_4 < 2^{\delta-4}, \text{ or } \lambda_5 < 2^{\delta-5}, \ \ldots, \text{ or } \lambda_\delta < 1.$$

Thus the theorem enunciated in § 10 is true for the equation

$$e^{-a_2 D_3}(0, \lambda_3, \lambda_4, \ldots, \lambda_\delta) = R.$$

And in just the same way it can be established for

$$e^{-a_r D_s}(\lambda_2, \lambda_3, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_\delta) = R \quad (s > r).$$

17. Let us now consider the equation

$$e^{-a_2 D_{s_1} - a_2 D_{s_2} - \ldots - a_2 D_{s_\eta}}(0, \lambda_3, \ldots, \lambda_\delta) \equiv [0, \lambda_3, \ldots, \lambda_\delta] = R$$
$$(s_1 < s_2 < \ldots < s_\eta).$$

By means of the perpetuant syzygies of the types A and B, discussed in §§ 11, 12, we obtained relations by which we can reduce our equation

when $\lambda_\sigma < 2^{\delta-\sigma}$, where σ is any one of the numbers 3, 4, ..., δ which is not included among the numbers $s_1, s_2, \ldots, s_\eta$.

We may then confine our attention to those syzygies which will give limitations to the value of λ_σ when σ is one of the numbers $s_1, s_2, \ldots, s_\eta$.

To fix our ideas let us put $s_1 = 3$, and consider the syzygies which will affect λ_3.

We obtain first certain syzygies of the type C,

$$\{(a_1 a_{r_1}) - (a_2 a_3)\}^{\lambda_{r_1}} (a_2 a_{s_2})^{\lambda_{s_2}} (a_2 a_{s_3})^{\lambda_{s_3}} \ldots (a_2 a_{s_\eta})^{\lambda_{s_\eta}} (a_1 a_{r_2})^{\lambda_{r_2}} \ldots$$

$$= \{(a_3 a_{r_1}) + (a_1 a_2)\}^{\lambda_{r_1}} \{(a_1 a_{s_2}) - (a_1 a_2)\}^{\lambda_{s_2}} \ldots \{(a_1 a_{s_\eta}) - (a_1 a_2)\}^{\lambda_{s_\eta}} (a_1 a_{r_2})^{\lambda_{r_2}} \ldots,$$

where $r_1, r_2, \ldots$ are just those of the numbers 4, 5, ..., δ, which are not included in the set $s_2, s_3, \ldots, s_\eta$.

Similarly we have

$$\{(a_1 a_{r_1}) - (a_2 a_3)\}^{\lambda_{r_1}} \{(a_1 a_{r_2}) - (a_2 a_3)\}^{\lambda_{r_2}} (a_2 a_{s_2})^{\lambda_{s_2}} \ldots (a_2 a_{s_\eta})^{\lambda_{s_\eta}} (a_1 a_{r_3})^{\lambda_{r_3}} \ldots$$

$$= \{(a_3 a_{r_1}) + (a_1 a_2)\}^{\lambda_{r_1}} \{(a_3 a_{r_2}) + (a_1 a_2)\}^{\lambda_{r_2}} \{(a_1 a_{s_2}) - (a_1 a_2)\}^{\lambda_{s_2}} \ldots$$

$$\{(a_1 a_{s_\eta}) - (a_1 a_2)\}^{\lambda_{s_\eta}} (a_1 a_{r_3})^{\lambda_{r_3}} \ldots .$$

Then we have a set of syzygies we will call syzygies of the type D: such are

$$\{(a_2 a_{s_2}) - (a_2 a_3)\}^{\lambda_{s_2}} (a_2 a_{s_3})^{\lambda_{s_3}} \ldots (a_2 a_{s_\eta})^{\lambda_{s_\eta}} (a_1 a_{r_1})^{\lambda_{r_1}} (a_1 a_{r_2})^{\lambda_{r_2}} \ldots$$

$$= (a_3 a_{s_2})^{\lambda_{s_2}} \{(a_1 a_{s_3}) - (a_1 a_2)\}^{\lambda_{s_3}} \ldots \{(a_1 a_{s_\eta}) - (a_1 a_2)\}^{\lambda_{s_\eta}} (a_1 a_{r_1})^{\lambda_{r_1}} (a_1 a_{r_2})^{\lambda_{r_2}} \ldots .$$

We obtain fresh syzygies by replacing any term $(a_2 a_s)$ on the left by $\{(a_2 a_s) - (a_2 a_3)\}$, and making the corresponding change on the right of $\{(a_1 a_s) - (a_1 a_2)\}$ into $(a_3 a_s)$. Or we may change on the left $(a_1 a_r)$ into $\{(a_1 a_r) - (a_2 a_3)\}$, and at the same time on the right $(a_1 a_r)$ into $\{(a_3 a_r) + (a_1 a_2)\}$.

In this way we obtain a set of syzygies which will give us the $2^{\delta-3}$ relations between our equations

$$e^{-a_3 D_{\sigma_1} - a_3 D_{\sigma_2} - \ldots - a_3 D_{\sigma_k}} [0, 0, \lambda_4, \ldots, \lambda_\delta] = R,$$

where $\sigma_1, \sigma_2, \ldots, \sigma_k$ are any, all or none of the numbers 4, 5, ..., δ.

Again, we have the syzygies of the type C,

$$\{(a_1 a_{r_1}) + (a_2 a_3)\}^{\lambda_{r_1}} \{(a_1 a_{r_2}) + (a_2 a_3)\}^{\lambda_{r_2}} (a_2 a_{s_2})^{\lambda_{s_2}} \ldots (a_2 a_{s_\eta})^{\lambda_{s_\eta}} (a_1 a_{r_3})^{\lambda_{r_3}} \ldots$$

$$= \{(a_2 a_{r_1}) + (a_1 a_3)\}^{\lambda_{r_1}} \{(a_2 a_{r_2}) + (a_1 a_3)\}^{\lambda_{r_2}} (a_2 a_{s_2})^{\lambda_{s_2}} \ldots (a_2 a_{s_\eta})^{\lambda_{s_\eta}} (a_1 a_{r_3})^{\lambda_{r_3}} \ldots,$$

for example. This will give the relation

$$e^{+a_3 D_{r_1} + a_3 D_{r_2}} [0, 0, \lambda_4, \ldots, \lambda_\delta] = R.$$

And so we obtain syzygies which yield

$$e^{a_3 D_{\sigma_1} + a_3 D_{\sigma_2} + \ldots + a_3 D_{\sigma_k}} [0, 0, \lambda_4, \ldots, \lambda_\delta] = R,$$

where $\sigma_1, \sigma_2, \ldots, \sigma_k$ are all or any of the numbers $r_1, r_2, r_3, \ldots$.

We have then certain syzygies which we shall include in the type D; an example of these is

$$\{(a_1 a_{r_1}) + (a_2 a_3)\}^{\lambda_{r_1}} \{(a_2 a_{s_2}) - (a_2 a_3)\}^{\lambda_{s_2}} (a_2 a_{s_3})^{\lambda_{s_3}} \ldots$$
$$(a_2 a_{s_\eta})^{\lambda_{s_\eta}} (a_1 a_{r_2})^{\lambda_{r_2}} (a_1 a_{r_3})^{\lambda_{r_3}} \ldots$$
$$= \{(a_2 a_{r_1}) + (a_1 a_3)\}^{\lambda_{r_1}} \{(a_1 a_{s_2}) - (a_1 a_3)\}^{\lambda_{s_2}} (a_2 a_{s_3})^{\lambda_{s_3}} \ldots$$
$$(a_2 a_{s_\eta})^{\lambda_{s_\eta}} (a_1 a_{r_2})^{\lambda_{r_2}} (a_1 a_{r_3})^{\lambda_{r_3}} \ldots .$$

This particular syzygy yields

$$e^{a_3 D_{r_1} - a_3 D_{s_2}} [0, 0, \lambda_4, \ldots, \lambda_\delta] = R.$$

The syzygies of which this is an example yield the set of relations

$$e^{a_3 D_{\rho_1} + a_3 D_{\rho_2} + \ldots + a_3 D_{\rho_k} - a_3 D_{\sigma_1} - a_3 D_{\sigma_2} - \ldots - a_3 D_{\sigma_k}} [0, 0, \lambda_4, \ldots, \lambda_\delta] = R,$$

where $\rho_1, \rho_2, \ldots, \rho_k$ are any of $r_1, r_2, r_3, \ldots$, and $\sigma_1, \sigma_2, \ldots, \sigma_k$ are any of $s_1, s_2, \ldots, s_\eta$.

Lastly, we have a set of syzygies we shall call syzygies of the type E. They are really forms of the Jacobian syzygy, an example of these is

$$(a_2 a_3) \{(a_2 a_{s_2}) - (a_2 a_3)\}^{\lambda_{s_2}} \{(a_2 a_{s_3}) - (a_2 a_3)\}^{\lambda_{s_3}} (a_2 a_{s_4})^{\lambda_{s_4}} \ldots$$
$$(a_2 a_{s_\eta})^{\lambda_{s_\eta}} (a_1 a_{r_1})^{\lambda_{r_1}} (a_1 a_{r_2})^{\lambda_{r_2}} \ldots$$
$$= (a_1 a_3) \{(a_1 a_{s_2}) - (a_1 a_3)\}^{\lambda_{s_2}} \{(a_1 a_{s_3}) - (a_1 a_3)\}^{\lambda_{s_3}} (a_1 a_{r_1})^{\lambda_{r_1}} (a_1 a_{r_2})^{\lambda_{r_2}} \ldots$$
$$(a_2 a_{s_4})^{\lambda_{s_4}} \ldots (a_2 a_{s_\eta})^{\lambda_{s_\eta}}$$
$$-(a_1 a_2)(a_3 a_{s_2})^{\lambda_{s_2}} (a_3 a_{s_3})^{\lambda_{s_3}} \{(a_1 a_{s_4}) - (a_1 a_2)\}^{\lambda_{s_4}} \ldots$$
$$\{(a_1 a_{s_\eta}) - (a_1 a_2)\}^{\lambda_{s_\eta}} (a_1 a_{r_1})^{\lambda_{r_1}} (a_1 a_{r_2})^{\lambda_{r_2}} \ldots,$$

whence $$e^{-a_3 D_{s_2} - a_3 D_{s_3}} [0, 1, \lambda_4, \ldots, \lambda_\delta] = R.$$

and so, in general, we have syzygies which yield

$$e^{-a_3 D_{\sigma_1} - a_3 D_{\sigma_2} - \ldots - a_3 D_{\sigma_k}} [0, 1, \lambda_4, \ldots, \lambda_\delta] = R,$$

where $\sigma_1, \sigma_2, \ldots, \sigma_k$ are any, all or none of $s_2, s_3, \ldots, s_\eta$.

18. We have to prove that the $2^{\delta - t + 1}$ equations

$$\text{(i)} \quad e^{-a_3 D_{\sigma_1} - a_3 D_{\sigma_2} - \ldots - a_3 D_{\sigma_k}} [0, 0, \lambda_4, \ldots, \lambda_\delta] = 0,$$

where $\sigma_1, \sigma_2, \ldots, \sigma_k$ are all, any or none of $t+1, t+2, \ldots, \delta$;

$$\text{(ii)} \quad e^{a_3 D_{\rho_1}+a_3 D_{\rho_2}+\ldots+a_3 D_{\rho_k}-a_3 D_{\sigma_1}-a_3 D_{\sigma_2}-\ldots-a_3 D_{\sigma_k}} [0, 0, \lambda_4, \ldots, \lambda_\delta] = 0,$$

where $\rho_1, \rho_2, \ldots, \rho_k$ are all or any of the numbers $r_1, r_2, r_3, \ldots$ which are contained in $t+1, t+2, \ldots, \delta$; and $\sigma_1, \sigma_2, \ldots, \sigma_k$ are all, any or none of the numbers $s_1, s_2, \ldots, s_\eta$ which are contained in $t+1, t+2, \ldots, \delta$;

$$\text{(iii)} \quad e^{-a_3 D_{\sigma_1}-a_3 D_{\sigma_2}-\ldots-a_3 D_{\sigma_k}} [0, 1, \lambda_4, \ldots, \lambda_\delta] = 0,$$

where $\sigma_1, \sigma_2, \ldots, \sigma_k$ are all, any or none of the numbers $s_1, s_2, \ldots, s_\eta$ which are contained in $t+1, t+2, \ldots, \delta$:

are just sufficient to express all forms

$$[0, \lambda_3, \lambda_4, \ldots, \lambda_t, \lambda_{t+1}, \ldots, \lambda_\delta],$$

for which $\lambda_3 < 2^{\delta-t+1}$, in terms of forms

$$[0, \mu_3, \lambda_4, \ldots, \lambda_t, \mu_{t+1}, \ldots, \mu_\delta],$$

where $\mu_3 \nless 2^{\delta-t+1}$, and the first of the differences

$$\lambda_\delta-\mu_\delta, \ \lambda_{\delta-1}-\mu_{\delta-1}, \ \ldots, \ \lambda_{t+1}-\mu_{t+1}$$

which does not vanish is positive.

The proof follows the lines of the proof of the Lemma of § 15, and we need not give it in full.

We assume that the theorem is true for a particular value of t, and then proceed to prove the next step. We have two cases here.

(i) $t = r$; then applying the theorem of § 9, we show that

$$e^{-a_3 D_r} [0, \lambda_3, \lambda_4, \ldots, \lambda_\delta] = R \quad \text{for} \quad \lambda_3 = 0, 1, \ldots, 2^{\delta-r}-1.$$

We obtain, in the same way, for the same values of λ_3,

$$e^{+a_3 D_r} [0, \lambda_3, \lambda_4, \ldots, \lambda_\delta] = R,$$

for the proof of the theorem of § 9 is not altered if the sign of certain of the operators is changed throughout. From these two equations we obtain the result by the reasoning of § 15.

(ii) $t = s$; our assumption gives at once

$$e^{a_3 D_s} [0, \lambda_3, \lambda_4, \ldots, \lambda_\delta] = R \quad \text{for} \quad \lambda_3 = 0, 1, \ldots, 2^{\delta-s+1}.$$

Then, applying the theorem of § 9, we find the truth of the statement of this paragraph.

Thus, in either case, the induction proceeds step by step, and, as the theorem is true for the simplest case of $t = \delta - 1$, it is always true.

19. We apply this result to the relations of § 17, and we at once obtain the truth of the theorem of § 10 for the equation

$$e^{-a_2 D_3 - a_2 D_{s_2} - \ldots - a_2 D_{s_\eta}} (0, \lambda_3, \ldots, \lambda_\delta) = R$$

so far as the argument λ_3 is concerned.

The proof follows the same lines for the arguments $\lambda_{s_2}, \ldots, \lambda_{s_\eta}$. But it is necessary now in order to complete the proof to add a fresh convention.

We have so far regarded the equations

$$e^{-a_r D_{\sigma_1} - a_r D_{\sigma_2} - \ldots - a_r D_{\sigma_h}} (\lambda_2, \lambda_3, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_\delta) = R$$

$$(r < \sigma_1 < \sigma_2 < \ldots < \sigma_h),$$

$$e^{-a_r D_{\tau_1} - a_r D_{\tau_2} - \ldots - a_r D_{\tau_k}} (\lambda_2, \lambda_3, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_\delta) = R$$

$$(r < \tau_1 < \tau_2 < \ldots < \tau_k),$$

as simultaneous when $\sigma_1 = \tau_1$.

We must now arrange all our equations in sequence according to the law that the first of the above equations precedes the second if the first of the numbers

$$\sigma_1 - \tau_1, \quad \sigma_2 - \tau_2, \quad \ldots,$$

which does not vanish is positive, and this rule will be made complete if we introduce the symbols σ_{h+1}, τ_{k+1}, each of which is supposed to be numerically greater than any given number.

Thus, when $h = 0$, we have the equation

$$(\lambda_2, \lambda_3 \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_\delta) = R,$$

for which σ_1 exceeds any given number, and which therefore precedes all the other equations at the moment under consideration.

We deal with our equations in regular order, beginning with the earliest in the sequence. Each equation will reduce a fresh form or else with the previous equations in the sequence it must give rise to a syzygy.

The truth of the theorem of § 10 is established now for every possible case, exactly as we have established it for those cases we have discussed.

20. Having arrived at the truth of the theorem of § 10, let us consider the equation

$$e^{-a_r D_{s_1} - a_r D_{s_2} - \ldots - a_r D_{s_\eta}} (\lambda_2, \lambda_3, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_\delta) = R$$

$$(r < s_1 < s_2 < \ldots < s_\eta).$$

In general it has been taken as one of $2^{\delta-s_1}$ equations which will reduce forms

$$(\lambda_2, \lambda_3, \ldots, \lambda_{r-1}, \lambda_r, \lambda_{r+1}, \ldots, \lambda_\delta) = R,$$

when

$$\lambda_r = 2^{\delta-s_1},\ 2^{\delta-s_1}+1,\ \ldots,\ 2^{\delta-s_1+1}-1.$$

We have in the last paragraph introduced a convention by which these $2^{\delta-s_1}$ equations are arranged in a definite sequence. We may then associate each equation with a definite form which it reduces. We shall suppose that the earliest equation will reduce the form with the lowest value of λ_r, and so on. This supposition gives consistent results, for the determinants of the coefficients involved are easily seen to be different from zero—in general. By this arrangement the equation

$$e^{-a_r D_{s_1} - a_r D_{s_2} - \ldots - a_r D_{s_\eta}} (\lambda_2, \lambda_3, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_\delta) = R$$

reduces the form

$$(\lambda_2, \lambda_3, \ldots, \lambda_{r-1}, 2^{\delta-s_1}+2^{\delta-s_2}+\ldots+2^{\delta-s_\eta}, \lambda_{r+1}, \ldots, \lambda_{s_1}-2^{\delta-s_1}, \ldots,$$
$$\lambda_{s_2}-2^{\delta-s_2}, \ldots);$$

i.e. it expresses this form in terms of later members of our sequence of forms and of products of forms of lower degree.

If $\lambda_s < 2^{\delta-s+1}$ or if $\lambda_\tau < 2^{\delta-\tau}$, where $\tau > r$ and is not one of $s_1, s_2, \ldots, s_\eta$, then this form has been reduced by a previous equation. But, in either of these cases, there is a syzygy by means of which this equation can be expressed in terms of previous equations, as we have shewn in our theorem of § 10.

Thus, to every equation we have a definite reduction or a syzygy.

21. Now let us review the perpetuant types of degree δ.

Firstly, they can all, reducible or irreducible, be expressed linearly in terms of the forms

$$(\lambda_2, \lambda_3, \ldots, \lambda_\delta),$$

and these forms are all linearly independent. *Secondly*, any product of perpetuant types of total degree δ can be expressed as a product of *two* perpetuants, neither of which is necessarily irreducible; and, when this product is expressed in terms of our standard forms of degree δ, it can be written, without ambiguity,

$$e^{-a_r D_{s_1} - a_r D_{s_2} - \ldots - a_r D_{s_\eta}} (\lambda_2, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_\delta)$$
$$(r < s_1 < s_2 < \ldots < s_\eta < \delta+1).$$

Thirdly, the complete discussion of the equations

$$e^{-a_r D_{s_1} - a_r D_{s_2} - \dots - a_r D_{s_r}} (\lambda_2, \dots, \lambda_{r-1}, 0, \lambda_{r+1}, \dots, \lambda_\delta) = R,$$

involves, firstly, the discovery of the laws of reducibility and irreducibility, and, secondly, the discovery of all the syzygies of the first kind.

The laws of reducibility established by Grace follow from this. And we have now shewn that all syzygies of the first kind can very simply be deduced from those of Stroh and the Jacobian form of syzygy.

III. *Forms of Finite Order.*

22. The discussion for forms of finite order follows identically the same lines as that for perpetuants. We express all covariants of degree δ in terms of the forms

$$(\lambda_2, \lambda_3, \dots, \lambda_\delta)$$

defined as in § 5. We then consider every possible product of *two* covariants of total degree δ, and we express it in terms of our standard forms. The equations which we get in this way will give us the laws of reducibility of our standard forms, and also will yield every syzygy for this degree.

The discussion is rendered more complicated by the fact that

$$(\lambda_2, \lambda_3, \dots, \lambda_\delta)$$

is no longer equal to the simple product

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \dots (a_1 a_\delta)^{\lambda_\delta},$$

but is equal to this plus a linear function of the fundamental forms.

If the set of inequalities

$$\lambda_2 \not> n_1, \; 2\lambda_2 + \lambda_3 \not> n_1 + n_2, \; 2\lambda_2 + 2\lambda_3 + \lambda_4 \not> n_1 + n_2 + n_3, \; \dots,$$

$$2\lambda_2 + 2\lambda_3 + 2\lambda_4 + \dots + 2\lambda_{\delta-1} + \lambda_\delta \not> n_1 + n_2 + n_3 + \dots + n_{\delta-1},$$

is not satisfied,

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \dots (a_1 a_\delta)^{\lambda_\delta}$$

is itself a fundamental form; and we must write

$$(\lambda_2, \lambda_3, \dots, \lambda_\delta) = 0.$$

The analysis for perpetuants must then be modified in two ways.

Firstly, the product $(s < s_1 < s_2 < \ldots < s_\eta < \delta+1)$

$$\left(\frac{a_{r_1}^{\lambda_{r_1}} a_{r_2}^{\lambda_{r_2}} \ldots a_{r_\epsilon}^{\lambda_{r_\epsilon}}}{a_1}\right)\left(\frac{a_{s_1}^{\lambda_{s_1}} a_{s_2}^{\lambda_{s_2}} \ldots a_{s_\eta}^{\lambda_{s_\eta}}}{a_s}\right)$$

is equal to a sum of forms, of which the *earliest* are

$$e^{-a_s D_{s_1} - a_s D_{s_2} - \ldots - a_s D_{s_\eta}} (\lambda_2, \ldots, \lambda_{s-1}, 0, \lambda_{s+1}, \ldots, \lambda_\delta),$$

in general; but which contains other terms too.

Secondly, if the numbers $\lambda_{s_1}, \lambda_{s_2}, \ldots, \lambda_{s_\eta}$ do not satisfy the set of inequalities

$$\lambda_{s_1} \not> n_s, \quad 2\lambda_{s_1} + \lambda_{s_2} \not> n_s + n_{s_1}, \quad 2\lambda_{s_1} + 2\lambda_{s_2} + \lambda_{s_3} \not> n_s + n_{s_1} + n_{s_2}, \quad \ldots,$$

$$2\lambda_{s_1} + 2\lambda_{s_2} + \ldots + 2\lambda_{s_{\eta-1}} + \lambda_{s_\eta} \not> n_s + n_{s_1} + \ldots + n_{s_{\eta-1}},$$

then

$$\left(\frac{a_{s_1}^{\lambda_{s_1}} a_{s_2}^{\lambda_{s_2}} \ldots a_{s_\eta}^{\lambda_{s_\eta}}}{a_s}\right) = 0,$$

in this case there is no equation.

Thus many of the equations obtained for the case of perpetuants do not exist for forms of finite order; the corresponding reductions either do not exist or else they are brought about by other equations. Thus, equations which for perpetuants yielded syzygies may now yield reductions. It will frequently be found that the reduction which corresponds to such an equation is most simply found by a consideration of what the corresponding perpetuant syzygy becomes when the orders of the quantics take the finite values of the case in hand.

The forms

$$(\lambda_2, \lambda_3, \ldots, \lambda_\delta)$$

are arranged in sequence according to the same law as for perpetuants. Also the law of sequence of equations is still adhered to. It is useful to remember that no form can be reducible for quantics of finite order, which is not so for perpetuants, and also that an equation which produces a reduction for perpetuants must reduce the same or an earlier form (if it exists at all) for quantics of finite order.

23. At the outset the question rises: Can we find an explicit expression for

$$(\lambda_2, \lambda_3, \ldots, \lambda_\delta)$$

in terms of $$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \dots (a_1 a_\delta)^{\lambda_\delta},$$

and the fundamental forms?

We proceed to find such an expression for the case when n_1 alone is finite and the orders of all the other quantics are infinite. In this case we observe that a fundamental form is simply a form

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \dots (a_1 a_\delta)^{\lambda_\delta},$$

for which $\lambda_2 > n_1$.

We proceed to prove the following theorem:—

When the orders $n_2, n_3, \dots, n_\delta$ of the quantics concerned are greater than the weight of the covariants under consideration, while the order n_1 is less than this quantity, the covariant

$$(\lambda_2, \lambda_3, \dots, \lambda_\delta)$$

may be represented by the sum

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \dots (a_1 a_\delta)^{\lambda_\delta}$$

$$+\Sigma(-)^i \binom{n_1-\lambda_2+i-1}{i-1} \binom{\lambda_3}{j_3} \binom{\lambda_4}{j_4} \dots \binom{\lambda_\delta}{j_\delta} (a_1 a_2)^{n_1+i} (a_1 a_3)^{j_3} (a_1 a_4)^{j_4} \dots (a_1 a_\delta)^{j_\delta},$$

where $$i = \Sigma\lambda - \Sigma j - n_1.$$

For simplicity we will take $\delta = 4$. And for this case we will prove the symbolical identity

$$\text{(I)}\quad (a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} (a_1 a_4)^{\lambda_4}$$

$$+\Sigma(-)^i \binom{n_1-\lambda_2+i-1}{i-1} \binom{\lambda_3}{j} \binom{\lambda_4}{\rho-i-j} (a_1 a_2)^{n_1+i} (a_1 a_3)^j (a_1 a_4)^{\rho-i-j}$$

$$= \sum_{\xi=0}^{\lambda_3-\rho} \binom{\rho-1+\xi}{\rho-1} (a_2 a_3)^\rho (a_1 a_2)^{\lambda_2+\xi} (a_1 a_3)^{\lambda_3-\rho-\xi} (a_1 a_4)^{\lambda_4}$$

$$+ \sum_{\xi=1}^{\lambda_4} \binom{\lambda_3}{j} \binom{\xi-1}{\rho-1-j} (a_2 a_3)^j (a_2 a_4)^{\rho-j} (a_1 a_2)^{\lambda_2+\lambda_3+\xi-\rho} (a_1 a_4)^{\lambda_4-\xi},$$

where $$\rho = \lambda_2+\lambda_3+\lambda_4-n_1.$$

The forms on the right are ordinary symbolical products which represent as they stand covariants of the quantics with which we are concerned. Let us assume the truth of this identity as it stands and then deduce that it is true when λ_4 is changed into λ_4+1 and n_1 into

n_1+1. It is to be noticed that this change leaves ρ unchanged. To do this multiply the supposed identity by $(a_1 a_4)$. Then when the order of the a_1 quantic is n_1+1, those terms under the sign of summation on the left for which $i=1$ are no longer fundamental, and those terms only.

From the identity

$$(a_2 a_3)^j (a_2 a_4)^{\rho-j} = \{(a_1 a_3)-(a_1 a_2)\}^j \{(a_1 a_4)-(a_1 a_2)\}^{\rho-j},$$

we obtain

$$(a_1 a_3)^j (a_1 a_4)^{\rho-j}$$
$$= (a_2 a_3)^j (a_2 a_4)^{\rho-j} - \sum_{i_1+i_2\neq 0} (-)^{i_1+i_2} \binom{j}{i_1}\binom{\rho-j}{i_2} (a_1 a_2)^{i_1+i_2} (a_1 a_3)^{j-i_1} (a_1 a_4)^{\rho-j-i_2}.$$

We make use of this result and the identity becomes

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} (a_1 a_4)^{\lambda_4+1}$$
$$+ \sum_{i=2} (-)^i \binom{n_1-\lambda_2+i-1}{i-1}\binom{\lambda_3}{j}\binom{\lambda_4}{\rho-i-j} (a_1 a_2)^{n_1+i} (a_1 a_3)^j (a_1 a_4)^{\rho+1-i-j}$$
$$+ \sum_{i_1+i_2\neq 0} \binom{\lambda_3}{j}\binom{\lambda_4}{\rho-1-j} (-)^{i_1+i_2} \binom{j}{i_1}\binom{\rho-j}{i_2} (a_1 a_2)^{n_1+1+i_1+i_2}$$
$$\times (a_1 a_3)^{j-i_1} (a_1 a_4)^{\rho-j-i_2}$$
$$= \sum_{\xi=0}^{\lambda_3-\rho} \binom{\rho-1+\xi}{\rho-1} (a_2 a_3)^{\rho} (a_1 a_2)^{\lambda_2+\xi} (a_1 a_3)^{\lambda_3-\rho-\xi} (a_1 a_4)^{\lambda_4+1}$$
$$+ \sum_{\xi=1}^{\lambda_4} \binom{\lambda_3}{j}\binom{\xi-1}{\rho-1-j} (a_2 a_3)^j (a_2 a_4)^{\rho-j} (a_1 a_2)^{\lambda_2+\lambda_3+\xi-\rho} (a_1 a_4)^{\lambda_4+1-\xi}$$
$$+ \sum \binom{\lambda_3}{j}\binom{\lambda_4}{\rho-1-j} (a_2 a_3)^j (a_2 a_4)^{\rho-j} (a_1 a_2)^{n_1+1}.$$

The right-hand side of our identity is already the same that we should get by writing λ_4+1 for λ_4, and n_1+1 for n_1 in the identity we want to prove. The coefficient of $(a_1 a_2)^{n_1+1+i} (a_1 a_3)^j (a_1 a_4)^{\rho-i-j}$ on the left is

$$(-)^{i+1} \binom{n_1-\lambda_2+i}{i}\binom{\lambda_3}{j}\binom{\lambda_4}{\rho-i-1-j}$$
$$+ (-)^i \sum_k \binom{\lambda_3}{k}\binom{\lambda_4}{\rho-1-k}\binom{k}{j}\binom{\rho-k}{\rho-i-j}.$$

Now $$\sum_k \binom{\lambda_3}{k}\binom{\lambda_4}{\rho-1-k}\binom{k}{j}\binom{\rho-k}{\rho-i-j}$$

$=$ the coefficient of $x^j y^{\rho-1} z^{\rho-i-j}$ in the expansion of

$$\{1+y(1+x)\}^{\lambda_3}\{1+y(1+z)\}^{\lambda_4}(1+z)$$

$=$ the coefficient of $x^j y^{\rho-1} z^{\rho-i-j}$ in the expansion of

$$\left\{1+\frac{xy}{1+y}\right\}^{\lambda_3}\left\{1+\frac{zy}{1+y}\right\}^{\lambda_4}(1+y)^{\lambda_3+\lambda_4}(1+z)$$

$$= \binom{\lambda_3}{j}\left\{\binom{\lambda_4}{\rho-i-j}\binom{\lambda_3+\lambda_4-\rho+i}{i-1}+\binom{\lambda_4}{\rho-i-j-1}\binom{\lambda_3+\lambda_4-\rho+i+1}{i}\right\}.$$

Hence the coefficient of

$$(a_1 a_2)^{n_1+1+i}(a_1 a_3)^j (a_1 a_4)^{\rho-i-j}$$

is $(-)^i\binom{\lambda_3}{j}\binom{\lambda_4}{\rho-i-j}\binom{n_1-\lambda_2+i}{i-1}$

$$+(-)^i\binom{\lambda_3}{j}\binom{\lambda_4}{\rho-i-j-1}\left\{\binom{n_1-\lambda_2+i+1}{i}-\binom{n_1-\lambda_2+i}{i}\right\}$$

$$= (-)^i\binom{\lambda_3}{j}\binom{n_1-\lambda_2+i}{i-1}\left\{\binom{\lambda_4}{\rho-i-j}+\binom{\lambda_4}{\rho-i-j-1}\right\}$$

$$= (-)^i\binom{n_1-\lambda_2+i}{i-1}\binom{\lambda_3}{j}\binom{\lambda_4+1}{\rho-i-j}.$$

The identity is true, then, when we replace λ_4 and n_1 by λ_4+1 and n_1+1.

If, then, it is true for certain values of λ_4 and n_1, it is still true if these values are both increased by unity, and therefore if they are both increased by any the same number.

(i) Let n_1 be greater than λ_4. Then, if the identity is true when $n_1-\lambda_4$ and 0 are written for n_1 and λ_4, it is true as it stands. It will be sufficient simply to discuss the case $\lambda_4 = 0$ and leave n_1 unaltered. The identity then becomes

(II) $(a_1 a_2)^{\lambda_2}(a_1 a_3)^{\lambda_3}$

$$+\Sigma(-)^i\binom{n_1-\lambda_2+i-1}{i-1}\binom{\lambda_3}{\rho-1}(a_1 a_2)^{n_1+i}(a_1 a_3)^{\rho-1}$$

$$= \sum_{\xi=0}^{\lambda_3-\rho}\binom{\rho-1+\xi}{\rho-1}(a_2 a_3)^{\rho}(a_1 a_2)^{\lambda_2+\xi}(a_1 a_3)^{\lambda_3-\rho-\xi}.$$

To prove this we write the right-hand side in the form

$$\sum_{\xi=0}^{\lambda_3-\rho}\binom{\rho-1+\xi}{\rho-1}\left\{(a_1a_3)-(a_1a_2)\right\}^{\rho}(a_1a_2)^{\lambda_2+\xi}(a_1a_3)^{\lambda_3-\rho-\xi}$$

$$=\sum_{\xi=0}^{\lambda_3-\rho}\binom{\rho-1+\xi}{\rho-1}\Sigma(-)^{\zeta}\binom{\rho}{\zeta}(a_1a_2)^{\lambda_2+\xi+\zeta}(a_1a_3)^{\lambda_3-\xi-\zeta}.$$

The coefficient of $\qquad (a_1a_2)^{\lambda_2+\eta}(a_1a_3)^{\lambda_3-\eta}$

is $(-)^{\eta}\left[\binom{\rho}{\eta}-\binom{\rho}{\rho-1}\binom{\rho}{\eta-1}+\ldots+(-)^{\lambda_3-\rho}\binom{\rho-1+\lambda_3-\rho}{\rho-1}\binom{\rho}{\eta-\lambda_3+\rho}\right].$

To find the value of this we shall prove the identity

$$\binom{\rho}{i}-\binom{\rho}{1}\binom{\rho}{i-1}+\binom{\rho+1}{2}\binom{\rho}{i-2}-\ldots+(-)^j\binom{\rho+j-1}{j}\binom{\rho}{i-j}$$

$$=(-)^j\binom{i-1}{j}\binom{\rho+j}{i}.$$

Assume that it is true as it stands and add one more term $(-)^{j+1}\binom{\rho+j}{j+1}\binom{\rho}{i-j-1}$ to each side.

The right-hand side becomes

$$(-)^{j+1}\frac{(\rho+j)!\,\{\rho i-(j+1)(\rho+j+1-i)\}}{(j+1)!\,(\rho+j+1-i)!\,(i-j-1)!\,i}=(-)^{j+1}\binom{i-1}{j+1}\binom{\rho+j+1}{i},$$

and so the induction proceeds step by step: for the identity is obvious for $j=0$.

Making use of this result we find that the coefficient of $(a_1a_2)^{\lambda_2+\eta}(a_1a_3)^{\lambda_3-\eta}$ is

$$(-)^{\eta+\lambda_3-\rho}\binom{\eta-1}{\lambda_3-\rho}\binom{\lambda_3}{\eta},$$

which is the same as the coefficient of the corresponding term on the left-hand side of the identity, for $\lambda_2+\lambda_3=n_1+\rho$. This coefficient is unity when η is zero, it is zero for $\eta=1, 2, \ldots, n_1-\lambda_2$, and its value is

$$(-)^i\binom{n_1-\lambda_2+i-1}{i-1}\binom{\lambda_3}{\rho-i}$$

for $$\eta=n_1-\lambda_2+i.$$

The identity (I) is then true if $\lambda_4=0$, and therefore whenever $n_1>\lambda_4$.

(ii) Let n_1 be equal to or less than λ_4. Then, if the identity is true

when 0 and λ_4-n_1 are written for n_1 and λ_4, it is true as it stands. It will be sufficient to discuss the case $n_1=0$. It is just as easy to take the case $n_1<\lambda_2$. Here the left-hand side of (I) becomes

$$(a_1a_2)^{\lambda_2}(a_1a_3)^{\lambda_3}(a_1a_4)^{\lambda_4}+(-)^{\lambda_2-n_1}\binom{-1}{\lambda_2-n_1-1}(a_1a_2)^{\lambda_2}(a_1a_3)^{\lambda_3}(a_1a_4)^{\lambda_4},$$

every other term under the sign of summation vanishes. The left-hand side is therefore zero. On the right there are no terms in the first sum, for $\lambda_3-\rho$ is negative, and in the second sum every coefficient is zero for $\xi-1<\rho-1-j$, since j must be less than λ_3. Thus (I) is true when $n_1<\lambda_2$. [In the same way we see that the general expression in the enunciation of our theorem

$$(a_1a_2)^{\lambda_2}\dots(a_1a_\delta)^{\lambda_\delta}$$
$$+\Sigma(-)^i\binom{n_1-\lambda_2+i-1}{i-1}\binom{\lambda_3}{j_3}\binom{\lambda_4}{j_4}\dots\binom{\lambda_\delta}{j_\delta}(a_1a_2)^{n_1+i}(a_1a_3)^{j_3}\dots(a_1a_\delta)^{j_\delta},$$

vanishes when $n_1<\lambda_2$.]

The identity (I) is then true when $n_1 \not> \lambda_4$; it is therefore true for all values of n_1 and λ_4.

Now the identity (I) expresses the sum of

$$(a_1a_2)^{\lambda_2}(a_1a_3)^{\lambda_3}(a_1a_4)^{\lambda_4},$$

and certain fundamental forms as a sum of symbolical products which represent actual covariants of the quantics under discussion. This sum of covariants is then the covariant we have named

$$(\lambda_2,\ \lambda_3,\ \lambda_4).$$

The theorem is then true for degree 4. Assuming that it has been proved for degree $\delta-1$, it can be proved for degree δ in just the same way that it has been proved for degree 4. The actual form of the covariants on the right of the identity is not given, and it is not required. It is sufficient that the right-hand side of the identity should contain only symbolical products which represent actual covariants of the quantics concerned. There is no difficulty in obtaining the expression, but it is troublesome to write out, and no advantage is gained by doing so.

24. When the orders of all the quantics are finite the case is not so simple. For the discussion of the covariants of degree 4 we require

the linear function of fundamental forms that must be added to

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3}$$

in order that the sum may really be a covariant of $a_{1_x}^{n_1}$, $a_{2_x}^{n_2}$, $a_{3_x}^{n_3}$. We shall prove that :—

The covariant

$$(\lambda_2, \lambda_3) = (a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} + \Sigma(-)^i \binom{\lambda_3-\rho_1-\rho_2+i-1}{i-1} \binom{\lambda_3-\rho_2}{\rho_1-i} (a_1 a_2)^{n_1-\rho_2+i} (a_1 a_3)^{\rho_1+\rho_2-i},$$

where $\rho_i = \lambda_2+\lambda_3-n_i$, *or* 0, *according as* $\lambda_2+\lambda_3 >$ *or* $< n_i$.

In the first place the terms under the sign of summation are all fundamental forms, for

$$2(n_1-\rho_2+i)+\rho_1+\rho_2-i = 2n_1+\rho_1-\rho_2+i = n_1+n_2+i > n_1+n_2,$$

since the coefficient is zero unless $i > 0$.

Moreover the index of $(a_1 a_2)$ never exceeds $n_1-\rho_2+\rho_1 = n_2$, for $i \not> \rho_1$.

From the identity (II) of the last paragraph, we have for the case $\rho_2 = 0$,

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} + \Sigma(-)^i \binom{\lambda_3-\rho_1+i-1}{i-1} \binom{\lambda_3}{\rho_1-i} (a_1 a_2)^{n_1+i} (a_1 a_3)^{\rho_1-i}$$

$$= \sum_{\xi=0}^{\lambda_3-\rho_1} \binom{\rho_1-1+\xi}{\rho_1-1} (a_2 a_3)^{\rho_1} (a_1 a_2)^{\lambda_2+\xi} (a_1 a_3)^{\lambda_3-\rho_1-\xi},$$

an identity which establishes our theorem in this case. We shall take this as it stands and suppose that n_2 has its least possible value $\lambda_2+\lambda_3$.

Now in this replace λ_3 by $\lambda_3-\rho_2$, keeping λ_2 and ρ_1 unaltered; then n_1 must be replaced by $n_1-\rho_2$, since $n_1 = \lambda_2+\lambda_3-\rho_1$, and n_2 must be replaced by $n_2-\rho_2$; we have

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3-\rho_2} + \Sigma(-)^i \binom{\lambda_3-\rho_1-\rho_2+i-1}{i-1} \binom{\lambda_3-\rho_2}{\rho_1-i} (a_1 a_2)^{n_1+i-\rho_2} (a_1 a_3)^{\rho_1-i}$$

$$= \sum_{\xi=0}^{\lambda_3-\rho_1-\rho_2} \binom{\rho_1-1+\xi}{\rho_1-1} (a_2 a_3)^{\rho_1} (a_1 a_2)^{\lambda_2+\xi} (a_1 a_3)^{\lambda_3-\rho_1-\rho_2-\xi}.$$

Now multiply this result through by $(a_1 a_3)^{\rho_2}$ and we have

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} + \Sigma(-)^i \binom{\lambda_3-\rho_1-\rho_2+i-1}{i-1} \binom{\lambda_3-\rho_2}{\rho_1-i} (a_1 a_2)^{n_1+i-\rho_2} (a_1 a_3)^{\rho_1+\rho_2-i}$$

$$= \sum_{\xi=0}^{\lambda_3-\rho_1-\rho_2} \binom{\rho_1-1+\xi}{\rho_1-1} (a_2 a_3)^{\rho_1} (a_1 a_2)^{\lambda_2+\xi} (a_1 a_3)^{\lambda_3-\rho_1-\xi}.$$

Since the right-hand side of this represents a covariant of the quantics concerned, it is $=(\lambda_2, \lambda_3)$. Q. E. D.

25. It will be sometimes useful to use the notation

$$(a_1 a_2)^{\lambda_2}(a_1 a_3)^{\lambda_3}+\Sigma(-)^i\binom{\lambda_3-\rho_1-\rho_2+i-1}{i-1}\binom{\lambda_3-\rho_2}{\rho_1-i}(a_1 a_2)^{n_1+i-\rho_2}(a_1 a_3)^{\rho_1+\rho_2-i}$$
$$\equiv (a_1 a_2)_{\lambda_2}(a_1 a_3)_{\lambda_3}.$$

Also we shall copy the index notation of ordinary algebra further by writing

$$\{(a_1 a_3)-(a_1 a_2)\}_\lambda \equiv \Sigma(-)_i\binom{\lambda}{i}(a_1 a_2)_i(a_1 a_3)_{\lambda-i},$$

and also by writing

$$(a_1 a_2)_\mu (a_1 a_3)_\nu \{(a_1 a_3)-(a_1 a_2)\}_\lambda \equiv \Sigma(-)^i\binom{\lambda}{i}(a_1 a_2)_{\mu+i}(a_1 a_3)_{\nu+\lambda-i}.$$

When we confine ourselves to the operations of symbolical algebra this notation will not involve any assumptions.

We will now prove that with this notation

(III) $$\{(a_1 a_3)-(a_1 a_2)\}_\lambda = \{(a_1 a_3)-(a_1 a_2)\}^\lambda.$$

In other words we shall prove that

$$\Sigma(-)^i\binom{\lambda}{i}(a_1 a_2)_i(a_1 a_3)_{\lambda-i}=\Sigma(-)^i\binom{\lambda}{i}(a_1 a_2)^i(a_1 a_3)^{\lambda-i}.$$

In fact

$$\Sigma(-)^i\binom{\lambda}{i}(a_1 a_2)_i(a_1 a_3)_{\lambda-i}-\Sigma(-)^i\binom{\lambda}{i}(a_1 a_2)^i(a_1 a_3)^{\lambda-i}$$
$$=\Sigma(-)^{i+}\binom{\lambda}{i}\binom{n_2-\rho_1-i+j-1}{j-1}\binom{n_2-i}{\rho_1-j}(a_1 a_2)^{n_1+j-\rho_2}(a_1 a_3)^{\rho_1+\rho_2-j}.$$

The coefficient of

$$(a_1 a_2)^{n_1+j-\rho_2}(a_1 a_3)^{\rho_1+\rho_2-j}$$

= the coefficient of $x^{j-1}y^{\rho_1-j}$ in the expansion of

$$\{(1+x)(1+y)-1\}^\lambda(1+x)^{j-\rho_1-\rho_2-1}(1+y)^{-\rho_2}(-)^j$$

= the coefficient of $x^{j-1}y^{\rho_1-j}$ in the expansion of

$$\{x+y+xy\}^\lambda(1+x)^{j-\rho_1-\rho_2-1}(1+y)^{-\rho_2}(-)^j$$

$=0$, unless $j-1+\rho_1-j \geqslant \lambda$,

i.e., unless $\lambda-n_1>\lambda$,

all the coefficients on the right are then zero, and hence (III) is identically true.

Consider now the difference

$$(a_1 a_2)_\mu (a_1 a_3)_\nu \{(a_1 a_3)-(a_1 a_2)\}_{\lambda-\mu-\nu}-(a_1 a_2)^\mu (a_1 a_3)^\nu \{(a_1 a_3)-(a_1 a_2)\}^{\lambda-\mu-\nu}$$

$$=\Sigma(-)^i \binom{\lambda-\mu-\nu}{i}(-)^j \binom{\lambda-\mu-i-\rho_1-\rho_2+j-1}{j-1}\binom{\lambda-\mu-i-\rho_2}{\rho_1-j}$$

$$\times (a_1 a_2)^{n_1+j-\rho_2}(a_1 a_3)^{\rho_1+\rho_2-j}.$$

The coefficient of $(a_1 a_2)^{n_1+j-\rho_2}(a_1 a_3)^{\rho_1+\rho_2-j}(-)^j$

= the coefficient of $x^{j-1}y^{\rho_1-j}$ in the expansion of

$$\{(1+x)(1+y)-1\}^{\lambda-\mu-\nu}(1+x)^{\nu-\rho_1-\rho_2+j-1}(1+y)^{\nu-\rho_2}$$

$= 0$, unless $\rho_1 > \lambda-\mu-\nu$,

i.e., unless $\mu+\nu > n_1$.

Hence

(IV) $(a_1 a_2)_\mu (a_1 a_3)_\nu \{(a_1 a_3)-(a_1 a_2)\}_{\lambda-\mu-\nu}$

$$=(a_1 a_2)^\mu (a_1 a_3)^\nu \{(a_1 a_3)-(a_1 a_2)\}^{\lambda-\mu-\nu},$$

unless $\mu+\nu > n_1$.

IV. *Covariants of Degree* 4.

26. These may all be represented as linear functions of the covariants defined by

$$(\lambda_2,\ \lambda_3,\ \lambda_4).$$

We shall suppose the quantics of which these are covariants are

$$a_{1_x}^{n_1},\ a_{2_x}^{n_2},\ a_{3_x}^{n_3},\ a_{4_x}^{n_4}.$$

Then we obviously must have

$$\lambda_2 \not> n_2,\quad \lambda_3 \not> n_3,\quad \lambda_4 \not> n_4.$$

Also, if the set of inequalities

$$\lambda_2 \not> n_1,\quad 2\lambda_2+\lambda_3 \not> n_1+n_2,\quad 2\lambda_2+2\lambda_3+\lambda_4 \not> n_1+n_2+n_3$$

is not satisfied, the form $(\lambda_2,\ \lambda_3,\ \lambda_4) = 0$.

Otherwise these forms are linearly independent.

The first step towards discussing the problem of reducibility is to

express all products of lower forms of total degree 4 in terms of these forms, just as we have done for perpetuants. We obtain thus a set of equations which we have to discuss.

As a basis of discussion the forms are arranged in sequence as in perpetuants; thus

$$(\lambda_2, \lambda_3, \lambda_4)$$

precedes

$$(\mu_2, \mu_3, \mu_4),$$

if the first of the differences

$$\lambda_4-\mu_4, \lambda_3-\mu_3, \lambda_2-\mu_2$$

which does not vanish is positive. We seek to express earlier members of the sequence in terms of later members and of products of forms.

27. Let us first suppose that the factor containing a_1 is of degree 3.

Then, by the theorem of § 24,

$$\left(\frac{a_2^{\lambda_2} a_3^{\lambda_3}}{a_1}\right) = (a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3}$$
$$+\Sigma(-)^i \binom{\lambda_3-\rho_1-\rho_2+i-1}{i-1} \binom{\lambda_3-\rho_2}{\rho_1-i} (a_1 a_2)^{n_1-\rho_2+i} (a_1 a_3)^{\rho_1+\rho_2-i}.$$

Hence

$$\left(\frac{a_2^{\lambda_2} a_3^{\lambda_3}}{a_1}\right) a_{4_x}^{n_4} = (\lambda_2, \lambda_3, 0),$$

for it can differ from this by fundamental forms only.

Again, if $\rho_1 = \lambda_2+\lambda_4-n_1$ and $\rho_2 = \lambda_2+\lambda_4-n_2$,

$$\text{(V)} \quad \left(\frac{a_2^{\lambda_2} a_4^{\lambda_4}}{a_1}\right) a_{3_x}^{n_3} = (\lambda_2, 0, \lambda_4)$$
$$+\Sigma(-)^i \binom{\lambda_4-\rho_1-\rho_2+i-1}{i-1} \binom{\lambda_4-\rho_2}{\rho_1-i} (n_1-\rho_2+i, 0, \rho_1+\rho_2-i).$$

This gives a reduction for $(\lambda_2, 0, \lambda_4)$, provided

$$2\lambda_2+\lambda_4 \not> n_1+n_2.$$

Again,

$$\text{(VI)} \quad \left(\frac{a_3^{\lambda_3} a_4^{\lambda_4}}{a_1}\right) a_{2_x}^{n_2} = (0, \lambda_3, \lambda_4)$$
$$+\Sigma(-)^i \binom{\lambda_4-\rho_1-\rho_3+i-1}{i-1} \binom{\lambda_4-\rho_3}{\rho_1-i} (0, n_1-\rho_3+i, \rho_1+\rho_3-i),$$

which gives a reduction for $(0, \lambda_3, \lambda_4)$, provided

$$2\lambda_3+\lambda_4 \ngtr n_1+n_3 \quad \text{and} \quad \lambda_3 \ngtr n_1.$$

28. If the factor containing a_1 is of degree 2, we have simply

$$\text{(VII)} \qquad (a_1 a_2)^{\lambda_2} (a_3 a_4)^{\lambda_4} = \Sigma (-)^i \binom{\lambda_4}{i} (a_1 a_2)^{\lambda_2} (a_1 a_3)^i (a_1 a_4)^{\lambda_4 - i}$$

$$= e^{-a_3 D_4} (\lambda_2, 0, \lambda_4),$$

using the same notation as for perpetuants.

Next we have

$$\text{(VIII)} \qquad (a_1 a_3)^{\lambda_3} (a_2 a_4)^{\lambda_4} = e^{-a_2 D_4} (0, \lambda_3, \lambda_4),$$

and then

$$\text{(IX)} \qquad (a_1 a_4)^{\lambda_4} (a_2 a_3)^{\lambda_3} = e^{-a_2 D_3} (0, \lambda_3, \lambda_4).$$

Finally, the factor containing a_1 may be of degree 1 only, then we have

$$\text{(X)} \quad \left(\frac{a_3^{\lambda_3} a_4^{\lambda_4}}{a_2}\right) a_{1_x}^{n_1} = e^{-a_2 D_3 - a_2 D_4} \Big[(0, \lambda_3, \lambda_4)$$

$$+ \Sigma (-)^i \binom{\lambda_4 - \rho_2 - \rho_3 + i - 1}{i-1} \binom{\lambda_4 - \rho_3}{\rho_2 - i} (0, n_2 - \rho_3 + i, \rho_2 + \rho_3 - i) \Big],$$

provided $\qquad 2\lambda_3+\lambda_4 \ngtr n_2+n_3, \quad \lambda_3 \ngtr n_2.$

Thus, we have obtained every possible reduction equation for degree 4. The equations are either the same as or modifications of the corresponding perpetuant equations.

In discussing the equations we shall confine ourselves to the case of most importance, viz., when

$$n_2 = n_3 = n_4 = n\,;$$

but the order n_1 may be any independent number.

29. As concerns λ_4, the only limit is the same as for perpetuants:

$$(\lambda_2, \lambda_3, \lambda_4)$$

is reducible if $\lambda_4 = 0$; otherwise we must go to λ_2 or λ_3.

For the limit of λ_3 for reducibility we have two equations, (V) and (VII).

From (V) we learn that

$$(\lambda_2, 0, \lambda_4) = R,$$

provided $\qquad 2\lambda_2+\lambda_4 \ngtr n+n_1.$

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1 Selwyn Gardens Cambridge —T. J. I'A. BROMWICH.

Putting the value of $(\lambda_2, 0, \lambda_4)$ obtained from (V) in (VII), we find

$$(\lambda_2,\ 1,\ \lambda_4-1) = R,$$

provided $$2\lambda_2+\lambda_4 \ngtr n+n_1.$$

If $2\lambda_2+\lambda_4 > n+n_1$, we have, from (VII),

$$(\lambda_2,\ 0,\ \lambda_4) = R$$

(for λ_4 cannot exceed $n_4 = n$ in any case).

Thus always $$(\lambda_2,\ 0,\ \lambda_4) = R,$$

and $$(\lambda_2,\ 1,\ \lambda_4) = R,$$

provided $$\lambda_4 < n, \quad \text{and} \quad 2\lambda_2+\lambda_4 \ngtr n+n_1-1.$$

30. Let us now discuss the reducibility limits of λ_2.

We have the following equations

(VI) $$(0,\ \lambda_3,\ \lambda_4) = R,$$

when $2\lambda_3+\lambda_4 \ngtr n+n_1$, $\lambda_3 \ngtr n_1$,

(VIII) $$(0,\ \lambda_3,\ \lambda_4)-\lambda_4\,(1,\ \lambda_3,\ \lambda_4-1)+\ldots = R,$$

when $\lambda_3 \ngtr n_1$,

(IX) $$(0, \lambda_3, \lambda_4)-\lambda_3\,(1, \lambda_3-1, \lambda_4)+\binom{\lambda_3}{2}(2, \lambda_3-2, \lambda_4)-\binom{\lambda_3}{3}(3, \lambda_3-3, \lambda_4) +\ldots = R,$$

when $\lambda_4 \ngtr n_1$,

(X) $$(0, \lambda_3, \lambda_4)-\lambda_3\,(1, \lambda_3-1, \lambda_4)+\binom{\lambda_3}{2}(2, \lambda_3-2, \lambda_4)-\binom{\lambda_3}{3}(3, \lambda_3-3, \lambda_4)+\ldots$$
$$-\lambda_4\,(1, \lambda_3, \lambda_4-1)+\lambda_3\lambda_4\,(2, \lambda_3-1, \lambda_4-1)-\binom{\lambda_3}{2}\lambda_4(3, \lambda_3-2, \lambda_4-1)+\ldots = R,$$

when $2\lambda_3+\lambda_4 \ngtr 2n$.

Taking these last two equations together, we see that (IX) is true when *either* $\lambda_4 \ngtr n_1$, *or* $2\lambda_3+\lambda_4 \ngtr 2n$. And that when we replace these conditions by the original condition of (IX) we may replace (X) by

(XI) $$(1, \lambda_3, \lambda_4-1)-\lambda_3\,(2, \lambda_3-1, \lambda_4-1)+\binom{\lambda_3}{2}(3, \lambda_3-2, \lambda_4-1)-\ldots = R,$$

when $\lambda_4 \ngtr n_1$, and $2\lambda_3+\lambda_4 \ngtr 2n$.

Let us first see what the equations give us just as they stand.

$(0, \lambda_3, \lambda_4)$ is reducible if any one of our equations exists. Hence we see that it is reducible unless $\lambda_3 > n_1$, $\lambda_4 > n_1$, and $2\lambda_3+\lambda_4 > 2n$.

The reduction of $(3, \lambda_3, \lambda_4)$ requires the coexistence of equations of each of the four types, and there is only one way in which it can be reduced. It is easy to see that it is not reducible unless

$$\lambda_3 \ngtr n-3, \quad \lambda_3 \ngtr n_1-3, \quad \lambda_4 \ngtr n-1, \quad \lambda_4 \ngtr n_1-1, \quad 2\lambda_3+\lambda_4 \ngtr 2n-5,$$
$$2\lambda_3+\lambda_4 \ngtr n+n_1-5.$$

The conditions of reducibility are more complicated when $\lambda_2 = 1$ or 2; it will be convenient to separate the discussion into two cases.

(α) $n_1 \geqslant n$.—The equations (VIII) and (IX) always exist; together they reduce $(1, \lambda_3-1, \lambda_4)$. Then $(1, \lambda_3, \lambda_4)$ is reducible if $\lambda_3 < n$.

From (VI) and (VIII) we have a reduction for $(1, \lambda_3, \lambda_4)$, provided $\lambda_4 < n$ and $2\lambda_3+\lambda_4 \ngtr n+n_1-1$.

Thus $(1, \lambda_3, \lambda_4) = R$, when $\lambda_3 < n$, or when

$$\lambda_4 < n \quad \text{and} \quad 2\lambda_3+\lambda_4 \ngtr n+n_1-1.$$

From the first two equations with (XI) we find that $(2, \lambda_3, \lambda_4) = R$, when $\lambda_3 \ngtr n-1$, $\lambda_4 \ngtr n-1$, $2\lambda_3+\lambda_4 \ngtr n+n_1-3$.

In this case we observe that $(0, \lambda_3, \lambda_4)$ is always reducible.

(β) $n_1 < n$.—Here $(1, \lambda_3, \lambda_4)$ may be reduced by (VI) and (VIII), in which case we have the conditions

$$\text{(i)} \quad \lambda_3 \ngtr n_1, \quad \lambda_4 \ngtr n-1, \quad 2\lambda_3+\lambda_4 \ngtr n+n_1-1;$$

or by (VIII) and (IX) in which case the conditions are

$$\text{(ii)} \quad \lambda_3 \ngtr n_1-1, \quad \lambda_4 \ngtr n_1;$$

or $$\text{(iii)} \quad \lambda_3 \ngtr n_1-1, \quad 2\lambda_3+\lambda_4 \ngtr 2n-2;$$

or else by (XI) when

$$\text{(iv)} \quad \lambda_4 \ngtr n_1-1, \quad 2\lambda_3+\lambda_4 \ngtr 2n-1.$$

Also $(2, \lambda_3, \lambda_4)$ may be reduced by (VI), (VIII) and (IX) when the conditions are

$$\text{(i)} \quad \lambda_3 \ngtr n_1-2, \quad \lambda_4 \ngtr n-1, \quad 2\lambda_3+\lambda_4 \ngtr n+n_1-3;$$

or, by (XI), (VIII) and (IX), when

$$\text{(ii)} \quad \lambda_3 \ngtr n_1-2, \quad \lambda_4 \ngtr n_1-1, \quad 2\lambda_3+\lambda_4 \ngtr 2n-3;$$

or else using (VI) and (VIII) to reduce the first term of (XI), we obtain the conditions

$$\text{(iii)}\quad \lambda_3 \not> n_1-1, \quad \lambda_4 \not> n_1-1, \quad 2\lambda_3+\lambda_4 \not> n+n_1-3.$$

31. It is necessary to examine equation (VI) a little more closely. The two conditions for its existence may be replaced by the single condition $\lambda_3 \not> n_1-\rho$.

When $\lambda_3 = n_1-\rho$, the equation takes the form

$$(0, \lambda_3, \lambda_4)-\rho_1(0, \lambda_3+1, \lambda_4-1) = R;$$

and when $\lambda_3 < n_1-\rho$, it takes the form

$$(0, \lambda_3, \lambda_4) = R;$$

where in each case R represents a linear function of products of forms and of forms $(0, \mu_3, \mu_4)$ for which $\mu_4 < \lambda_4-1$.

A difficulty apparently arises when we use (VI) and (VIII) in conjunction in the case $\lambda_3 = n_1-\rho$; for eliminating $(0, \lambda_3, \lambda_4)$, we have

$$\rho_1(0, \lambda_3+1, \lambda_4-1)-\lambda_4(1, \lambda_3, \lambda_4-1)+\ldots = R,$$

giving a reduction for $(0, \lambda_3+1, \lambda_4-1)$ instead of for $(1, \lambda_3, \lambda_4-1)$.

But in this case $(0, \lambda_3+1, \lambda_4-1)$ is reduced by another equation of the type (VIII), unless $\rho = 0$, and the reduction of $(1, \lambda_3, \lambda_4-1)$ then follows.

When $\rho = 0$, $\quad 2(\lambda_3+1)+(\lambda_4-1) \not> 2n-(\lambda_4-1)$,

and hence, from (IX), we have

$$(0, \lambda_3+1, \lambda_4-1)-(\lambda_3+1)(1, \lambda_3, \lambda_4-1)+\ldots = R.$$

Then, taking these equations in conjunction, we obtain the reductions exactly as stated in the last paragraph.

32. We have so far discussed our equations without any reference to the reductions already obtained when $\lambda_3 < 2$ or $\lambda_4 < 1$. Thus some of our forms will be reduced twice over. In the case of perpetuants the result of equating the different reductions was shewn to lead to a syzygy in every case. Now we shall find that it may lead to a syzygy or else it may lead to the reduction of a form not previously reduced.

Let us turn to equation (VI). Put $\lambda_3 = 0$ and use (V), thus

$$\text{(XII)}\quad R = (0, 0, \lambda_4)+\Sigma(-)^i \binom{\lambda_4-\rho_1+i-1}{i-1}\binom{\lambda_4}{\rho_1-i}(0, n_1+i, \rho_1-i)$$

$$-(0, 0, \lambda_4)-\Sigma(-)^i \binom{\lambda_4-\rho_1+i-1}{i-1}\binom{\lambda_4}{\rho_1-i}(n_1+i, 0, \rho_1-i),$$

giving a reduction for $(0, n_1+1, \lambda_4-n_1-1)$ instead of a syzygy when $\lambda_4 > n_1+1$; it should be noted here that $\lambda_4 \not> n$.

Now this is already reduced by (IX) since $2(n_1+1)+\lambda_4-n_1-1 \not> 2n$. Also we have a reduction for the form $(1, n_1, \lambda_4-n_1-1)$ which occurs in this equation from (VI) and (VIII). Thus we obtain a reduction for $(2, n_1-1, \lambda_4-n_1-1)$. This is the final reduction when $\lambda_4 > 2n_1$, but if $\lambda_4 \not> 2n_1$, we can use an equation of the type (XI), and so reduce the form $(3, n_1-2, \lambda_4-n_1-1)$. These forms were not reduced in § 30.

The reduction when $\lambda_3 = 1$ is given by (VII). To find what (VI) gives us in this case, put $\lambda_2 = 0$ in (VII) and use (VI) for each term, thus (assuming $\rho = 0$)

$$a_{1_x}^{n_1} a_{2_x}^{n_2} (a_3 a_4)^{\lambda_4}$$

$$= \Sigma(-)^i \binom{\lambda_4}{i}(0, i, \lambda_4-i)$$

$$= \Sigma(-)^i \binom{\lambda_4}{i}\left[\left(\frac{a_3^i a_4^{\lambda_4-i}}{a_1}\right) a_{2_x}^{n_2} - \Sigma(-)^j \binom{\lambda_4-i-\rho_1+j-1}{j-1}\binom{\lambda_4-i}{\rho_1-j}(0, n_1+j, \rho_1-j)\right]$$

$$= \Sigma(-)^i \binom{\lambda_4}{i}\left(\frac{a_3^i a_4^{\lambda_4-i}}{a_1}\right) a_{2_x}^{n_2},$$

since the coefficient of $(0, n_1+j, \rho_1-j)$ is zero. Thus in this case we only get a syzygy of a very obvious nature.

When ρ is not zero, we have only the case $\lambda_4 = n$, and then (VI) gives the reduction of $(0, 1, n)$ which has not been reduced by (VII).

When $\lambda_4 = 0$, (VI) only gives an obvious syzygy.

33. The equation (VIII) gives syzygies just as in the case of perpetuants when $\lambda_4 = 0$ or 1, or $\lambda_3 = 0$.

When $\lambda_3 = 1$, we reduced the equation in the perpetuant theory by

means of the syzygy

$$\{(a_2a_4)-(a_1a_3)\}^{\lambda_4+1}-\{(a_3a_4)-(a_1a_2)\}^{\lambda_4+1}=0.$$

This holds good as it stands when $\lambda_4 \not> n-1$, and $\lambda_4 \not> n_1-1$. But it still furnishes an identity when $\lambda_4 > n_1-1$ and $\lambda_4 \not> n-1$.

We write this identity

$$\sum_{i=0}^{n_1}(-)^i\binom{\lambda_4+1}{i}\left[(a_2a_4)^{\lambda_4+1-i}(a_1a_3)^i-(a_3a_4)^{\lambda_4+1-i}(a_1a_2)^i\right]$$

$$=(-)^{n_1}\left[\binom{\lambda_4+1}{n_1+1}\{(a_2a_4)^{\lambda_4-n_1}(a_1a_3)^{n_1+1}-(a_3a_4)^{\lambda_4-n_1}(a_1a_2)^{n_1+1}\}\right.$$
$$\left.-\binom{\lambda_4+1}{n_1+2}\{(a_2a_4)^{\lambda_4-n_1-1}(a_1a_3)^{n_1+2}-(a_3a_4)^{\lambda_4-n_1-1}(a_1a_2)^{n_1+2}\}+\dots\right].$$

Now, from (XII), we have (changing λ_4 into λ_4+1)

$$\binom{\lambda_4+1}{n_1+1}\{(0,\ n_1+1,\ \lambda_4-n_1)-(n_1+1,\ 0,\ \lambda_4-n_1)\}$$
$$-(n_1+1)\binom{\lambda_4+1}{n_1+2}\{(0,\ n_1+2,\ \lambda_4-n_1-1)-(n_1+2,\ 0,\ \lambda_4-n_1-1)\}+\dots=R.$$

Hence on subtraction we obtain a syzygy if $\lambda_4 \not> n_1+1$; and a reduction for

$$(0,\ n_1+2,\ \lambda_4-n_1-1),$$

when $$\lambda_4 \not> n-1.$$

The reduction equation is

$$\text{(XIII)}\quad \binom{\lambda_4+1}{n_1+1}\{[e^{-a_2D_4}-1](0,\ n_1+1,\ \lambda_4-n_1)$$
$$-[e^{-a_3D_4}-1](n_1+1,\ 0,\ \lambda_4-n_1)\}$$
$$-\binom{\lambda_4+1}{n_1+2}\{[e^{-a_2D_4}-(n_1+1)](0,\ n_1+2,\ \lambda_4-n_1-1)$$
$$-[e^{-a_3D_4}-(n_1+1)](n_1+2,\ 0,\ \lambda_4-n_1-1)\}+\dots=R.$$

With the help of (IX), this in general will reduce the form

$$(1,\ n_1+1,\ \lambda_4-n_1-1)$$

when $\lambda_4 > 2n_1$; but if otherwise we can use (XI) also and so reduce $(2,\ n_1,\ \lambda_4-n_1-1)$.

We must examine (XIII) further, owing to the presence of an excep-

tion. Expanding, we obtain

$$\text{(XIV)}\quad \sum_{i=1} (-)^i \binom{\lambda_4+1}{n_1+i}\left[\binom{n_1+i-1}{i-1}-1\right][(0,\ n_1+i,\ \lambda_4-n_1-i+1)$$
$$-(n_1+i,\ 0,\ \lambda_4-n_1-i+1)]$$
$$-\sum_{j=1}^{n_1}\sum_{i=0}^{\lambda_4-n_1-j}(-)^{i+j}\frac{(\lambda_4+1)!}{j!\,(n_1+1+i)!\,(\lambda_4-n_1-j-i)!}$$
$$[(j,\ n_1+1+i,\ \lambda_4-n_1-i-j)-(n_1+1+i,\ j,\ \lambda_4-n_1-i-j)] = R.$$

When $n_1 = 1$, the left-hand side of (XIV) becomes

$$\sum_{i=1}^{\lambda_4} (-)^i \binom{\lambda_4+1}{i+1}(i-1)\, e^{-a_2D_3-a_2D_4}(0,\ 1+i,\ \lambda_4-i).$$

And since $2(1+i)+\lambda_4-i \not> 2n$ (for $\lambda_4 \not> n-1$) we can use (X), and thus obtain a syzygy. This furnishes then no extra reduction when $n_1 = 1$.

We have yet to consider the case $\lambda_4 = n$, that is the equation

$$e^{-a_2D_4}\,(0,\ 1,\ n) = R.$$

34. The equation (IX) gives syzygies which are quite obvious when $\lambda_4 = 0$ or $\lambda_3 < 2$.

For $\lambda_3 = 2$, we use the syzygy

$$\{(a_1a_4)-(a_2a_3)\}^{\lambda_4+2}+\{(a_1a_4)+(a_2a_3)\}^{\lambda_4+2}$$
$$=\{(a_3a_4)+(a_1a_2)\}^{\lambda_4+2}+\{(a_1a_3)+(a_2a_4)\}^{\lambda_4+2},$$

which reduces the equation when

$$\lambda_4 \not> n-2 \quad\text{and}\quad \lambda_4 \not> n_1-2.$$

The equation exists only when $\lambda_4 \not> n_1$. We can shew then that this furnishes a syzygy whenever our equation exists and $\lambda_4 \not> n-2$. For

$$\begin{aligned}0 = {}& \{(a_1a_4)-(a_2a_3)\}^{\lambda_4+2}+\{(a_1a_4)+(a_2a_3)\}^{\lambda_4+2}\\ &-\{(a_3a_4)+(a_1a_2)\}^{\lambda_4+2}-\{(a_2a_4)+(a_1a_3)\}^{\lambda_4+2}\\ ={}& P+2(a_1a_4)^{\lambda_4+2}-(a_1a_2)^{\lambda_4+2}-(a_1a_3)^{\lambda_4+2}\\ &-(\lambda_4+2)(a_1a_2)^{\lambda_4+1}(a_3a_4)-(\lambda_4+2)(a_1a_3)^{\lambda_4+2}(a_2a_4)\end{aligned}$$

(where P is used here and elsewhere to denote products of covariants)

$$\begin{aligned}&= P+\{(a_1a_2)+(a_2a_4)\}^{\lambda_4+2}+\{(a_1a_3)+(a_3a_4)\}^{\lambda_4+2}-(a_1a_2)^{\lambda_4+2}\\ &\quad-(a_1a_3)^{\lambda_4+2}-(\lambda_4+2)(a_1a_2)^{\lambda_4+1}(a_3a_4)-(\lambda_4+2)(a_1a_3)^{\lambda_4+1}(a_2a_4)\\ &= P+(\lambda_4+2)(a_2a_3)\{(a_1a_2)^{\lambda_4+1}-(a_1a_3)^{\lambda_4+1}\}\\ &= P,\end{aligned}$$

giving us a syzygy for all cases $\lambda_3 = 2$, $\lambda_4 \not> n_1$.

For $\lambda_3 = 3$, we use the syzygy

$$\begin{aligned}0 = \; & \{(a_1a_4)+(a_2a_3)\}^{\lambda_4+3} - \{(a_1a_4)-(a_2a_3)\}^{\lambda_4+3} \\ & -\{(a_1a_3)+(a_2a_4)\}^{\lambda_4+3} + \{(a_1a_2)+(a_3a_4)\}^{\lambda_4+3} \\ & +\{(a_1a_3)-(a_2a_4)\}^{\lambda_4+3} - \{(a_1a_2)-(a_3a_4)\}^{\lambda_4+3} \\ = \; & P+2(\lambda_4+3)\{(a_1a_4)^{\lambda_4+2}(a_2a_3)+(a_1a_3)^{\lambda_4+2}(a_4a_2)+(a_1a_2)^{\lambda_4+2}(a_3a_4)\} \\ = \; & P+2(\lambda_4+3)\left[\{(a_1a_2)+(a_2a_4)\}^{\lambda_4+2}(a_2a_4)-\{(a_1a_3)+(a_3a_4)\}^{\lambda_4+2}(a_3a_4)\right. \\ & \qquad\qquad \left. -(a_1a_3)^{\lambda_4+2}(a_2a_4)+(a_1a_2)^{\lambda_4+2}(a_3a_4)\right] \\ = \; & P+4\binom{\lambda_4+3}{2}\left[(a_1a_2)^{\lambda_4+1}(a_2a_4)^2-(a_1a_3)^{\lambda_4+1}(a_3a_4)^2\right] \\ & +2\binom{\lambda_4+3}{1}\left[(a_1a_2)^{\lambda_4+2}\{(a_2a_4)+(a_3a_4)\}-(a_1a_3)^{\lambda_4+2}\{(a_2a_4)+(a_3a_4)\}\right].\end{aligned}$$

If $\lambda_4 = n_1$, we obtain

$$\begin{aligned}&(n_1+2)\{(n_1+1,\,0,\,2)-(0,\,n_1+1,\,2)\} \\ &-2(n_1+1)\{(n_1+2,\,0,\,1)-(0,\,n_1+2,\,1)\} = P.\end{aligned}$$

And using (XII) we find we have a relation between products of covariants only, *i.e.*, a syzygy.

If $\lambda_4 = n_1-1$, we obtain

$$(n_1+1,\,0,\,1)-(0,\,n_1+1,\,1) = P,$$

and again using (XII) we have a syzygy.

If $\lambda_4 < n-1$, there is a syzygy without the help of (XII). Thus we obtain a syzygy from (IX) in every case when $\lambda_3 < 4$, or $\lambda_4 < 1$, provided $\lambda_4 \not> n-2$.

We have still to consider the cases $\lambda_4 = n-1$ or n. In fact we have to consider the four equations

$$e^{-a_2D_3}(0,\,2,\,n-1) = R, \quad e^{-a_2D_3}(0,\,3,\,n-1) = R,$$

$$e^{-a_2D_3}(0,\,2,\,n) = R, \qquad e^{-a_2D_3}(0,\,3,\,n) = R,$$

where it must be remembered in each case that $\lambda_4 \not> n_1$.

35. The equation (X) gives obvious syzygies when $\lambda_3 < 2$ or $\lambda_4 < 2$. For the other cases the syzygies

$$\{(a_2a_4)-(a_2a_3)\}^w = (a_3a_4)^w,$$

and $\{(a_2a_4)-(a_2a_3)\}^{w-1}(a_2a_3)=-(a_3a_4)^{w-1}(a_1a_2)-(a_3a_4)^{w}+(a_3a_4)^{w-1}(a_1a_4)$,

may be used, as for perpetuants; provided the weight w is not greater than n. When the weight is greater than n we find ourselves with five equations to deal with of just the same type as those of equation (IX).

We are thus left with ten equations to consider, four of weight $n+1$, four of weight $n+2$, and two of weight $n+3$.

36. For weight $n+1$ the equations, in the case of perpetuants were reduced by means of syzygies obtained from the symbolical identity

$$\text{(XV)}\quad A_1(n+1)\left[(a_2a_3)\{(a_2a_4)-(a_2a_3)\}^n-(a_1a_3)\{(a_1a_4)-(a_1a_3)\}^n+(a_1a_2)(a_3a_4)^n\right]$$
$$+A_2\left[\{(a_2a_4)-(a_2a_3)\}^{n+1}-(a_3a_4)^{n+1}\right]$$
$$+A_3\left[\{(a_3a_4)-(a_1a_2)\}^{n+1}-\{(a_2a_4)-(a_1a_3)\}^{n+1}\right]$$
$$+A_4\left[\{(a_3a_4)+(a_1a_2)\}^{n+1}-\{(a_1a_4)-(a_2a_3)\}^{n+1}\right]$$
$$+A_5\left[\{(a_2a_4)+(a_1a_3)\}^{n+1}-\{(a_1a_4)+(a_2a_3)\}^{n+1}\right]$$
$$+(A_2-A_3-A_4)\left[(a_3a_4)^{n+1}-\{(a_1a_4)-(a_1a_3)\}^{n+1}\right]$$
$$+(-A_2+A_3-A_5)\left[(a_2a_4)^{n+1}-\{(a_1a_4)-(a_1a_2)\}^{n+1}\right]$$
$$+(-)^n\{-A_1(n+1)+A_2-A_4+(-)^nA_5\}\left[(a_2a_3)^{n+1}-\{(a_1a_3)-(a_1a_2)\}^{n+1}\right]=0.$$

From § 25 we see that

$$\{(a_2a_4)-(a_2a_3)\}^{n+1}$$
$$=\sum_{i=1}^{n}(-)^i\binom{n+1}{i}(a_2a_3)_i(a_2a_4)_{n+1-i}+(a_2a_4)^{n+1}+(a_3a_2)^{n+1}-\{1-(-)^n\}(a_2a_3)^n(a_2a_4);$$

and that

$$(a_2a_3)\{(a_2a_4)-(a_2a_3)\}^n$$
$$=\sum_{i=0}^{n-1}(-)^i\binom{n}{i}(a_2a_3)_{i+1}(a_2a_4)_{n-i}-(a_3a_2)^{n+1}-(-)^n(a_2a_3)^n(a_2a_4),$$

where $(a_2a_3)_i(a_2a_4)_{n+1-i}$ is an actual covariant of the three quantics concerned: in these we replace

$$(a_2a_3)^n(a_2a_4)\quad\text{by}\quad(a_2a_3)^n(a_1a_4)-\{(a_1a_3)-(a_1a_2)\}^n(a_1a_2),$$

and then substitute in (XV).

In the result the coefficient of each of $(a_3a_4)^{n+1}$, $(a_2a_4)^{n+1}$, $(a_1a_4)^{n+1}$, $(a_2a_3)^{n+1}$, $(a_1a_3)^{n+1}$ is zero. And, in fact, the identity is a syzygy as it stands for all values of the five constants when $n_1 \geqslant n$.

If $n_1 \not> n$, we need the following results from § 25,

$$(a_1a_3)\{(a_1a_4)-(a_1a_3)\}^n-\Sigma(-)^i\binom{n}{i}(a_1a_3)_{i+1}(a_1a_4)_{n-i}+(a_3a_1)^{n+1}$$

$$=(-)^n\sum_{i=1}^{n-n_1+1}(-)^i\binom{-n-3+n_1+i}{i-1}\binom{-1}{n+1-n_1-i}(a_1a_3)^{n_1+i-1}(a_1a_4)^{n+2-n_1-i}$$

$$=(-)^{n_1}\sum_{i=1}^{n-n_1+1}(-)^i\binom{n+1-n_1}{i-1}(a_1a_3)^{n_1+i-1}(a_1a_4)^{n+2-n_1-i};$$

and

$$\{(a_1a_4)-(a_1a_3)\}^{n+1}-\Sigma(-)^i\binom{n+1}{i}(a_1a_3)_i(a_1a_4)_{n+1-i}-(a_1a_4)^{n+1}-(a_3a_1)^{n+1}$$

$$=\sum_{i=1}^{n-n_1+1}(-)^i\left[\binom{n_1+i-2}{i-1}\binom{n}{n_1+i-1}-(-)^{n_1}\binom{n+1-n_1}{i-1}\right]$$
$$\times(a_1a_3)^{n_1+i-1}(a_1a_4)^{n+2-n_1-i}.$$

Making use of these results in (XV), and of the corresponding result for $\{(a_1a_4)-(a_1a_2)\}^{n+1}$, we obtain from (XV) in the notation of this paper,

(XVI)

$$-A_1(n+1)(-)^{n_1}\sum_{i=1}^{n-n_1+1}(-)^i\binom{n+1-n_1}{i-1}(0,\ n_1+i-1,\ n+2-n_1-i)$$

$$-(A_2-A_3-A_4)\sum_{i=1}^{n-n_1+1}(-)^i\left[\binom{n_1+i-2}{i-1}\binom{n}{n_1+i-1}-(-)^{n_1}\binom{n+1-n_1}{i-1}\right]$$
$$\times(0,\ n_1+i-1,\ n+2-n_1-i)$$

$$-(A_2-A_3+A_5)\left\{\binom{n}{n_1}-(-)^{n_1}\right\}(n_1,\ 0,\ n+1-n_1)$$

$$+\sum_{i=1}^{n-n_1}\binom{n+1}{n_1+i}\{A_5-(-)^{n_1+i}A_3\}\,e^{-a_2D_4}(0,\ n_1+i,\ n+1-n_1-i)$$

$$-\sum_{i=1}^{n-n_1}\binom{n+1}{n_1+i}\{A_5+(-)^{n_1+i}A_4\}\,e^{-a_2D_3}(0,\ n+1-n_1-i,\ n_1+i)=P.$$

It is evident that we only get syzygies (with the help of the regular equations) when $n_1=n$.

In general when all the A's except A_1 are zero, we find

$$(0,\ n_1,\ n+1-n_1)-(n+1-n_1)(0,\ n_1+1,\ n-n_1)+\ldots=R.$$

From $A_1 = A_4 = A_5 = 0$, $A_2 = A_3$, we obtain

$$\binom{n+1}{n_1+1} e^{-a_2 D_4}(0, n_1+1, n-n_1)$$

$$-\binom{n+1}{n_1+2} e^{-a_2 D_4}(0, n_1+2, n-n_1-1)+\ldots = R.$$

From

$$A_3 = A_4 = A_5 = 0 \quad \text{and} \quad A_1(n+1)(-)^{n_1} + A_2 \left\{ \binom{n}{n_1} - (-)^{n_1} \right\} = 0,$$

we obtain $$\left\{ \binom{n}{n_1} - (-)^{n_1} \right\} (n_1, 0, n+1-n_1)$$

$$+\left[n_1 \binom{n}{n_1+1} - (n+1-n_1)\binom{n}{n_1} \right] (0, n_1+1, n-n_1) + \ldots = R.$$

This is all we get in general, for (XVI) does not help us, as a rule, unless A_4 and A_5 are zero, as it is easy to see. Further the three equations are all we require; since when $n_1 < n-2$, the equations

$$e^{-a_2 D_3}(0, 2, n-1) = R, \qquad e^{-a_2 D_3}(0, 3, n-2) = R$$

no longer exist.

Thus, when $n_1 < n-2$, we find three new reductions which easily may be shewn to be those for $(n_1, 1, n-n_1)$, $(2, n_1-1, n-n_1)$, the third being $(1, n_1, n-n_1)$, if $n > 2n_1-1$; but, if $n \leqslant 2n_1-1$, this is already reduced and our third reduction is $(3, n_1-2, n-n_1)$.

When $n_1 = n-1$, we have to look for five reductions or syzygies; the three equations obtained for the general case enable us to express each of $(0, n, 1)$, $(0, n-1, 2)$, and $(n-1, 0, 2)$ as a sum of products. Substitute their values in (XVI); and it reduces to

$$-\binom{n+1}{n} \{A_5 + (-)^n A_4\} e^{-a_2 D_3}(0, 1, n) = P.$$

But using (VI) we find that

$$\begin{aligned} e^{-a_2 D_3}(0, 1, n) &= (0, 1, n) - (1, 0, n) \\ &= (n-1)(0, n-1, 2) - (n-2)(0, n, 1) - (n-1)(n-1, 0, 2) + P \\ &= P. \end{aligned}$$

And hence the extra equations give two syzygies here.

When $n_1 = n-2$, we have one extra equation to obtain.

It is plain that we must have $A_5 - (-)^n A_4 = 0$ in (XVI). We find our

equation by putting $A_1 = A_3 = 0$, $A_2 = A_4 = (-)^n$, $A_5 = 1$. And by means of this we can obtain the reduction of the extra form $(2, n-2, 1)$.

37. For weight $n+2$, we find that the equation

$$e^{-a_2D_3-a_2D_4}(0, 2, n) = R$$

is required for the ordinary reductions, unless $n_1 \geqslant n$. The equation

$$e^{-a_2D_3}(0, 2, n) = R$$

exists only when $n_1 \geqslant n$, and is then required for the reduction of $(1, 1, n)$. The equation

$$e^{-a_2D_3}(0, 3, n-1) = R$$

exists only when $n_1 \geqslant n-1$; and

$$e^{-a_2D_3-a_2D_4}(0, 3, n-1) = R,$$

which exists only when $n \geqslant 5$ is required for ordinary reductions when $n_1 < 4$.

Thus we require three reductions or syzygies when $n_1 \geqslant n$, two when $n_1 = n-1$, one only when $n-1 > n_1 > 3$, and none when $n_1 \leqslant 3$.

We replace n by $n+1$ in (XV); and observing that by § 25 we have

$$\begin{aligned}\{(a_2a_4)-(a_2a_3)\}^{n+2} &= \sum_{i=2}^{n}(-)^i\binom{n+2}{i}(a_2a_3)_i(a_2a_4)_{n+2-i}+(a_2a_4)^{n+2}-(n+2)(a_2a_4)^{n+1}(a_2a_3)\\ &\quad+(a_3a_2)^{n+2}+(n+2)(a_3a_2)^{n+1}(a_2a_4)\\ &\quad+\{n^2-1-(-)^n(2n+3)\}(a_2a_3)^{n-1}(a_2a_4)^3\\ &\quad-\{n^2-n-3-(-)^n(3n+4)\}(a_2a_3)^n(a_2a_4)^2,\end{aligned}$$

and

$$\begin{aligned}(a_2a_3)\{(a_2a_4)-(a_2a_3)\}^{n+1} &= \sum_{i=1}^{n-1}(-)^i\binom{n+1}{i}(a_2a_3)_{i+1}(a_2a_4)_{n+1-i}+(a_2a_3)(a_2a_4)^{n+1}\\ &\quad-(a_3a_2)^{n+2}-(n+1)(a_3a_2)^{n+1}(a_2a_4)\\ &\quad-\{n-1-(-)^n(2n+1)\}(a_2a_3)^{n-1}(a_2a_4)^3\\ &\quad+\{n-2-(-)^n(3n+1)\}(a_2a_3)^n(a_2a_4)^2.\end{aligned}$$

In these we replace

$$(a_2 a_3)^{n-1} (a_2 a_4)^3$$

by $(a_2 a_3)^{n-1}(a_1 a_4)^3$

$$-\{(a_1 a_3)-(a_1 a_2)\}^{n-1}\{3(a_1 a_2)(a_1 a_4)^2-3(a_1 a_2)^2(a_1 a_4)+(a_1 a_2)^3\},$$

and

$$(a_2 a_3)^n (a_2 a_4)^2$$

by $$(a_2 a_3)^n (a_1 a_4)^2-\{(a_1 a_3)-(a_1 a_2)\}^n \{2(a_1 a_2)(a_1 a_4)-(a_1 a_2)^2\},$$

and then substitute in our new identity.

In order that the identity may yield a relation between actual covariants, the constants must satisfy the conditions

$$\text{(XVII)} \qquad A_1-A_2+A_3+A_5=0, \qquad A_1-A_3+A_4=0.$$

When $n_1 \geqslant n+2$, we find if n is even no syzygies, but reductions for the forms $(3, n-3, 2)$, $(2, n-1, 1)$, $(3, n-2, 1)$; and if n is odd there is a syzygy and the forms $(3, n-3, 2)$, $(2, n-1, 1)$ only are reducible.

When $n_1 = n+1$, and n is even, our identity furnishes reductions for $(3, n-3, 2)$, $(1, n, 1)$ and $(2, n-1, 1)$; but when n is odd there is a syzygy and reductions only for $(3, n-3, 2)$ and $(1, n, 1)$.

When $n_1 = n$, there are no syzygies, the reductions are $(n-1, 1, 2)$, $(2, n-2, 2)$, $(3, n-3, 2)$, when n is even, and $(2, n-2, 2)$, $(3, n-3, 2)$, $(1, n, 1)$ when n is odd.

When $n_1 = n-1$, we expect only two results from our identity, and we find that the constants must satisfy the additional condition $A_4+A_5 = 0$. And whether n is odd or even we find the new reductions to be $(3, n-4, 3)$ and $(1, n-1, 2)$.

When $n_1 < n-1$, we have one reduction only to look for, and we must have $A_4 = 0 = A_5$; and therefore $2A_1 = A_2 = 2A_3$. We find then a reduction for $(3, n_1-3, n-n_1+2)$, when $n_1 \nless \frac{n+3}{2}$, but for $(2, n_1-2, n-n_1+2)$, when $n_1 < \frac{n+3}{2}$; and no new reduction at all when $n_1 < 4$.

38. Lastly, when the weight is $n+3$, we find that the equation

$$e^{-a_2 D_3}(0, 3, n) = R,$$

which only exists when $n_1 \geqslant n$, is always required for the reduction of $(1, 2, n)$. The equation

$$e^{-a_2 D_3 - a_2 D_4}(0, 3, n) = R$$

is also required for the ordinary reductions unless $n_1 \geqslant n \geqslant 6$. To obtain the reduction or syzygy corresponding to this last case, we replace n by $n+2$ in (XV) and proceed as before; then we find that the constants must satisfy the two conditions (XVII), and also the further conditions $A_3+A_4=0$ and $A_3-A_5=0$; whence

$$\frac{A_1}{2}=\frac{A_2}{4}=\frac{A_3}{1}=\frac{A_4}{-1}=\frac{A_5}{1}.$$

When $n_1 > n$ *and* n *is odd*, the form $(3, n-4, 4)$ is reduced.

When $n_1 > n+1$ *and* n *is even*, the form $(3, n-3, 3)$ is reduced.

When $n_1 = n+1$ *and* n *is even*, the form $(2, n-2, 3)$ is reduced.

When $n_1 = n$, the form $(2, n-3, 4)$ is reduced, whether n be even or odd.

39. We can now sum up our results. As before stated, $(0, \lambda_3, \lambda_4)$ is reducible unless

$$\lambda_3 > n_1, \quad \lambda_4 > n_1, \quad \text{and} \quad 2\lambda_3+\lambda_4 > 2n;$$

it is therefore always reducible when $n_1 \geqslant n$.

The reducibility limits of $(1, \lambda_3, \lambda_4)$ are illustrated in Fig. 1; where

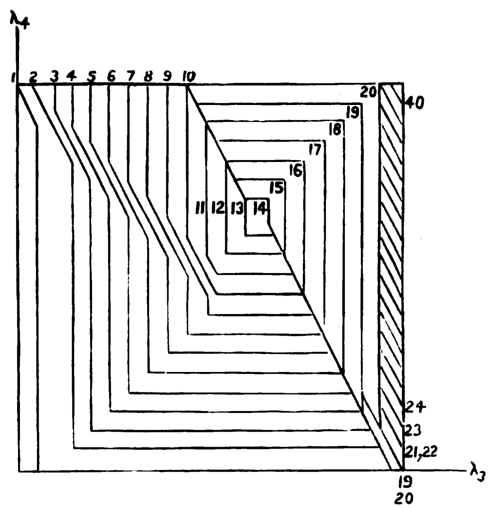

FIG. 1.

contours are drawn for different values of n_1 when $n = 20$, the form corresponding to any point (λ_3, λ_4) either on or on the origin side of the contour being reducible. The character of the contour changes according

to the value of n_1; thus the reducibility limits are, when

(i) $n_1 \leqslant \frac{n}{2}$, $\lambda_3 \not> n_1-1$ or $\lambda_3 = n_1$ or n_1+1 and $2\lambda_3+\lambda_4 \not> n+n_1$

or $\lambda_4 \not> n_1-1$, $2\lambda_3+\lambda_4 \not> 2n-1$.

(ii) $\frac{2n}{3} \geqslant n_1 > \frac{n}{2}$, $\lambda_3 \not> n_1-1$, $2\lambda_3+\lambda_4 \not> 2n-2$,

or $\lambda_4 \not> n_1-1$, $2\lambda_3+\lambda_4 \not> 2n-1$.

(iii) $n-2 \geqslant n_1 > \frac{2n}{3}$, $2\lambda_3+\lambda_4 \not> 2n-2$ or $\lambda_3 \not> n_1-1$, $\lambda_4 \not> n_1$

or $\lambda_4 \not> n_1-1$, $2\lambda_3+\lambda_4 \not> 2n-1$.

(iv) $n_1 = n-1$, a modification is introduced owing to the reducibility of $(1, n-1, 2)$; we have then

$n_1 = n-1$ or n, $2\lambda_3+\lambda_4 \not> 2n-2$ or $\lambda_3 \not> n_1-1$, $\lambda_4 \not> n_1$

or $\lambda_4 \not> n_1-1$, $2\lambda_3+\lambda_4 \not> 2n$.

(v) $n_1 = n+1$, $\lambda_3 \not> n-1$ or $2\lambda_3+\lambda_4 \not> 2n+1$.

(vi) $n_1 > n+1$, $\lambda_3 \not> n-1$ or $2\lambda_3+\lambda_4 \not> n+n_1-1$.

(vii) $n_1 > 2n$, every form is reducible.

The reducibility limits of $(2, \lambda_3, \lambda_4)$ and $(3, \lambda_3, \lambda_4)$ are traced in Figs. 2 and 3. It will be seen that in both these cases there is part

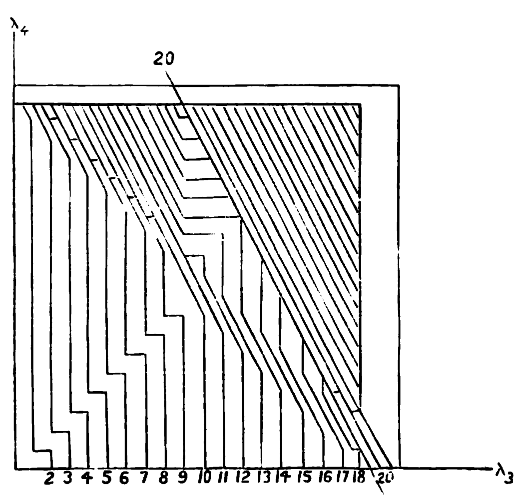

FIG. 2.

of the figure which corresponds to forms irreducible for all values of n_1.

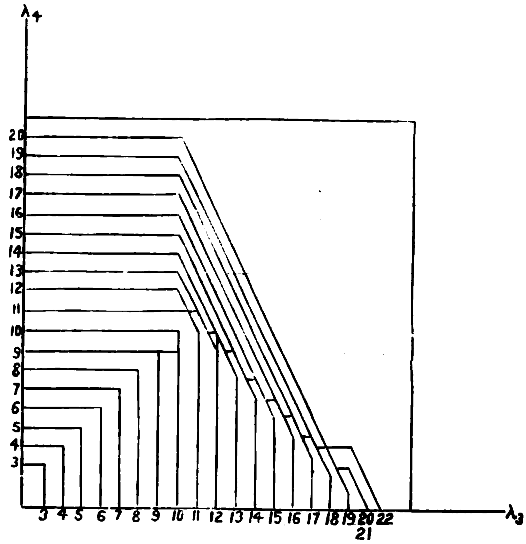

FIG. 3.

40. It is noteworthy that our special cases introduce the reductions of $(n_1, 1, n-n_1)$ when $n_1 < n$; and of $(n-1, 1, 2)$ when $n_1 = n$, and is even, which must be added to the reductions given in § 29.

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[*The names of Authors are given and the numbers of the pages on which the titles of the papers appear. An Abstract of a paper will usually be found on a page soon after that containing the title.*]

INDEX OF PAPERS PUBLISHED IN THIS VOLUME ARRANGED ACCORDING TO NAMES OF AUTHORS.

[*The numbers refer to pages.*]

INDEX ACCORDING TO SUBJECTS.

[*The numbers refer to pages.*]

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The London Mathematical Society.

RECORDS OF PROCEEDINGS AT MEETINGS

SESSION NOVEMBER, 1913–JUNE, 1914.

Thursday, November **13*th*, 1913.**

ANNUAL GENERAL MEETING.

Prof. A. E. H. LOVE, President, in the Chair.

Present twenty-three members and a visitor.

The minutes of the last meeting were read and confirmed.

Messrs. A. Korn and R. E. Powers were elected members.

Profs. A. Hurwitz, M. Noether, P. Painlevé, C. Segre, W. Voigt were elected honorary members.

The Treasurer, Sir Joseph Larmor, presented his Report, and on the motion of Lt.-Col. Cunningham, seconded by Mr. A. B. Grieve, it was received.

Dr. J. G. Leathem was reappointed Auditor.

On the motion of the President, seconded by Sir Joseph Larmor, it was agreed that a letter of condolence be sent to the relations of the late Mr. S. Roberts, a former President of the Society.

The President moved, and Dr. Hobson seconded, the following resolution, which was carried unanimously :—

That the London Mathematical Society hereby places on record its sense of the deep debt of gratitude which it owes to Sir Joseph Larmor for his management of its financial affairs during the twenty-one years of his tenure of the office of Treasurer, and tenders to him its hearty thanks for the work that he has done in that capacity.

The members of Council and Officers for the new Session were elected as follows:—President, Prof. A. E. H. Love; Vice-Presidents, Dr. H. F. Baker and Prof. W. Burnside; Treasurer, Dr. A. E. Western; Secretaries, Mr. J. H. Grace and Dr. T. J. I'A. Bromwich; other members of the Council, Mr. S. Chapman, Mr. E. Cunningham, Mr. A. L. Dixon, Prof. L. N. G. Filon, Prof. E. W. Hobson, Mr. J. H. Jeans, Mr. J. E. Littlewood, Prof. H. M. Macdonald, Major P. A. MacMahon, Mr. H. W. Richmond.

The following papers were communicated:—

The Skew Isogram Mechanism: Mr. G. T. Bennett.

Tauberian Theorems concerning Power Series and Dirichlet's Series whose Coefficients are Positive: Messrs. G. H. Hardy and J. E. Littlewood.

Note on Lambert's Series: Mr. G. H. Hardy.

(i) The Connexion between Surfaces whose Lines of Curvature are Spherical and Surfaces whose Inflectional Tangents belong to Linear Complexes, (ii) Surfaces whose Systems of Inflectional Tangents belong to Systems of Linear Complexes: Mr. J. E. Campbell.

On Integration with respect to a Function of Bounded Variation: Prof. W. H. Young.

The Computation of Cotes's Numbers, and their Values up to $n = 20$: Prof. W. W. Johnson.

Some Ruler Constructions for the Covariants of a Binary Quantic: Mr. S. G. Soal.

Analogues of Orthocentric Tetrahedra in Higher Space: Mr. T. C. Lewis.

Closed Linkages and Poristic Polygons: Col. R. L. Hippisley.

ABSTRACTS.

Tauberian Theorems concerning Power Series and Dirichlet's Series whose Coefficients are Positive: Messrs. G. H. Hardy and J. E. Littlewood.

It was proved by Lasker and Pringsheim that, if

$$f(x) = \Sigma a_n x^n,$$

and $$s_n = a_0 + a_1 + \ldots + a_n \sim A n^a L(n),$$

where $$L(n) = (\log n)^{a_1} (\log \log n)^{a_2} \ldots,$$

the indices $a, a_1, a_2, \ldots$ being such that $n^a L(n) \to \infty$, then

$$f(x) \sim \frac{A\Gamma(a+1)}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right)$$

as $x \to 1$. The principal object of the paper is to show that *the converse also is true when the coefficients a_n are positive.* Various generalisations are made, and analogous theorems proved for ordinary Dirichlet's series.

Note on Lambert's Series: Mr. G. H. Hardy.

Generalisation of theorems proved by the author (*Math. Annalen*, Vol. 64) and Knopp (*Crelle's Journal*, Vol. 142) concerning the behaviour of a "Lambert's series"

$$\Sigma a_n \frac{x^n}{1-x^n},$$

convergent for $|n| < 1$, as x tends to a rational point $e^{2p\pi i/q}$ on the circle of convergence.

(i) The Connexion between Surfaces whose Lines of Curvature are Spherical and Surfaces whose Inflectional Tangents belong to Linear Complexes, (ii) Surfaces whose Systems of Inflectional Tangents belong to Systems of Linear Complexes: Mr. J. E. Campbell.

Much study has been devoted to the system of surfaces characterized by the property that their lines of curvature are plane or spherical, and references will be made in Forsyth's *Differential Geometry*, Chapter IX. My own knowledge of the subject is almost entirely derived from Darboux's *Théorie Générale des Surfaces*, Livre IV, Ch. IX and XI, and I have been guided in my investigations by the results there presented. It is well known that Lie's contact transformation, by which spheres are transformed into straight lines, connects the geometry of lines of curvature on the surface with the geometry of asymptotic lines on another surface. I do not know, however, that any attempt has been made to apply this theory to the investigation of the properties of surfaces, whose lines of curvature are spherical, by investigating the properties of surfaces, whose inflectional tangents belong to systems of linear congruences. The first of the two papers here presented treats of Lie's contact transformation, and shows how the knowledge of either class of surface involves that of the other. The second investigates the properties of surfaces whose inflectional tangents belong to systems of linear complexes. The two papers are connected, but either can be read independently of the other. I have

tried to write them that they may be intelligible to one who has very little knowledge of the theory of surfaces, and therefore there is much in them, and especially in the first, that lays no claim to originality, except possibly in method.

The Computation of Cotes's Numbers and their Values up to $n = 20$: Prof. W. W. Johnson.

This paper contains the computed values of Cotes's numbers for values of n from 11 to 20, in continuation of the values up to $n = 10$ given by Cotes, and published in the *Harmonia Mensurarum*.

Attention is called to some curious cases, occurring in the computation of the divisibility of certain sums, whereby the denominators of the fractional values are rendered considerably smaller than might have been expected.

Some Ruler Constructions for the Covariants of a Binary Quantic: Mr. S. G. Soal.

In this paper a binary quantic and its covariants are represented by points which are taken to lie on a conic S.

It is first shown how to determine by ruler the polars of any point P of S with respect to the system of conics which pass through the marking points of the $C_{2,4}$ of a sextic.

One conic of the system has for a pole and polar pair any point P of S, and the line which denotes the $C_{2,2}$ of the polar 5-points of P with respect to the sextic.

The polar reciprocal with respect to S of this conic intersects S in the marking points of the covariant $C_{4,4}+I_2C_{2,4}$.

The next step is to construct a certain joint quartic covariant of a quadratic and a sextic linear in the coefficients of each.

If for the quadratic is taken the unique $C_{3,2}$ of the sextic, this joint covariant becomes the $C_{4,4}+I_2C_{2,4}$ determined above.

The reversal of this process leads to a ruler construction for the unique $C_{3,2}$ of the sextic.

The remaining quadratic and quartic covariants are then readily constructed.

In connection with the sextic the following result is of interest :—If the connector of the polar 2-points of each of the marking points of a sextic with respect to the remaining five points be constructed, then the

six lines so obtained touch a conic which is harmonically circumscribed to a system of covariant conics associated with the Hessian of the sextic.

Next, using the septimic and octavic as illustrations, there is indicated a chain of ruler constructions for the $C_{2,2}$ of a $(2n+1)$-ic based upon the two following results:—

(i) If P be one of the marking points of a $2n$-ic, and p be the line which represents the $C_{2,2}$ of the polar $(2n-3)$-points of P with respect to the remaining $(2n-1)$-points, then $[P, p]$ is a pole and polar pair with respect to a member of the system of conics through the unique $C_{2,4}$ of the $2n$-ic.

(ii) If P be one of the marking points of a $(2n+1)$-ic, and Q be the conjugate of P with respect to the four-point system through the unique $C_{2,4}$ of the remaining $2n$ points, then Q is a point of the line which represents the $C_{2,2}$ of the $(2n+1)$-ic.

Prof. Morley's elegant construction for the $C_{2,2}$ of a quintic may be exhibited as a special case of this result.

Analogues of Orthocentric Tetrahedra in Higher Space: Mr. T. C. Lewis.

The results obtained geometrically in the author's paper published in the September and October numbers of the *Proceedings* of the Society, Vol. 12, pp. 474–483, are now proved analytically by the use of penta-spherical coordinates, or corresponding coordinates when the space considered is of more than three dimensions. These coordinates are explained by M. Gaston Darboux in *La Théorie Générale des Surfaces*, Livre II, Ch. VI, p. 213; see also the same writer's *Systèmes Orthogonaux et les Coordonnées Curvilignes*, Livre I, Ch. VI, p. 121.

An independent investigation of this system of coordinates is given, based on its connexion with an orthocentric tetrahedron or higher analogue, a connexion not noted by M. Gaston Darboux, but one which makes the system naturally suitable for application to the study of the geometry of such orthocentric figures in space of any dimensions. The application of the method for this purpose is believed to be new.

While the homogeneous equation of the first degree represents an n-sphere or n-plane, the homogeneous equation of the second degree will in general represent a (hyper)-cyclide which reduces to a (hyper)-quadric on the fulfilment of certain conditions. This opens out a further field of investigation, which is being pursued.

Closed Linkages and Poristic Polygons: Col. R. L. Hippisley.

This is an article pointing out certain consequences arising out of the connection between closed linkages and poristic polygons which was briefly alluded to by the author in a previous paper (*Proc. London Math. Soc.*, Ser. 2, Vol. 11, p. 29). The results arrived at have a special bearing on the "Theory of the Transformation of Elliptic Functions." It is shown that the subsidiary polygons formed by joining the vertices of the original polygon in every possible way are connected together by an endless chain of linkages and also by a double system of inversions.

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RECORDS OF PROCEEDINGS AT MEETINGS

SESSION NOVEMBER, 1913–JUNE, 1914.

Thursday, December 11th, 1913.

Prof. A. E. H. LOVE, President, in the Chair.

Present twenty members and a visitor.

The minutes of the last meeting were read and confirmed.

Mr. T. L. Wren was nominated for membership.

Mr. S. B. Maclaren was elected a member.

The President presented the Treasurer's Report. On the motion of Lt.-Col. Cunningham, seconded by Prof. Nicholson, the Report was received.

The President alluded to the deaths of Sir R. S. Ball, Prince Camille de Polignac, and Mr. Morgan Jenkins, the last of whom was for nearly thirty years Secretary to the Society. It was unanimously agreed that a letter of condolence should be sent to the widow of Mr. Jenkins.

The following papers were communicated:—

On the Linear Integral Equation: Prof. E. W. Hobson.

Generalized Hermite Functions and their Connexion with the Bessel Functions: Mr. H. E. J. Curzon.

Limiting Forms of Long Period Tides: Mr. J. Proudman.

On the Number of Primes of the same Residuacity: Lt.-Col. Cunningham.

Some Results on the Form near Infinity of Real Continuous Solutions of a certain Type of Second Order Differential Equation: Mr. R. H. Fowler.

The Potential of a Uniform Convex Solid possessing a Plane of Symmetry with Application to the Direct Integration of the Potential of a Uniform Ellipsoid: Dr. S. Brodetsky.

The Dynamical Theory of the Tides in a Polar Basin: Mr. G. R. Goldbrough.

Proof of the Complementary Theorem: Prof. J. C. Fields.

ABSTRACTS.

On the Linear Integral Equation: Prof. E. W. Hobson.

In this paper the integral equation

$$f(s) = \phi(s) - \lambda \int_a^b K(s, t)\, \phi(t)\, dt$$

is treated of with a view to removing as far as possible the restrictions on the nature of the nucleus $K(s, t)$ and the given function $f(s)$. Throughout, the definition of a definite integral due to Lebesgue is employed. When $K(s, t)$ is summable and limited in the square for which it is defined, it is shewn that Fredholm's solution is the only summable solution of the integral equation. The case in which $K(s, t)$ and a finite number of the repeated nuclei are unlimited is investigated by a method in which Lalesco's theory of the canonical sub-groups of the resolvant of a limited nucleus is employed; a more general result than that obtained by Poincaré is thus established. Certain cases are considered in which Fredholm's solution is applicable when the nucleus and all the repeated nuclei are unlimited.

Generalized Hermite Functions and their connexion with the Bessel Functions: Mr. H. E. J. Curzon.

This forms the sequel to a preceding paper in which a connexion was found between the Hermite functions and the Legendre functions. In a memoir, read before the Royal Society in 1908, Mr. Cunningham discusses the properties of certain ω-functions occurring as the result of

searching for all solutions of

$$\nabla^2 u = \frac{\partial u}{\partial t}$$

that have the form $f(t)\,\phi(r^2/t)\,\Theta$, where Θ depends only on angular coordinates. Or, slightly transforming the fundamental differential equation for these ω-functions, an equation that arises is

$$\frac{d^2z}{d\xi^2}+\frac{2\mu}{\xi}\frac{dz}{d\xi}-2\xi\frac{dz}{d\xi}+2(\nu-\mu)z=0, \tag{i}$$

an equation that deforms into Hermite's equation on making μ zero. In the present paper this equation (i) is solved by means of two definite integrals which I call $H_{\nu,\mu}(\xi)$ and $K_{\nu,\mu}(\xi)$, a function $W_{\nu,\mu}(\xi)$ being related to these functions when they are not independent solutions of (i) in the same way that the function $Y_n(x)$ is related to the ordinary Bessel functions when n is integral. Connexions are established between the generalised Hermite functions and the Bessel functions of the type

$$H_{\nu,\mu}(x)=\frac{e^{\frac{1}{2}\pi i(\nu-\frac{1}{2})}\,\Gamma\left(\dfrac{z+\mu+1}{\nu}\right)}{\pi x^{\mu-\frac{1}{2}}}\int_{\infty}^{-\infty}e^{-t^2}\,t^{-\nu-\frac{1}{2}}\,J_{\mu-\frac{1}{2}}(-2xti)\,dt,$$

the path of integration avoiding the origin by means of a small semicircle above the real axis. The relation is worked out between the H and K functions and Mr. Cunningham's ω-function, and various other properties of these generalised Hermite functions are discussed, apart from their connexions with the Bessel functions.

Limiting Forms of Long Period Tides: Mr. J. Proudman.

This paper contains a general discussion of the limiting forms of forced tides on a rotating globe, as the period of the disturbing force tends to become infinite, and of the use of such as approximations to tides of long period.

The method adopted for the former purpose is to use the limiting forms of the general equations of forced motion, and to add as conditions any properties of the general motion which are independent of the period, so long as it is finite, but do not follow as consequences of the limiting forms of the equations themselves. These properties are obtained and used in a form which involves "relative circulations."

The area of the surface of the water is divided into regions of three different types, according to the properties of $h \sec\theta$ (h being the depth

and θ the co-latitude). These regions are first discussed separately, and then the solutions are combined.

In all cases but one the limiting forms are found to exist uniquely.

A few examples are given in which the limiting forms are calculated for such disturbing forces as occur in terrestrial tides. The chief of these refers to a polar sea bounded by any parallel of latitude.

In the discussion of the possibility of applying limiting forms as approximations, an attempt is made to apply to tidal theory results which have been established only for systems with a finite number of degrees of freedom.

Using the tide-tables for the Indian ports it is concluded that free oscillations of long period may exist, as the theory is found not to apply.

A suggestion is made to use the Lake Victoria Nyanza to determine the amount of the earth's yielding to tidal forces.

An appendix is added on the general equations of tidal motion for seas on a yielding nucleus.

On the Number of Primes of the same Residuacity: Lt.-Col. Cunningham.

This paper presents the results of a count of the numbers (μ, m) of odd primes (p), which satisfy the congruences

$$y^{1/\nu(p-1)} \equiv +1, \quad y^{1/n(p-1)} \equiv +1 \quad (\text{mod } p),$$

for certain bases (y), through certain ranges (R) of the natural numbers, where

$\nu =$ the maximum factor of $(p-1)$ possible to y, $n =$ *any* factor of ν;

μ is the number with ν, m is the number with n.

The counts of μ were made for all values of ν for the eight bases

$$y = 2, 3, 5, 6, 7, 10, 11, 12,$$

for the range $R = 1$ to 10^4; and also for $y = 2$ and 10 for each successive range of 10000 up to 10^5. The counts of m were made for all values of $n = 1$ to 40 for the same bases and ranges as for μ.

Comparing μ, m with the number (M) of primes (p) of form

$$p = (\nu\varpi + 1),$$

$(n\varpi+1)$ in the same range, the chief results were the following as *approximate* rules (in *large ranges* of numbers)

$$\nu = 1 \text{ (}y\text{ a primitive root)}, \quad \mu = \frac{1}{2.5} \text{ to } \frac{1}{2.8} M.$$

$$\nu = 2 \text{ (}y\text{ a quadratic root)}, \quad \mu = \frac{1}{3.2} \text{ to } \frac{1}{3.6} M.$$

$$m = \frac{1}{n} M, \text{ nearly } \left(\text{but usually } < \frac{1}{n} M\right) \text{ [except as below].}$$

$$m = \frac{1}{n} 2M, \text{ nearly } \left(\text{but usually } < \frac{1}{n} 2M\right), \text{ with } n = 8i, \text{ for base 2.}$$

$$m = \frac{1}{n} 2M, \text{ nearly; in some cases, with } n = ky, \text{ when } y > 2.$$

Some Results on the Form near Infinity of Real Continuous Solutions of a certain Type of Second Order Differential Equations: Mr. R. H. Fowler.

This paper is the outcome of an attempt to obtain for the differential equation

$$P(xyy'y'') = 0,$$

where P is a polynomial with real coefficients, results analogous to those obtained by Mr. Hardy (*Proc. London Math. Soc.*, Ser. 2, Vol. 10, pp. 451–468) for the equation

$$P(xyy') = 0.$$

The following theorem is proved:—*If $y = y(x)$ is any real continuous solution (with real continuous differential coefficients of the first two orders) of the equation*

$$y'' = P(x, y),$$

where P is a polynomial in x and y, then either there exists a K such that

$$y(x) = O(x^k),$$

or else there exists real numbers A and ρ, of which ρ is rational, such that

$$y(x) = e^{Ax^{\rho}(1+\epsilon)} \quad (\epsilon \to 0).$$

The latter type cannot exist unless the degree of P in y is unity.

Various generalizations of this theorem are considered; in particular it is shown that the theorem remains true when a rational function of x and y is substituted for $P(x, y)$.

The Potential of a Uniform Convex Solid possessing a Plane of Symmetry, with Application to the Direct Integration of the Potential of a Uniform Ellipsoid: Dr. S. Brodetsky.

The difficulty of finding the potential of a uniform solid at an external point consists in the fact that the integral representing the potential involves limits which are complicated functions of position. This renders the direct integration impossible, except in very few special cases. Thus the potential at an external point of a uniform ellipsoid has hitherto been calculated only by means of special methods and devices. The nearest approach to a direct method is the discontinuous factor used by Dirichlet, Kronecker, and Hobson (see *Proc. London Math. Soc.*, Old Series, Vol. XXVII). The object of this paper is to devise a method of integration that shall overcome this difficulty and give us a general direct integrating process.

The potential of a uniform straight rod of line density m is

$$m \log (1+e)/(1-e),$$

where e is the excentricity of the ellipse having the ends of the rod as foci and passing through the point at which the potential is calculated. We split up the solid into elementary rods perpendicular to the plane of symmetry, and we find its potential at any point in the form

$$V = \sigma \iint r\,dr\,d\phi \log (1+e)/(1-e),$$

the double integral being taken for all the rods in the body, and r, ϕ being measured in the plane of symmetry. The body under consideration being convex, it follows easily that there is only one maximum value for e corresponding to any given point at which the potential is to be found; further, that the rods giving any particular value of e less than the maximum lie on a convex cylinder surrounding the rod giving the maximum value of e. Still measuring in the plane of symmetry with the projection of the rod giving maximum e as a new origin for r', ϕ', we get

$$V = -\sigma \int_0^E \int_0^{2\pi} r'\,dr'\,d\phi' \log(1+e)/(1-e),$$

E being the maximum value of e. We now take e as an independent variable instead of r', and integrate partially with respect to e. We obtain, after dropping a vanishing term,

$$V = \sigma \int_0^E \int_0^{2\pi} \frac{r'^2}{1-e^2}\,de\,d\phi'.$$

For an internal point (ξ, η, ζ), $E = 1$, and the equation for r'^2 in terms of e and ϕ' is

$$\frac{1}{e^2} f(r' \cos\phi' + \xi,\ r' \sin\phi' + \eta) = \zeta^2 + r'^2/(1-e^2),$$

the equation to the surface of the solid being

$$z^2 = f(x, y).$$

For an external point, we first have to find the coordinates (ξ_0, η_0) of the rod giving maximum E, from the equations

$$\frac{1}{e^2} f(x, y) = \xi^2 + \{(x-\xi)^2 + (y-\eta)^2\}/(1-e^2),$$

$$\frac{1}{e^2}\, \partial f/\partial x = 2(x-\xi)/(1-e^2), \qquad \frac{1}{e^2}\, \partial f/\partial y = 2(y-\eta)/(1-e^2).$$

These equations give us E, ξ_0, η_0, and we get for r'^2 the equation

$$\frac{1}{e^2} f(r' \cos\phi' + \xi_0,\ r' \sin\phi' + \eta_0)$$
$$= \xi^2 + \{(r' \cos\phi' + \xi_0 - \xi)^2 + (r' \sin\phi' + \eta_0 - \eta)^2\}/(1-e^2).$$

The method does not simplify the process in the case of an internal point. But for an external point the method of this paper introduces appreciable simplification. Having solved the equations giving E, ξ_0 and η_0, and having found r'^2 in terms of e and ϕ', the integration is quite straightforward, and the limits are simple and well defined. It is worth noticing that E is really a solution of

$$\int_0^{2\pi} r'^2 d\phi' = 0.$$

The uniform ellipsoid can be treated very simply indeed by this method, as it is found that in this special case we do not need the actual values of ξ_0, η_0, and E is found by equating $\int_0^{2\pi} r'^2 d\phi'$ expressed in terms of e to zero.

Incidentally, the analysis used in this paper leads to some interesting properties of confocal surfaces, including confocal conicoids. In the case of confocal conicoids these properties can be obtained by elementary methods, but I do not think they have been noticed before.

Proof of the Complementary Theorem: Prof. J. C. Fields.

In a recent paper (*Proc. London Math. Soc.*, Ser. 2, Vol. 12, pp. 218–

235) the writer deduced, with reference to an arbitrary fundamental equation $f(z, u) = 0$, the expression

$$ni_\kappa + \sum_{s=1}^{r_\kappa} \tau_s^{(\kappa)} \nu_s^{(\kappa)} - \tfrac{1}{2} \sum_{s=1}^{r_\kappa} \left(\mu_s^{(\kappa)} - 1 + \frac{1}{\nu_s^{(\kappa)}}\right) \nu_s^{(\kappa)},$$

for the number of the conditions imposed by a set of orders of coincidence $\tau_1^{(\kappa)}, \ldots, \tau_{r_\kappa}^{(\kappa)}$ on the general rational function of (z, u), which can be represented in the form $(z-a_\kappa)^{-i_\kappa}((z-a_\kappa, u))$, the integer i_κ being taken sufficiently large. Representing any pair of complementary bases by (τ) and $(\bar{\tau})$, we know that 0 is the value of the principal residue relative to the value $z = \infty$ in the products of the general rational function built on the basis $(\bar{\tau})$ by the general rational functions conditioned by the partial bases $(\tau)'$ and $(\tau)^{(\infty)}$ respectively. This fact, combined with the result cited above, enables us to derive the inequality

$$N_{\bar{\tau}} \leqslant N_\tau - n + \sum_\kappa \sum_{s=1}^{r} \tau_s^{(\kappa)} \nu_s^{(\kappa)} - \tfrac{1}{2} \sum_\kappa \sum_{s=1}^{r_\kappa} \left(\mu_s^{(\kappa)} - 1 + \frac{1}{\nu_s^{(\kappa)}}\right) \nu_s^{(\kappa)},$$

where N_τ and $N_{\bar{\tau}}$ represent the numbers of arbitrary constants involved in the general rational functions built on the bases (τ) and $(\bar{\tau})$ respectively. Interchange of (τ) and $(\bar{\tau})$ in this inequality gives us a second inequality. Addition of the corresponding sides of the two inequalities shews us the inadmissibility of the unequal sign in either of the inequalities. The two inequalities then both become equalities from whose combination we immediately derive the complementary theorem

$$N_\tau + \tfrac{1}{2} \sum_\kappa \sum_{s=1}^{r_\kappa} \tau_s^{(\kappa)} \nu_s^{(\kappa)} = N_{\bar{\tau}} + \tfrac{1}{2} \sum_\kappa \sum_{s=1}^{r_\kappa} \bar{\tau}_s^{(\kappa)} \nu_s^{(\kappa)}.$$

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THE LONDON MATHEMATICAL SOCIETY.

Financial Report by the Treasurer for the year 1912–13 (Nov. 14th, 1912, to Nov. 13th, 1913).

GENERAL CASH ACCOUNT OF THE SOCIETY. Dr.

Trust Funds.

	£	s.	d.	£	s.	d.	
1. Lord Rayleigh's Fund	1000	0	0 ...				Terminable Annuity of £50. 8s. 8d. (Class B, deduction for Sinking Fund) Great Indian Peninsula Railway.
				200	0	0	Guaranteed 3 per Cent. Stock of same Railway Co.
2. De Morgan Medal Fund	102	5	9	60	0	0	Great Western Railway 5 per Cent. Preference Stock.

Signed on behalf of the Council by A. E. H. LOVE,
A. E. WESTERN.

I report to the Members that I have obtained all the information and explanation I required as Auditor, and that I have examined the above Accounts with the books, and that, to the best of my information and of the explanations given to me, such Accounts are properly drawn up so as to exhibit a true and correct view of the state of the affairs of the Society as shown by the books of the Society.

18*th November*, 1913.

J. G. LEATHEM.

The London Mathematical Society.

RECORDS OF PROCEEDINGS AT MEETINGS

SESSION NOVEMBER, 1913–JUNE, 1914.

Thursday, January 22nd, 1914.

Prof. A. E. H. LOVE, President, in the Chair.

Present ten members and a visitor.

The minutes of the last meeting were read and confirmed.

Mr. T. L. Wren was elected and admitted a member of the Society.

The following papers were communicated:—

(i) A Generalisation of the Euler-Maclaurin Sum Formulæ, (ii) The Deduction of the Formulæ of Mechanical Quadrature from the Generalised Euler-Maclaurin Sum Formulæ, (iii) A Generalisation of certain Sum Formulæ in the Calculus of Finite Differences: Mr. S. T. Shovelton.

On Binary Forms: Dr. A. Young.

On Darboux's Method of Solution of Partial Differential Equations of the Second Order: Mr. J. R. Wilton.

The President made an informal communication "On the Potential due to the Distribution on an Electrified Circular Disc."

ABSTRACTS.

A Generalisation of the Euler-Maclaurin Sum Formulæ: Mr. S. T. Shovelton.

The formula is obtained by using the operator $\phi(\Delta)$, where

$$\phi(\Delta) \equiv 1 - \frac{\Delta}{2} + \frac{\Delta^2}{3} - \frac{\Delta^4}{4} + \dots,$$

and it is shown that

$$\Sigma_a^b v_x = \int_a^b v_x + \phi_1(0)(v_b - v_a) + \frac{\phi_2(0)}{2!}(v'_b - v'_a) + \dots + \frac{\phi_n(0)}{n!}(v_b^{(n-1)} - v_a^{(n-1)}) \dots,$$

where $$\phi_n(0) = \phi(\Delta)\, 0^n.$$

A more general formula of summation is then proved in the form

$$h(v_{a+\kappa h} + v_{a+h+\kappa h} + \dots + v_{b+\kappa h-h})$$
$$= \int_a^b v_x dx + h\phi_1(\kappa)(v_b - v_a) + \frac{h^2\phi_2(\kappa)}{2!}(v'_b - v'_a) + \dots,$$

where the remainder after $2(n+1)$ terms is

$$-\frac{h^{2n+1}}{(2n)!}\left[\int_0^{1-\kappa} \phi_{2n}(z+\kappa) \sum_{r=1}^{r=(b-a)h} v_{a+rh-zh}^{(2n)}\, dz \right.$$
$$\left. + \int_{1-\kappa}^{1} \phi_{2n}(z+\kappa-1) \sum_{r=1}^{r=(b-a)h} v_{a+rh-zh}^{(2n)}\, dz\right],$$

$\phi_n(\kappa)$ is $\phi(\Delta)\kappa^n$, and $\phi_n(\kappa) - \phi_n(0)$ is shown to be the Bernoullian function of degree n. By putting $v_x = e^x$ it follows that $\phi_n(\kappa)$ is the coefficient of h^n in $\dfrac{he^{\kappa h}}{e^h - 1}$, and that

$$\phi_{2n}(0) = (-1)^{n-1} B_n.$$

Defining a new function $\phi_n^r(\kappa)$ by the equation

$$\phi_n^r(\kappa) = [\phi(\Delta)]^r \kappa^n,$$

it is shown that

$$h^r D^r u_{x+\kappa h} = \Delta^r u_x + h\phi_1^r(\kappa)\Delta^r u'_x + \frac{h^2\phi_2^r(\kappa)}{2!}\Delta^r u''_x + \dots,$$

from which we get

$$h^r \Sigma^r u_{x+\kappa h} = \int^{(r)} u_x (dx)^r + h\phi_1^r(\kappa) \int^{(r-1)} u_x (dx)^{r-1} + \dots.$$

The function $\phi_m^r(\kappa)$ possesses properties analogous to those of the Bernoullian function—amongst others $\phi_m^r\left(\dfrac{r}{2}\right) = 0$ if m is odd,

$$\phi_m^r(\kappa) = (-1)^m \phi_m^r(r - \kappa),$$

and $(-1)^m \phi^r_{2m}\left(\frac{r}{2}\right)$ is positive. Thus taking $\kappa = \frac{r}{2}$ in the series for $D^r u_{x+\kappa h}$, we have

$$h^r D^r u_{x+\frac{1}{2}rh} = \Delta^r u_x + \frac{h^2 \phi^r_2\left(\frac{r}{2}\right)}{2!} \Delta^r u''_x + \dots,$$

and it is shown that

$$x^r \operatorname{cosec}^r x = 1 - \frac{2^2 \phi^r_2\left(\frac{r}{2}\right)}{2!} x^2 + \frac{2^4 \phi^r_4\left(\frac{r}{2}\right)}{4!} x^4 - \dots.$$

If u_x is put equal to e^x we find

$$\frac{h^r e^{\kappa h}}{(e^h - 1)^r} = 1 + h\phi^r_1(\kappa) + \frac{h^2}{2!}\phi^r_2(\kappa) + \dots,$$

and it follows that

$$\frac{x^r \cos(2\kappa - r)x}{\sin^r x} = 1 - \frac{2^2 \phi^r_2(\kappa)}{2!} x^2 + \frac{2^4 \phi^r_4(\kappa)}{4!} x^4 - \dots,$$

and $$\frac{x^r \sin(2\kappa - r)x}{\sin^r x} = 2\phi^r_1(\kappa)\, x - \frac{2^3 \phi^r_3(\kappa)}{3!} x^3 + \dots.$$

The function $\phi^r_{2m}\left(\frac{r}{2}\right)$ satisfies the reduction formula

$$(r-1)(r-2)\,\phi^r_{2m}\left(\frac{r}{2}\right)$$
$$= (r-2m-1)(r-2m-2)\,\phi^{r-2}_{2m}\left(\frac{r}{2} - 1\right) - \frac{m(2m-1)}{2}(r-2)^2 \phi^{r-2}_{2m-2}\left(\frac{r}{2} - 1\right),$$

from which a table of its values can be formed. Also

$$r\phi^{r+1}_m(\kappa) = (r-m)\,\phi^r_m(\kappa) + m(\kappa - r)\,\phi^r_{m-1}(\kappa),$$

from which a table of the values of $\phi^r_m(0)$ can be formed.

The expansions of $\tan x$ and $\sec x$ are found, and from the latter it appears that

$$E_{2n} = (-1)^{n-1} \frac{2^{4n+2}\phi_{2n+1}\left(\frac{1}{4}\right)}{(2n+1)}$$
$$= \frac{(-1)^{n-1}}{(2n+1)} \left\{1 - 2(2n+1) + {}^{2n+1}C_2\, 4^2 B_1 - {}^{2n+1}C_4\, 4^4 B_2 + \dots\right\}.$$

Other expansions are

$$x^r \cot^r x = 1 + \Sigma(-1)^n \frac{x^{2n}}{(2n)!} \left\{ 2^{2n} \phi_{2n}^r(0) + r 2^{2n-1} . 2n \phi_{2n-1}^{r-1}(0) \right.$$
$$\left. + {}^rC_2 . 2^{2n-2} (2n)^{(2)} \phi_{2n-2}^{r-2}(0) + \dots \right\},$$

and also can be expressed in terms of $\phi_{2n}^{2m}(m)$, $\phi_{2n-2}^{2n-2}(m-1)$, ..., when r is of the form $2m$ or $2m+1$,

$$\tan^2 x = \left\{ \Delta^r - 2r \frac{\Delta^{r+1}}{4} + \frac{2r(2r+3)}{2!} \frac{\Delta^{r+2}}{4^2} \right.$$
$$\left. - \frac{2r(2r+4)(2r+5)}{3!} \frac{\Delta^{r+3}}{4^3} \dots \right\} \begin{matrix} (-1)^{\frac{1}{2}r} \cos(4x . \\ (-1)^{\frac{1}{2}(r-1)} \sin(4x . 0) \end{matrix},$$

according as r is even or odd, the Δ's operating on the powers of zero,

$$\sec^r x = \left\{ 1 - r \frac{\Delta}{4} + \frac{r(r+3)}{2!} \frac{\Delta^2}{4^2} - \dots \right\} \left[1 + r \frac{\Delta}{4} + \frac{r(r-4)}{2!} \frac{\Delta^2}{4^2} + \dots \right]$$
$$\times \cos(4x . 0).$$

The Deduction of the Formulæ of Mechanical Quadrature from the Generalised Euler-Maclaurin Sum Formulæ: Mr. S. T. Shovelton.

Several of the well-known formulæ, such as Weddle's and G. F. Hardy's, are very easily obtained by the use of the sum formula, and new ones are given, one of which is

$$\int_0^{10} v_x dx = \tfrac{5}{126} [8(v_0 + v_{10}) + 35(v_1 + v_2 + v_7 + v_9) + 15(v_2 + v_4 + v_6 + v_8) + 36 v_5],$$

which has an error of less than (1—7)-th of Weddle when applied to the same range.

The generalised sum formula can be used to give approximate values of definite integrals in almost endless variation, but few of the results are of much practical value. A formula which is useful when fourth differences are small is

$$\int_0^{10} v_x dx = \tfrac{5}{6} [13(v_2 + v_4 + v_6 + v_8) - 20(v_3 + v_7)].$$

A Generalisation of certain Sum Formulæ in the Calculus of Finite Differences: Mr. S. T. Shovelton.

In this paper an investigation is given for the expression of $\Sigma^r a^x \phi(x + \kappa h)$ in terms of the differential coefficients of $\phi(x)$. The result

can be best expressed in the form

$$(a^h-1)^r\, a^x\, \phi(x+\kappa h)$$
$$=\Delta^r a^x \phi(x)+h\theta_1^r(\kappa)\,\Delta^r a^x \phi'(x)+\ldots+\frac{h^n}{n!}\theta_n^r(\kappa)\,\Delta^r a^x \phi^{(n)}(x)+R_n,$$

where Δ refers to intervals of h in x, and $\theta_n^r(\kappa)$ is defined by the equation

$$\theta_n^r(\kappa)=[\theta(\Delta)]^r\kappa^n=\left[1-\frac{a^h}{a^h-1}\Delta+\left(\frac{a^h}{a^h-1}\right)^2\Delta^2-\ldots\right]^r\kappa^n,$$

in which the differences refer to intervals of unity in κ. The remainder assumes a somewhat untractable form for general values of r, but when r is unity is equal to

$$-\frac{h^{n+1}}{n!}\left[\int_0^{1-\kappa}\theta_n(\kappa+z)\,a^{x+h}\phi^{(n+1)}(x+h-zh)\,dz\right.$$
$$\left.+\int_{1-\kappa}^1\theta_n(\kappa+z-1)\,a^x\phi^{(n+1)}(x+h-zh)\,dz\right].$$

When $a=-1$ and h is an odd integer, $\theta_n^r(\kappa)$ possesses properties analogous to those of $\phi_n^r(\kappa)$ defined in my paper on the Euler-Maclaurin formula. With these values we obtain

$$2\left[\phi(a+\kappa h)-\phi(a+h+\kappa h)+\phi(a+2h+\kappa h)+\ldots\right.$$
$$\left.+(-1)^{(p-1)}\phi(a+\overline{p-1+\kappa}\,h)\right]$$
$$=\{\phi(a)-(-1)^p\phi(a+ph)\}+\ldots+\frac{h^n\theta_n(\kappa)}{n!}[\phi^{(n)}(a)-(-1)^p\phi^{(n)}(a+ph)]+R_n,$$

and R_n may be made to vanish by suitably choosing n when $\phi(x)$ is a polynomial in x.

From the general result given above it follows that

$$\frac{(a-1)^r\,e^{\kappa h}}{(ae^h-1)^r}=1+h\theta_1^r(\kappa)+\frac{h^2}{2!}\theta_2^r(\kappa)+\ldots,$$

where a is written for a^h. When $a=-1$, the most useful value of κ is $\frac{r}{2}$, for $\theta_{2n+1}^r\left(\frac{r}{2}\right)=0$ and $\theta_{2n}^r\left(\frac{r}{2}\right)$ can be expressed in terms of the corresponding functions of $2n$ and $(2n+2)$ of order $(r-1)$. $\theta_{2n}^r\left(\frac{r}{2}\right)$ can also be calculated directly from

$$\theta_{2n}^r\left(\frac{r}{2}\right)=\frac{1}{2^{2n}}\left[1-r\frac{\delta^2}{2}+\frac{r(r+1)}{2!}\frac{\delta^4}{4}-\ldots\right]0^{2n},$$

where $$\delta=\frac{\Delta}{E^{\frac{1}{2}}}.$$

We readily deduce that

$$\sec^r x = \left[1 - \frac{r}{1}\,\frac{\delta^2}{2} + \frac{r(r+1)}{2!}\,\frac{\delta^4}{4} - \dots\right] \cos(x\,.\,0),$$

and that

$$\tan^r x = \delta^r \left[1 - \frac{r}{2}\,\frac{\delta^2}{4} + \frac{r(r+2)}{2\,.\,4}\,\frac{\delta^4}{16} - \dots\right]\left[\frac{x^r\,.\,O^r}{r!} - \frac{2^2 x^{r+2}\,O^{r+2}}{(r+2)!} + \dots\right].$$

Hence $$E_{2n} = (-1)^n \left[1 - \frac{\delta^2}{2} + \frac{\delta^4}{4} - \dots\right] O^{2n},$$

and $$B_n = \frac{(-1)^{n-1}\,n}{2(2^{2n}-1)} \left[\delta - \frac{1}{2}\,\frac{\delta^3}{4} + \frac{1\,.\,3}{2\,.\,4}\,\frac{\delta^5}{16} - \dots\right] O^{2n-1}.$$

Tables of the values of $\delta^{2m}\,O^{2n}$ and $\delta^{2m+1}\,O^{2n+1}$ can be readily constructed from the equation

$$\delta^p\,O^q = \frac{p^2}{4}\,\delta^p\,O^{q-2} + p(p-1)\,\delta^{p-2}\,O^{q-2}.$$

On Binary Forms: Dr. A. Young.

In a paper on perpetuants, Grace applied the symbolical method to discover the irreducible types; his result was that all perpetuants can be expressed in terms of the forms

$$(a_1 a_2)^{\lambda_2}\,(a_1 a_3)^{\lambda_3} \dots (a_1 a_\delta)^{\lambda_\delta},$$

where $$\lambda_r \geqslant 2^{\delta - r} \quad (r = 2, 3, \dots, \delta).$$

The attempt is made to solve the same problem here for irreducible covariant types of quantities of finite order, by multiplying all such covariants by the symbolical expression $a_{1_x}^{w-n_1}$, where w is the weight of the covariant $\gamma a_{1_x}^{n_1}$ is one of the quantics. We can thus express all our covariants as linear functions of the forms

$$(a_1 a_2)^{\lambda_2}\,(a_1 a_3)^{\lambda_3} \dots (a_1 a_\delta)^{\lambda_\delta}.$$

By means of this expression we find a certain set of covariant types of degree δ, which we write

$$(\lambda_2, \lambda_3, \dots, \lambda_\delta)$$

completely defined by the arguments, which are linearly independent, and in terms of which all can be linearly expressed.

The method used by Grace is not suitable here. So a suitable method of dealing with the covariants is developed. For the sake of the analysis the case of perpetuants is first discussed, and Grace's results are obtained.

Also the complete system of syzygies of the first kind is obtained for perpetuants.

On Darboux's Method of Solution of Partial Differential Equations of the Second Order: Mr. J. R. Wilton.

By a slight modification of Darboux's method of obtaining intermediate integrals of any given order in the case of the equation

$$r+f(x,\, y,\, z,\, p,\, q,\, s,\, t) = 0,$$

we may assume, as the form of an integral of the n-th order ($n > 2$),

$$p_{1,\,n-1}+mp_{0,\,n}+v = 0,$$

where
$$p_{hk} = \frac{\partial^{h+k} z}{\partial x^h \partial y^k},$$

m is either root of the characteristic equation, and v is a function of $x, y, z, p, q, \ldots, p_{1,\,n-2}, p_{0,\,n-1}$. If, however, $n = 2$, the form of the integral is

$$u+v\,(x,\, y,\, z,\, p,\, q) = 0,$$

where
$$\frac{\partial u}{\partial s} = \mu, \quad \frac{\partial u}{\partial t} = m\mu.$$

It is found that the number of independent equations which must be satisfied by v is two if n is greater than two, but when $n = 2$ the number depends on the nature of the given equation.

The assumption of this form of the intermediate integral leads directly to the determination of special integrals, such as Serret's integral of

$$rt-s^2+a^2(1+p^2+q^2)^2 = 0.$$

It will also be found to involve rather less labour than is usual in the determination of general intermediate integrals when such exist.

If, in the two equations which v must satisfy, when $n > 2$, we put

$$\frac{\partial v}{\partial \xi} = -\frac{\partial u}{\partial \xi}\Big/\frac{\partial u}{\partial v},$$

where ξ is any one of the independent variables, they become homogeneous and may be denoted by $\Delta_1 u = 0$, $\Delta_2 u = 0$. If $\Delta_{12} u = \Delta_1 \Delta_2 u - \Delta_2 \Delta_1 u$, it is found that, if $n > 3$,

$${}_1\Delta_{12} u \equiv \Delta_1 \Delta_{12} u - \Delta_{12} \Delta_1 u = 0,$$

where $\Delta_1 u = 0$ is that one of the two equations which involves no differential coefficients of u other than $\frac{\partial u}{\partial v}$, $\frac{\partial u}{\partial p_{0,n}}$, $\frac{\partial u}{\partial p_{1,n-1}}$. This fact considerably diminishes the number of independent conditions which must be satisfied in order that the equations for v should possess a common integral.

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The London Mathematical Society.

RECORDS OF PROCEEDINGS AT MEETINGS.

SESSION NOVEMBER, 1913–JUNE, 1914.

Thursday, February 12*th*, 1914.

Prof. H. F. BAKER, Vice-President, in the Chair.

Present seventeen members and a visitor.

The minutes of the last meeting were read and confirmed.

Messrs. W. E. H. Berwick and A. G. Veitch were elected members.

Professor S. B. McLaren was admitted into the Society.

The Secretaries reported that during the last Session the Society had lost seven members and that eight new members had been elected; thus the total number of members was 306 at the end of that Session as against 305 at the beginning.

The following papers were communicated :—

Exhibition and Explanation of some Models illustrating Kinematics: Mr. G. T. Bennett.

Formulæ for the Spherical Harmonic $P_n^{-m}(\mu)$, when $1-\mu$ is a Small Quantity: Prof. H. M. Macdonald.

The Representation of the Symmetrical Nucleus of a Linear Integral Equation: Prof. E. W. Hobson.

Fitting of Polynomials by the Method of Least Squares (Second Paper): Dr. W. F. Sheppard.

The Differential Geometry of Point-Transformations between Two Planes: Mr. H. Bateman.

ABSTRACTS.

On the Representation of the Symmetrical Nucleus of a Linear Integral Equation: Prof. E. W. Hobson.

This paper is concerned with the relation between the symmetrical nucleus of the integral equation

$$f(s) = \phi(s) - \lambda \int_a^b K(s, t)\, \phi(t)\, dt,$$

and the series $$\sum_{n=1} \frac{\phi_n(s)\, \phi_n(t)}{\lambda_n};$$

where λ_n denotes a characteristic number, and $\phi_n(s)$ the corresponding normal characteristic function. Cases are considered in which $K(s, t)$ has discontinuities.

It is shewn that a nucleus $K(s, t)$, such that $\int_a^b \{K(s, t)\}^2 dt$ is a limited function of s, exists, such as to have prescribed characteristic functions and numbers $\{\phi_n(s)\}$, $\{\lambda_n\}$; provided the series $\sum_{n=1} \left\{\frac{\phi_n(s)}{\lambda_n}\right\}^2$ converges for each value of s to a value which is a limited function of s, in (a, b). It is also shewn that the function $K(s, t)$, determined in accordance with the prescribed conditions is unique, except for equivalent functions not differing essentially from it. Several theorems are deduced from this result, and a simple proof of Mercer's theorem is obtained, that when $K(s, t)$ is continuous, and all the numbers $\{\lambda_n\}$ are positive, the series $\sum_{n=1} \frac{\phi_n(s)\, \phi_n(t)}{\lambda_n}$ converges uniformly to $K(s, t)$.

By means of an extension of a well known theorem due to Hilbert, to the case of a discontinuous nucleus, an extension of Mercer's theorem is obtained, which applies to all nuclei that are not infinitely discontinuous on the line $s = t$.

It is shewn that, in the case in which the repeated function of $K(s, t)$ is continuous, the nucleus $K(s, t)$ may be expressed as the sum of two functions $K^{(1)}(s, t)$, $K^{(2)}(s, t)$ that are orthogonal to one another, and are such that $K^{(1)}(s, t)$ has for its sole characteristic numbers those characteristic numbers of $K(s, t)$ that are positive, whereas $K^{(2)}(s, t)$ has for its sole characteristic numbers those belonging to $K(s, t)$ that are negative. The characteristic function corresponding to a characteristic number of $K^{(1)}(s, t)$, or of $K^{(2)}(s, t)$, is the same as that of $K(s, t)$ corresponding to the same characteristic number.

Fitting of Polynomials by Method of Least Squares: Dr. W. F. Sheppard.

If $\delta_1, \delta_2, \ldots, \delta_m$ are quantities containing errors, and ϵ_f is the improved value of δ_f, obtained by adding to it a linear compound of the δ's after δ_j, and choosing the coefficients so as to make the mean square of error of the total expression a minimum, then (i) the mean product of errors of ϵ_f and δ_t $(t > j)$ is zero, and (ii) the improved value of any linear compound of the δ's is the corresponding linear compound of the ϵ's. These two propositions give, by general reasoning, certain relations between the ϵ's for different values of j and f, and also a simple formula for the mean product of errors of ϵ_f and ϵ_g as the sum of $j-f+1$ or $j-g+1$ terms. Also, if $u_1, u_2, \ldots, u_m$ are connected with the δ's by linear relations, there are linear relations connecting the coefficients in the expression for ϵ_f in terms of the u's with the mean products of error of ϵ_f and the ϵ's.

These results are applied to the case (*Proceedings*, Ser. 2, Vol. 12, p. xliv) in which the u's satisfy the condition that their errors are independent and have all the same mean square, and the δ's are the advancing or central differences of the u's. Some questions in reference to the more general case are also investigated.

The Differential Geometry of Point-Transformations between Two Planes: Mr. H. Bateman.

In the neighbourhood of a given place any such transformation is equivalent to a projective one, in general, *i.e.*, unless the Jacobian vanishes or is infinite. The directions through a point are, in fact, altered in a projective manner. In this paper the approximation is carried a step further.

The pencil of lines through a point P in the original plane Π are mapped into a system of curves through the corresponding point p in the new plane π, and for three of the lines through P the curve arising has an inflexion at p. These are called inflexional lines, and from them spring inflexional curves everywhere tangent to an inflexional line. The properties of these are discussed and transformations are found for which the inflexional curves have assigned properties, *e.g.*, are such that at every point two of them coincide. Transformations which leave areas unaltered are discussed, and the results compared with the theory of the motion of an incompressible fluid in two dimensions.

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RECORDS OF PROCEEDINGS AT MEETINGS

SESSION NOVEMBER, 1913–JUNE, 1914.

Thursday, March 12th, 1914.

Prof. A. E. H. LOVE, President, in the Chair.

Present sixteen members and a visitor.

The minutes of the last meeting were read and confirmed.

The following papers were communicated:—

On the Rational Solutions of the Equation $X^3+Y^3+Z^3=0$ in Quadratic Fields: Prof. W. Burnside.

On Orthoptic and Isoptic Loci of Plane Curves: Prof. H. Hilton and Miss R. E. Colomb.

Normal Coordinates in Dynamical Systems: Dr. T. J. I'A. Bromwich.

On the Zeroes of Riemann's Zeta-Function: Mr. G. H. Hardy.

ABSTRACTS.

On the Rational Solutions of the Equation $X^3+Y^3+Z^3=0$ in Quadratic Fields: Prof. W. Burnside.

In this note attention was drawn to the fact that every solution of the equation

$$X^3+Y^3+Z^3=0,$$

in which X, Y, Z belong to the same quadratic field, while no one of them is zero, is obtained from the identity

$$(6k)^3+[3+\sqrt{-3(1+4k^3)}]^3+[3-\sqrt{-3(1+4k^3)}]^3\equiv 0,$$

on multiplying each term by $[\alpha+\beta\sqrt{-3(1+4k^3)}]^3$, and then taking for α, β, k any rational numbers, the value -1 for k being omitted.

If two solutions for which the relation

$$\frac{X}{X'} = \frac{Y}{Y'} = \frac{Z}{Z'}$$

do not hold, in whatever order X', Y', Z' are taken, be called distinct, the equation either admits no solution in an assigned quadratic field, or it admits an infinite number of distinct solutions.

On Orthoptic and Isoptic Loci of Plane Curves: Prof. Harold Hilton and Miss R. E. Colomb.

Little appears to have been written on this subject except that the degree of the orthoptic locus of a given curve has been found both in the general case and when the curve has two- or three-point contact with the line at infinity. In this paper are found all Plücker's numbers for the orthoptic locus in these cases. The nature of the intersections of the orthoptic locus with the given curve, and of the multiple points of the orthoptic locus are also investigated. For instance, if the given curve is the general curve of degree n and class m, with ι inflexions, the orthoptic locus is of degree $m(m-1)$ and class $m(m+n-3)$ with $m\iota$ cusps.

It is obvious that in general the orthoptic locus of a given curve is complicated, but, if the curve is specialized, the orthoptic locus may be fairly simple. This is especially the case if the curve has the line at infinity as a multiple tangent, so that pairs of points of contact form a harmonic range with the circular points. There are respectively 2, 3, 3, 3, 14, 38 types of curve with orthoptic locus, a straight line, a circle, a parabola, a conic, a cubic. a quartic.

A brief account is given of similar properties of isoptic loci in the simplest cases which can occur.

It is readily seen that we can deduce the properties of the locus of the intersection of two tangents, one drawn to each of two given curves and inclined at a given angle, and of the locus of the intersection of two normals to a given curve inclined at a given angle.

The results of the investigation are illustrated by several examples.

Normal Coordinates in Dynamical Systems: Dr. T. J. I'A. Bromwich.

Let the motion of a dynamical system performing small oscillations be

given by the differential equations

$$\left.\begin{array}{l} e_{11}x_1+e_{12}x_2+\ldots+e_{1n}x_n = 0 \\ e_{21}x_1+e_{22}x_2+\ldots+e_{2n}x_n = 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ e_{n1}x_1+e_{n2}x_2+\ldots+e_{nn}x_n = 0 \end{array}\right\} \quad (1)$$

The notation is that used by Lord Rayleigh (*Theory of Sound*, Vol. 1, § 82), namely, e_{rs} denotes the differential operator

$$e_{rs} = a_{rs}D^2+b_{rs}D+c_{rs};$$

but the symmetrical relation $e_{rs} = e_{sr}$ is not required, so that gyrostatic terms may be present, and the forces acting need not be conservative.

Then the solution of (1) is given by contour integrals

$$x_r = \frac{1}{2\pi\iota}\int e^{\lambda t}\xi_r\,d\lambda, \quad (2)$$

where $\xi_1, \xi_2, \ldots, \xi_n$ are functions of λ derived by solving the equations

$$\lambda_{11}\xi_1+\lambda_{12}\xi_2+\ldots+\lambda_{1n}\xi_n = p_1, \ \ldots, \quad (3)$$

where $$\lambda_{rs} = a_{rs}\lambda^2+b_{rs}\lambda+c_{rs},$$

and $p_1, p_2, \ldots, p_n$ are linear combinations of the initial displacements and velocities, such that

$$p_1 = \{a_{11}(\lambda+\delta)+h_{11}\}x_1+\{a_{12}(\lambda+\delta)+h_{12}\}x_2+\ldots$$
$$+\{a_{1n}(\lambda+\delta)+h_{1n}\}x_n, \ \ldots. \quad (4)$$

In the formulæ (4), $x_1, x_2, \ldots$ denote the *initial* displacements, $\delta x_1, \delta x_2, \ldots$ the *initial* velocities; and in the integrals (2), the path of integration in the λ-plane surrounds all the poles of the functions ξ, which are, of course, the roots of the determinant $|\lambda_{rs}| = 0$.

If the system is subject to additional impressed forces of the type $P_1e^{\mu t}, P_2e^{\mu t}, \ldots, P_ne^{\mu t}$, so that equations (1) become

$$e_{11}x_1+e_{12}x_2+\ldots+e_{1n}x_n = P_1e^{\mu t}, \ \ldots \quad (5)$$

(where μ may be real, imaginary, or complex), the solution corresponding to *zero* initial displacements and velocities is given by writing

$$p_1 = P_1/(\lambda-\mu), \ \ldots, \quad (6)$$

the path of integration now enclosing $\lambda = \mu$ in addition to all the roots of $|\lambda_{rs}| = 0$.

The solutions (2) can be shewn to reduce to the known solutions in terms of normal coordinates, when the equations (1) are specialised by the omission of one of the sets of letters (a), (b), (c); and thus (2) gives the extension to the general problem of the method of normal coordinates, which (as ordinarily presented) can only be used in special cases.

The extension to a continuous system (in which n tends to infinity) is also considered: the functions ξ_r then appear as integrals, instead of sums, and the resulting contour-integral reduces when $t = 0$ to the integral used by Prof. A. C. Dixon (*Proc. London Math. Soc.*, Ser. 2, Vol. 3, 1905; *Phil. Trans.*, 1911).

On the Zeroes of Riemann's Zeta-Function: Mr. G. H. Hardy.

It has been shown recently by Bohr and Landau (*Comptes Rendus*, January 12th, 1914) that, however small be the positive number δ, the majority of the zeroes of $\zeta(s) = \zeta(\sigma + ti)$ lie in the region $\frac{1}{2} - \delta < \sigma < \frac{1}{2} + \delta$. The object of this paper is to show that an infinity of the zeroes lie on the line $\sigma = \frac{1}{2}$, and so to advance one step further towards the proof of the "Riemann hypothesis" that *all* the zeroes, except the real negative zeroes $-2, -4, -6, \ldots$, lie on this line.

It may be observed that there is a numerical method, due to Lindelöf and Gram, and developed by Backlund (*Öfversigt af Finska Vetenskaps-Societetens Förhandlingar*, Vol. 54), by which it has been shown that there are exactly twenty-nine zeroes on the line $\sigma = \frac{1}{2}$ between $t = 0$ and $t = 100$; and that Backlund has also shown that there are no other complex zeroes whose imaginary part is positive and less than 100. These results are of great interest as evidence of the probable truth of Riemann's hypothesis; but they do not prove even that there are an infinity of zeroes on the critical line. Such a proof is supplied in the present paper.

The method adopted depends on Mellin's formula

$$e^{-y} = \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} \Gamma(u)\, y^{-u}\, du,$$

where $\mathfrak{R}(y) > 0$ and $\kappa > 0$. This formula is applied to the evaluation of a definite integral containing a function $\Xi(t)$ introduced by Riemann and closely connected with $\zeta(s)$. If $\zeta(s)$ has not infinitely many roots for which $\sigma = \frac{1}{2}$, $\Xi(t)$, which is real for real values of t, is ultimately of constant sign; and it is shown that this hypothesis leads to a contradiction.

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The London Mathematical Society.

RECORDS OF PROCEEDINGS AT MEETINGS

SESSION NOVEMBER, 1913–JUNE, 1914.

Thursday, April 23rd, 1914.

Prof. A. E. H. LOVE, President, in the Chair.

Present seven members.

Messrs. C. Jordan and J. Proudman were elected members.

The President referred to the death of Mr. G. M. Minchin, who had been a member of the Society since 1875.

The following paper was communicated:—

On a Modified Form of Pure Reciprocants possessing the Property that the Algebraical Sum of the Coefficients is Zero: Major P. A. MacMahon.

And Major MacMahon made an informal communication on "Lattice and Prime Lattice Permutations."

ABSTRACTS.

On a Modified Form of Pure Reciprocants possessing the Property that the Algebraical Sum of the Coefficients is Zero: Major P. A. MacMahon.

Instead of writing $$a_s = \frac{1}{(s+2)!}\frac{d^s y}{dx^s},$$

the author writes $$b_s = \frac{(s+1)!}{(2s+2)!}\frac{d^s y}{dx^s},$$

and represents a pure reciprocant as a homogeneous and isobaric function of

$$b_0, b_1, b_2, \ldots.$$

The algebraic sum of the coefficients in every pure reciprocant now vanishes, and the actual coefficients have much smaller numerical expressions.

Lattice and Prime Lattice Permutations: Major P. A. MacMahon.

We consider any assemblage of letters

$$\alpha^p \beta^q \gamma^r \ldots,$$

where $p, q, r, \ldots$ are in descending order of magnitude.

Certain permutations possess the property that if a line be drawn between *any two* letters, the letters to the left of the line are in some permutation of the assemblage

$$\alpha^{p'} \beta^{q'} \gamma^{r'} \ldots,$$

where $p', q', r', \ldots$ are in descending order of magnitude. These have been termed "lattice permutations" for a reason that has been given by the author in previous papers.

Consider in particular the assemblage

$$\alpha^p \beta^q.$$

It has been shewn that there are

$$\frac{(p+q)!}{(p+1)!\,q!}(p-q+1)$$

lattice permutations. This number in the coefficients of $x^p y^q$ is the expansion of

$$\frac{1-\dfrac{y}{x}}{1-x-y},$$

a crude generating function because it involves many terms which are not required. The exact generating function is obtained from the notion of prime lattice permutations. If we write down the lattice permutations of the assemblage

$$\alpha^2 \beta^2,$$

we find them to be

$$\alpha\alpha\beta\beta,$$

$$\alpha\beta . \alpha\beta,$$

and it is noticed that the second of these is divisible into two lattice permutations of smaller assemblages (as shewn by the dot), whereas the first

$\alpha\alpha\beta\beta$ is not so divisible. $\alpha\alpha\beta\beta$ is a prime lattice permutation, and $\alpha\beta.\alpha\beta$ a composite lattice permutation.

Similarly of the assemblage $\alpha^3\beta^3$ we find five lattice permutations, two of which are prime and three composite, viz.,

$$\alpha\alpha\alpha\beta\beta\beta,$$
$$\alpha\alpha\beta\alpha\beta\beta,$$

are prime, and

$$\alpha\alpha\beta\beta.\alpha\beta,$$
$$\alpha\beta.\alpha\alpha\beta\beta,$$
$$\alpha\beta.\alpha\beta.\alpha\beta,$$

composite.

If N_{xy} denote the generating function of the lattice permutations, and P_{xy} that of the prime permutations, slight reflection shews that

$$N_{xy}=\frac{1}{1-P_{xy}}.$$

Consider first the case $p=q$, it is seen that every lattice permutation of the assemblage

$$\alpha^p\beta^p$$

produces a prime permutation of the assemblage

$$\alpha^{p+1}\beta^{p+1},$$

by prefixing α and affixing β. Thus

Lattice Permutations.	Prime Permutations.
1	$\alpha\beta$
$\alpha\beta$	$\alpha\alpha\beta\beta$
$\alpha\alpha\beta\beta$	$\alpha\alpha\alpha\beta\beta\beta$
$\alpha\beta\alpha\beta$	$\alpha\alpha\beta\alpha\beta\beta$
$\alpha\alpha\alpha\beta\beta\beta$	$\alpha\alpha\alpha\alpha\beta\beta\beta\beta$
$\alpha\alpha\beta\alpha\beta\beta$	$\alpha\alpha\alpha\beta\alpha\beta\beta\beta$
$\alpha\alpha\beta\beta\alpha\beta$	$\alpha\alpha\alpha\beta\beta\alpha\beta\beta$
$\alpha\beta\alpha\alpha\beta\beta$	$\alpha\alpha\beta\alpha\alpha\beta\beta\beta$
$\alpha\beta\alpha\beta\alpha\beta$	$\alpha\alpha\beta\alpha\beta\alpha\beta\beta$
$\vdots$	$\vdots$

if therefore for the case $p=q$, u_{xy} be the generating function for the lattice permutations, xyu_{xy} is the generating function of the prime permutations.

Hence
$$u_{xy} = \frac{1}{1 - xyu_{xy}},$$

leading to
$$u_{xy} = \frac{1}{2xy} - \frac{1}{2xy}\sqrt{(1-4xy)}.$$

If $p \neq q$, there is only one additional prime permutation, viz., α, so that

$$x + xyu_{xy},$$

enumerates the prime permutations, and if v_{xy} be the function which enumerates the lattice permutations

$$v_{xy} = \frac{1}{1 - x - xyu_{xy}},$$

where
$$u_{xy} = \frac{1}{1 - xyu_{xy}}.$$

Eliminating u_{xy} or substituting the found expression for u_{xy}, we find

$$v_{xy} = \frac{1}{2x}\,\frac{\sqrt{(1-4xy)} + 2x - 1}{1 - x - y}.$$

In the case of the assemblage $\alpha^p \beta^q \gamma^r$,

the crude enumerating generating function is

$$\frac{\left(1 - \frac{y}{x}\right)\left(1 - \frac{z}{x}\right)\left(1 - \frac{z}{y}\right)}{1 - x - y - z}.$$

The attempt to form the exact generating function through the medium of the prime permutations has not yet proved successful. The theory of the prime permutations requires investigation.

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The London Mathematical Society.

RECORDS OF PROCEEDINGS AT MEETINGS

SESSION NOVEMBER, 1913–JUNE, 1914.

Thursday, May 14th, 1914.

Prof. A. E. H. LOVE, President, in the Chair.

Present seventeen members and two visitors.

The following papers were communicated :—

Diffraction by a Straight Edge: Prof. H. M. Macdonald.
Quadratic Forms and Factorization of Numbers: Hon. H. F. Moulton.
On the Reduction of Sets of Intervals: Prof. W. H. Young and Mrs. Young.
Diffraction of Tidal Waves on Flat Rotating Sheets of Water: Mr. J. Proudman.
The Algebraic Theory of Modular Systems: Dr. F. S. Macaulay.

ABSTRACTS.

Quadratic Forms and Factorization of Numbers: Hon. H. F. Moulton.

This paper describes a method for greatly diminishing the labour of finding whether a number M, which has one known representation X^2+DY^2, has any other $X'^2+DY'^2$, and of so finding the factors of M.

Take first the case where $X^2+4Y^2=M$. The ordinary methods of elimination show that if $M\equiv p^2 \bmod k$, where k is a prime, then X and also $X'\equiv \pm n \bmod k$, where n has one of a limited number of values, at most $\frac{k+1}{4}+1$. This paper shows that these values can be divided into two groups, and that if it is known that k is a residue of each factor of M, the value of $X' \bmod k$ must lie in the same group as that of $X \bmod k$.

It is also shown that the same method may apply to representations $M=X^2+DY^2$ if $D/k=1$, but in this case the various possible expressions of the factors m, m' of M by forms of determinant $-D$ must be considered, *e.g.*, $m=ax^2+2bxy+cy^2$, $m'=a'x'^2+2b'x'y'+c'y^2$, and the criterion for whether the X's lie in the same or different groups, is whether am and $a'm'$ are both residues or both non-residues of k. It is shown that the grouping is the same for all values of D such that $D/k=1$ if k is an odd prime, but if k is composite and $=k_1k_2$, these are different groupings according as $D/k_1=D/k_2=\pm 1$. The method can also be applied with certain modifications to representations of the form $M=AX^2+BY^2$, where $AB/k=1$. The grouping need only be tabulated for cases where $M\equiv 1 \bmod k$, as that for other cases can be derived by multiplication.

The fundamental theorem on which the method is based is that if a number M has two factors m, m' expressible as $ax^2+2bxy+cy^2$, ..., where $ac-b^2=a'c'-b'^2=D$, then $aa'M$ has two representations

$$X^2+DY^2,\quad X'^2+DY'^2,$$

and
$$am=\frac{1}{4(a'x'+b'y')^2}[(X+X'^2)+D(Y-Y'^2)]$$

$$=\frac{1}{4Dy'^2}[(X-X'^2)+D(Y+Y'^2)],$$

$a'm'$ has two similar values. If then

$$D/k=1 \quad\text{and}\quad am/k=a'm'/k=\pm 1,$$

then
$$[(X\pm X'^2)+D(Y\pm Y'^2)]/k=\pm 1,$$

according as
$$am/k=\pm 1.$$

The grouping for odd primes up to 25 is as follows, a semicolon showing the division of the groups.

Number ...	3	5	7	11	13	17	19	23
Grouping...	1 ; 0	1 ; 0	1, 0 ; 2	1, 5 ; 0, 3	1, 6 ; 0, 2	0, 1, 3 ; 4, 6	1, 2, 7 ; 0, 3, 4	0, 1, 8, 11 ; 4, 9, 10

Diffraction of Tidal Waves on Flat Rotating Sheets of Water: Mr. J. Proudman.

The methods of this paper are very similar to those already well known for two-dimensional problems in the diffraction of sound and electric waves.

Only free tidal motion of sheets of uniform depth is considered.

A complete solution is obtained for the case of the diffraction of a plane wave by a circular island, but the remaining solutions are all approximations. They are based on Lord Rayleigh's approximate theory of diffraction, and the method of conjugate functions is introduced so that Schwarz's method for conformal transformations becomes available. The problems considered are those of the diffraction of a plane wave by an elliptic island, by a semi-elliptic cape, by a rectangular bay, and by a passage between one sea and another.

On the Reduction of Sets of Intervals: Prof. W. H. Young and Mrs. Young.

It has been already pointed out that before sets of intervals were studied for their own sake, various writers had had occasion to make use of them, and had in this way virtually obtained the Heine-Borel theorem. Without entering again into these matters, a new moment is introduced, and it is shown how the careful study of the early documents leads to a new theorem, including the Heine-Borel theorem as a special case. The proof of this theorem is obtained by retaining all that is essential in Heine's proof of the property of uniform continuity, and rejecting that which is accessory.

The new theorem is not only more general than the Heine-Borel theorem, it leads to results unobtainable by application of the latter theorem alone. From it we deduce as easy corollaries, beside that already mentioned, Lusin's kindred theorem, used in the proof of his result that a continuous function cannot have an infinite differential coefficient at a set of points of positive content; Young's first lemma, used in our proof that a function with bounded derivates is the integral of any one of them; the tile theorem, used in various theorems on integration, as well as the earliest theorem on the reduction of a perfectly general set of overlapping integrals, given in 1902 in the *Proceedings* of this Society. These various and scattered theorems, and perhaps others also, are in this way classified systematically together, and proved without any use being made of the idea of transfinite ordinal types, or Cantor's numbers.

In the course of the work it was found that there was a flaw in the deduction of Lebesgue's lemma from Young's first lemma, given in a former number of the Society's *Proceedings*. Lebesgue's lemma remains therefore dependent on Cantor's numbers. The same error occurred in that one of the two proofs of Young's second lemma which was independent of Contor's numbers; a new proof is accordingly here supplied of a simpler nature. Thus the proof of all Lebesgue's results on derivates and their integrals and the extensions of these results already given without Cantor's numbers remain valid.

On the Algebraic Theory of Modular Systems: Dr. F. S. Macaulay.

The principal object of this paper is to supplement the account of the theory of modular systems given in the *Encyclopädie der Mathematischen Wissenschaften*. It also includes some properties which have not been published previously.

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The London Mathematical Society.

RECORDS OF PROCEEDINGS AT MEETINGS.

SESSION NOVEMBER, 1913–JUNE, 1914.

Thursday, June 11th, 1914.

Prof. A. E. H. LOVE, President, in the Chair.

Present thirteen members.

The President announced that the Council had awarded the De Morgan medal to Prof. Sir J. Larmor.

The following papers were communicated :—

A Problem of Diophantine Approximation : Mr. R. H. Fowler.

Some Theorems by Mr. S. Ramanujan : Mr. G. H. Hardy.

Proof of the general Borel-Tauber Theorem : Messrs. G. H. Hardy and J. E. Littlewood.

Theorems relating to Functions defined implicitly, with Applications to the Calculus of Variations : Prof. E. W. Hobson.

On the Differentiation of a Surface Integral at a Point of Infinity : Dr. J. G. Leathem.

On Mersenne's Numbers : Mr. R. E. Powers.

Free and Forced Longitudinal Tidal Motion in a Lake : Mr. J. Proudman.

ABSTRACTS.

A Problem of Diophantine Approximation: Mr. R. H. Fowler.

In a recent paper,* Messrs. G. H. Hardy and J. E. Littlewood have investigated the distribution of the points

$$(a^\nu\theta) \quad (\nu = 1, 2, \ldots, n),$$

where $(a^\nu\theta)$ is the fractional part of $a^\nu\theta$, and a is an integer, over the interval (0, 1) for large values of n. This is almost the same problem as the investigation of the distribution of the first n digits in the decimal expression for θ in the scale of a, and it is from this point of view, which is in that case the more interesting, that they solve the problem.

In this paper, I extend some of their results to the set of points $(\lambda_\nu\theta)$, resulting from any sequence of positive numbers $\lambda_1\lambda_2\ldots\lambda_n\ldots$ which satisfy the inequalities

$$\lambda_n/\lambda_{n-1} \geqslant \beta^{n^{-1+\xi}} \quad (\xi > 0,\ \beta > 1)$$

for all values of $n \geqslant n_0$. Roughly speaking, it may be said that my results hold for any sequence for which the increase of the n-th term is sufficiently regular, and faster than that of

$$\exp(n^\xi),$$

for some positive value of ξ.

The following is the principal theorem of the paper:—

THEOREM.—*If* $\{\lambda_n\}$ *be any sequence of positive numbers satisfying*

$$\lambda_n/\lambda_{n-1} \geqslant \beta^{n^{-1+\xi}} \quad (\xi > 0,\ \beta > 1),$$

if δ *be the length of any interval included in the interval* (0, 1), *and if* Δ_n *be the number of the first* n *numbers* $(\lambda_\nu\theta)$ *that fall inside* δ, *then*

$$\Delta_n \sim \delta n,$$

for all θ's *between* 0 *and* 1 *which do not belong to a set of measure zero.*

I conclude the paper by considering the extension of this theorem to the m-dimensional set of points

$$(\lambda_n\theta_1),\ (\lambda_n\theta_2),\ \ldots,\ (\lambda_n\theta_m).$$

Theorems relating to Functions defined implicitly, with Applications to the Calculus of Variations: Prof. E. W. Hobson.

* "Some Problems of Diophantine Approximation," *Acta Math.*, Vol. 37, pp. 155–190.

This paper contains extensions of the known theorems relating to the determination of values of a set of real variables $u_1, u_2, \ldots, u_n$, defined explicitly by means of a set of equations connecting them with m real variables $v_1, v_2, \ldots, v_n$; when one system of corresponding values of the two sets of variables is known.

The cases are treated in which the equations involve a set of parameters as well as the variables, and in which a set of systems of corresponding values of the two sets of variables is known, instead of only one such system. The results obtained are applied to proofs of theorems concerning the existence of Weierstrassian fields in the problems of the calculus of variations.

On the Differentiation of a Surface Integral at a Point of Infinity: Dr. J. G. Leathem.

The subject is the differentiation of a surface integral on a curved surface with respect to a displacement $\delta\lambda$ of a point O at which the subject of integration f has an infinity. Use is made of a lemma which constitutes a somewhat general relation between a surface integral and a line integral round the boundary curve T. Subject to certain restrictions on the form of f a differentiation formula is obtained in the shape

$$\frac{d}{d\lambda}\int^T f\,dS = \lim_{\eta\to 0}\left\{\int_\eta^T \frac{\partial f}{\partial\lambda}\,dS - \int_\eta f\cos(\delta\lambda, \nu)\,ds\right\},$$

where η is a vanishing cavity round O, and ν is the direction of the normal to η. If the original integral is absolutely convergent, the form of η is arbitrary; otherwise it is subject to restriction.

By way of illustration the theorem is employed to determine the tangential force at a point (i) in a sheet of gravitating matter of given surface density, (ii) in a double sheet. Incidentally a method of discussing the convergence and discontinuities of the potential due to a double sheet is explained.

On Mersenne's Numbers: Mr. R. E. Powers.

The purpose of Mr. Powers's paper is to show that the Mersenne number $2^{107}-1$ is prime, four mistakes having now been found in Mersenne's classification, viz., 2^p-1 proved composite for $p = 67$, and prime for $p = 61$, 89 and 107, contrary to his assertion. That $2^{107}-1$ is

a prime number is shown by means of the following theorem, which was proved by Lucas in 1878:—

If $N = 2^{4q+3}-1$ ($4q+3$ *prime*, $8q+7$ *composite*), *and we calculate the residues* (*modulo* N) *of the series*

$$3,\ 7,\ 47,\ 2207,\ \ldots,$$

each term of which is equal to the square of the preceding, diminished by two units: the number N *is prime if the residue* 0 *occurs between the* $(2q+1)$-*th and the* $(4q+2)$-*th term;* N *is composite if none of the first* $(4q+2)$ *residues is* 0.

The 106th term of the above series is congruent to 0 (modulo $2^{107}-1$), consequently the latter is a prime number.

Free and Forced Longitudinal Tidal Motion in a Lake: Mr. J. Proudman.

Following Prof. Chrystal, the solution of this problem is made to depend on that of the differential equation

$$\frac{d^2V}{dx^2}+\frac{\lambda}{p(x)}V = F(x), \tag{1}$$

between $x = 0$ and $x = a$, when V vanishes at these limits. Here λ is a constant, and $p(x)$, $F(x)$ are functions of x subject to certain very general conditions.

Take

$$I_n(\xi, \eta) = \frac{1}{a}\int_{s_n=\xi}^{\eta}\int_{s_{n-1}=\xi}^{s_n}\cdots\int_{s_2=\xi}^{s_3}\int_{s_1=\xi}^{s_2}\frac{(\eta-s_n)(s_n-s_{n-1})\ldots(s_2-s_1)(s_1-\xi)}{p(s_1)\,p(s_2)\ldots p(s_{n-1})\,p(s_n)} \times ds_1\,ds_2\ldots ds_{n-1}\,ds_n, \tag{2}$$

for $n > 0$, with $I_0(\xi, \eta) = (\eta-\xi)/a$, where $0 \leqslant \xi \leqslant \eta \leqslant a$.

Take also $$R(\xi, \eta, \lambda) = \sum_{n=0}^{\infty}(-\lambda)^n I_n(\xi, \eta). \tag{3}$$

Sufficient conditions will be given in the paper for the existence of (2) and the convergence of (3).

The *free modes* are then given by

$$V = R(0, x, \lambda_n), \tag{4}$$

where λ_n is such as to satisfy

$$R(0, a, \lambda_n) = 0, \tag{5}$$

and the *forced motion* is given by

$$V = -\frac{a}{R(0, a, \lambda)} \left\{ R(x, a, \lambda) \int_0^x R(0, s, \lambda) F(s)\, ds \right.$$
$$\left. + R(0, x, \lambda) \int_x^a R(s, a, \lambda) F(s)\, ds \right\}. \quad (6)$$

In the paper these statements will be proved, and the solution shown to arise naturally when the equation (1) is regarded as the limiting form of a certain difference equation. The connection of the solution with that of a particular Fredholm's equation will be indicated and various other relations proved.

It appears probable that, with a reasonable amount of labour, the solution can be applied, by means of approximate methods, to such forms of $p(x)$ as are provided by a bathymetric survey of a lake. This is at present being investigated.

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